General Equilibrium Theory

Elena del Mercato – Masters M2 MMMEF & APE

***

Paris – November 6, 2023

General presentation

The general equilibrium theory studies the interactions among heterogeneous agents on commodity and financial markets.

The course begins with an outline of the main properties of a competitive equilibrium in the classical Arrow-Debreu model.

The course then focuses on advanced questions arising from market imperfections and financial markets, such as:

- Externalities,
- Incomplete financial markets.

Part on "General equilibrium and Externalities"

Elena L. del Mercato

18 heures from Monday, September 11 to Monday, November 6.

- An overview of general equilibrium.
- Competitive equilibrium. Existence of an equilibrium. An overview of the other basic results (efficiency, regular economies).
- Competitive equilibrium with externalities. An overview of the basic results (existence, regular economies).
- Externalities and market failure; perfect internalization, Pareto improving policies.

Part on "General equilibrium and Financial Markets"

Jean-Marc Bonnisseau

18 heures from Monday, November 6 to Monday, December 18.

- The two period economy with uncertainty; risk aversion.
- Arrow securities, financial markets; real, nominal and numeraire assets.
- Absence of arbitrage opportunities, existence of present value vector, uniqueness, risk neutral probability.
- Complete and incomplete financial markets.
- Existence of a financial equilibrium for nominal and numeraire asset; generic existence for real assets.
- Behavior of the firms and incomplete markets.

Grading

- Attendance.
- Homework: one for Part 1, one for Part 2.
- Midterm exam on Part 1: one-hour written exam on a research paper.
- Final exam: written exam on both Parts 1 and 2.

Pedagogical space of the course

1. https://cours.univ-paris1.fr/
2. Click on UFR27.
3. In the list of Master courses, click on the link:

   General Equilibrium Theory - M2 MMMEF and M2 APE

Overview of an equilibrium model

- General formulation of an equilibrium model (structure of a well defined world) $\longrightarrow$ "Pure" Abstract Theory.
- Specific formulation $\longrightarrow$ For instance, a competitive or Walrasian model of pure exchange, i.e., distribution of privately owned commodities through competitive markets.

See the PDF file available on the pedagogical space of the course.

First question one could ask about an equilibrium model:

$\longrightarrow$ Existence of solutions.

Existence of a competitive equilibrium

Several approaches to prove the existence of a competitive equilibrium:

- Apply Brouwer's Fixed Point Theorem (functions):
  $\longrightarrow$ continuity, strict convexity, strict monotonicity.
- Apply Kakutani's Fixed Point Theorem (correspondences):
  $\longrightarrow$ upper semicontinuity, convexity, some kind of desirability.
- Apply the Homotopy Theorem (differentiable setting):
  $\longrightarrow$ differentiability, Karush-Kuhn-Tucker conditions, path-following proof.

- $I < +\infty$ individuals/consumers, $i = 1, ..., I$.
- $L < +\infty$ commodities/goods, $\ell = 1, ..., L$. $\mathbb{R}^L$ is the commodity space.
- The quantity of commodity ℓ consumed by individual i is $x_{i\ell} \in \mathbb{R}$, $x_i = (x_{i1}, ..., x_{i\ell}, ..., x_{iL}) \in \mathbb{R}^L$ is the consumption of individual i.
- $X_i \subseteq \mathbb{R}^L$ is the consumption set of individual i, $X = \prod_{i=1}^{I} X_i$.

  (A.1) X_i is closed, convex and bounded from below.
- $x = (x_1, ..., x_i, ..., x_I) \in X$ is an allocation.

- The unit price of commodity ℓ is $p_\ell \in \mathbb{R}$, $p = (p_1, ..., p_\ell, ..., p_L) \in \mathbb{R}^L$ is a price system.
- Individual i owns an initial endowment $e_{i\ell} \in \mathbb{R}$ of commodity ℓ, $e_i = (e_{i1}, ..., e_{i\ell}, ..., e_{iL}) \in \mathbb{R}^L$ is the initial endowment of individual i.
- $p \cdot e_i \in \mathbb{R}$ is the wealth of individual i at the price p.
- Individual i has a preference relation $\succeq_i$ on X_i,

  (A.2) $\succeq_i$ is complete, transitive and continuous on X_i.

  Under (A.1)-(A.2), the individual preferences of individual i are represented by a continuous utility function:

$$u_i : x_i \in X_i \longrightarrow u_i(x_i) \in \mathbb{R}.$$

- $\mathcal{E} = (X_i, u_i, e_i)_{i=1,...,I}$ is the economy.

Competitive equilibrium

$(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ is a competitive equilibrium of the economy $\mathcal{E}$ if:

1. for every individual i, x_i^* solves the utility maximization problem at the equilibrium price p^*, i.e.,

$$\begin{array}{ll} \max\limits_{x_i \in X_i} & u_i(x_i) \\ \text{subject to} & p^* \cdot x_i \leq p^* \cdot e_i \end{array}$$

2. $(x_1^*, ..., x_i^*, ..., x_I^*)$ satisfies Market Clearing Conditions, i.e.,

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i.$$

Presentation of basic mathematical tools, referring to the Mathematical Appendix available on the pedagogical space of the course:

- Closed, bounded, compact, convex sets.
- Linearity, concavity, quasi-concavity, strict quasi-concavity.
- Brouwer's Fixed Point Theorem.

A simple existence theorem $\longrightarrow$ Using the excess demand approach and the Brouwer's Fixed Point Theorem.

Main ideas.

For every individual i, consider the canonical utility maximization problem $(\text{UMP})_i$ at the price p and the initial endowment e_i:

$$(\text{UMP})_i \quad \max_{x_i \in X_i} \quad u_i(x_i) \quad \text{subject to} \quad p \cdot x_i \leq p \cdot e_i$$

Denote $x_i(p, e)$ the set of solutions of problem $(\text{UMP})_i$.

Assume that for all i, $x_i(p, e)$ is **well-defined (i.e., non-empty), single-valued, and continuous on prices**.

Hence, define the individual excess demand:

$$z_i(p) = x_i(p, e_i) - e_i,$$

and the excess demand $z(p) = \sum_{i=1}^{I} [x_i(p, e_i) - e_i]$.

By Market Clearing Conditions, p^* is an equilibrium price if:

$$z(p^*) = \sum_{i=1}^{I} [x_i(p^*, e_i) - e_i] = 0,$$

where $x_i^* = x_i(p^*, e_i)$ for all i.

The idea is to construct a continuous mapping f by perturbing the prices p, so that, at any fixed point p^* of f, one gets $z(p^*) \leq 0$.

To prove the existence of a fixed point p^*, one applies the Brouwer's Fixed Point Theorem to the mapping f.

One then shows that the excess demand $z(p^*)$ is equal to zero.

Main difficulties:

- $x_i(p, e)$ is not necessarily well-defined (i.e., non-empty), single-valued, and continuous on prices.
- $z(p^*) \leq 0$ does not necessarily implies that $z(p^*) = 0$.

In order to overcome these difficulties, in addition to the standard continuity of utilities, one assumes that utilities are strictly increasing and strictly quasi-concave on $\mathbb{R}^L_+$, and $e_i \gg 0$ for all i.

Notice that, for instance, linear and Leontief utilities are **not** strictly quasi-concave.

Proposition

Assume that $x_i(p, e)$ *is non-empty. Show that if* u_i *is strictly quasi-concave, then* $x_i(p, e)$ *is single-valued (i.e.,* $x_i(p, e)$ *is a singleton).*

Proof. It is left as an exercise.

Presentation of basic mathematical tools, referring to the Mathematical Appendix available on the pedagogical space of the course:

- u_i is striclty increasing, u_i is increasing, u_i satisfies local non-satiation.

A less demanding existence theorem $\longrightarrow$ Using Kakutani's fixed point theorem.

Three main references:

- Arrow and Debreu (1954), Existence of an equilibrium for a competitive economy, *Econometrica*.
- Debreu (1959), Theory of Value: An Axiomatic Analysis of Economic Equilibrium, New York: Wiley & Sons, Inc.
- Florenzano (2003), General Equilibrium Analysis: Existence and Optimality Properties of Equilibria, Kluwer Academic Publishers.

Basic assumptions. For all $i \in \mathcal{I} = \{1, ..., I\}$,

- (B.1) $X_i = \mathbb{R}^L_+$ and $e_i \geq 0$.
- (B.2) **(Continuity of preferences)** u_i is continuous on X_i.
- (B.3) **(Convexity of preferences)** u_i is quasi-concave on X_i.
- (B.4) **(Local non-satiation)** u_i is *locally non-satiated* on X_i, i.e. for every $x_i \in X_i$ and for every open neighborhood N of x_i there exists $x_i' \in N \cap X_i$ such that $u_i(x_i') > u_i(x_i)$.

Presentation of advanced mathematical tools, referring to the Mathematical Appendix available on the pedagogical space of the course:

- Upper semicontinuous correspondences.
- Kakutani's Fixed Point Theorem.
- Berge's Theorem.

Quasi-equilibrium

$(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ is a quasi-equilibrium of the economy $\mathcal{E}$ if $p^* \neq 0$ and

1. for every individual i, x_i^* solves the following minimization problem at p^*:

$$\min_{x_i \in X_i} \quad p^* \cdot x_i$$
$$\text{subject to} \quad u_i(x_i) \geq u_i(x_i^*)$$

2. for every individual i, $p^* \cdot x_i^* = p^* \cdot e_i$.

3. $(x_1^*, ..., x_i^*, ..., x_I^*)$ satisfies Market Clearing Conditions, i.e.,

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i$$

Theorem

For all $i \in \mathcal{I}$, assume **Assumptions (B.1), (B.2), (B.3), (B.4)**. *Then, there exists a quasi-equilibrium $(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ of the economy $\mathcal{E}$.*

Proof.

Step 1. Compactify the economy. Consider the non-empty, convex and compact set:

$$F = \{x = (x_1, ..., x_i, ..., x_I) \in \mathbb{R}_+^{LI} : \sum_{i \in \mathcal{I}} x_i \leq \sum_{i \in \mathcal{I}} e_i\}$$

There is some convex compact cube K of $\mathbb{R}^L$ with center 0 such that

$$F \subseteq \mathrm{Int}(K^I),$$

where $K^I = \prod_{i \in \mathcal{I}} K$.

Define the following sets, that are **non-empty, convex and compact** by construction.

$$\forall\, i \in \mathcal{I},\ C_i = X_i \cap K \text{ and } C = \prod_{i \in \mathcal{I}} C_i$$

The non-empty convex compact set of prices is defined as

$$P = \{p \in \mathbb{R}^L \colon \|p\| \leq 1\}$$

Step 2. Correspondences for a quasi-equilibrium.

- For all $i \in \mathcal{I}$, γ_i is the **budget correspondence** from P to C_i,

$$\gamma_i(p) = \{x_i \in C_i \colon p \cdot x_i \leq p \cdot e_i\}$$

and δ_i is the **quasi-budegt correspondence** from P to C_i,

$$\delta_i(p) = \{x_i \in C_i \colon p \cdot x_i < p \cdot e_i\}$$

- For all $i \in \mathcal{I}$, φ_i is the **quasi-demand correspondence** from P to C_i,

$$\varphi_i(p) = \left\{ x_i \in C_i \colon \begin{array}{ll} (1) & x_i \in \gamma_i(p), \text{ and} \\ (2) & \{x_i' \in C_i \colon u_i(x_i') > u_i(x_i)\} \cap \delta_i(p) = \emptyset \end{array} \right\}$$

- ψ is called the **Walrasian Auctioneer correspondence** from C to P,

$$\psi(x) = \{p^* \in P \colon p^* \text{ solves the problem (WA)}\},$$

where (WA) is the following maximization problem:

$$\begin{array}{ll} \max & p \cdot \sum_{i \in \mathcal{I}} (x_i - e_i) \\ \text{subject to} & p \in P \end{array}$$

- θ is the correspondence from $P \times C$ to $P \times C$,

$$\theta(p, x) = \psi(x) \times \prod_{i \in \mathcal{I}} \varphi_i(p)$$

Step 3. We prove the following lemma.

Lemma (FP)

There exists a fixed point $(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ *of* θ *with* $p^* \neq 0$.

Proof of Lemma (FP). Define the following correspondences $\widetilde{\gamma}_i$, $\widetilde{\delta}_i$, and $\widetilde{\varphi}_i$ from P to C_i. These correspondences slightly differ from γ_i, δ_i, and φ_i, because the individual wealth is increased by $\dfrac{(1-\|p\|)}{I}$. For all $i \in \mathcal{I}$ and for all $p \in P$:

- $\widetilde{\gamma}_i(p) = \left\{ x_i \in C_i : p \cdot x_i \leq p \cdot e_i + \dfrac{(1-\|p\|)}{I} \right\}$
- $\widetilde{\delta}_i(p) = \left\{ x_i \in C_i : p \cdot x_i < p \cdot e_i + \dfrac{(1-\|p\|)}{I} \right\}$
- $\widetilde{\varphi}_i(p) = \left\{ x_i \in C_i : \begin{array}{ll} (1) & x_i \in \widetilde{\gamma}_i(p), \text{ and} \\ (2) & \{x_i' \in C_i : u_i(x_i') > u_i(x_i)\} \cap \widetilde{\delta}_i(p) = \emptyset \end{array} \right\}$

Remark that if $\|p\| = 1$, then $\widetilde{\gamma}_i(p)$, $\widetilde{\delta}_i(p)$, and $\widetilde{\varphi}_i(p)$ coincide with $\gamma_i(p)$, $\delta_i(p)$, and $\varphi_i(p)$, respectively.

Consider now the correspondence $\widetilde{\theta}$ from $P \times C$ to $P \times C$,

$$\widetilde{\theta}(p, x) = \psi(x) \times \prod_{i \in \mathcal{I}} \widetilde{\varphi}_i(p)$$

where ψ is the **Walrasian Auctioneer correspondence**.

- For every $x \in C$, $\psi(x)$ is **non-empty**, because of the Weierstrass Theorem. Furthermore, $\psi(x)$ is obviously **convex**.
- Consider the correspondence ϕ from C to P defined by $\phi(x) = P$ for all $x \in C$. It is an easy matter to check that ϕ is continuous (i.e., lower and upper semicontinuous) on C. Then, the correspondence ψ is **upper semicontinuous** on C, by the Berge Theorem (see the Mathematical Appendix).

- For every $p \in P$, $\widetilde{\gamma}_i(p) \neq \emptyset$ since $e_i \in C_i = X_i \cap K$ and $(1 - \|p\|) \geq 0$. Furthermore, $\widetilde{\gamma}_i(p)$ is obviously convex and compact.

- For every $p \in P$, $\widetilde{\varphi}_i(p)$ is **non-empty**. Indeed, by the Weierstrass Theorem, there exists some x_i^* that maximizes u_i over the set $\widetilde{\gamma}_i(p)$. It is an easy matter to check that $x_i^* \in \widetilde{\varphi}_i(p)$. Furthermore, $\widetilde{\varphi}_i(x)$ is **convex**, since $\widetilde{\gamma}_i(p)$ is convex and u_i is **quasi-concave**.

- The correspondence $\widetilde{\varphi}_i$ is **upper semicontinuous** on P. Indeed, let $p^* \in P$ and $(p^n, x_i^n)_{n \in \mathbb{N}}$ be a sequence such that for every $n \in \mathbb{N}$, $p^n \in P$, $x_i^n \in \widetilde{\varphi}_i(p^n)$, and

$$(p^n, x_i^n) \longrightarrow (p^*, x_i^*)$$

We show below that $x_i^* \in \widetilde{\varphi}_i(p^*)$.

Since $x_i^n \in \widetilde{\gamma}_i(p^n)$, then $x_i^n \in C_i$ and $p^n \cdot x_i^n \leq p^n \cdot e_i + \frac{(1-\|p^n\|)}{I}$.

Taking the limit over n, $x_i^* \in C_i$ since C_i is closed, and $p^* \cdot x_i^* \leq p^* \cdot e_i + \frac{(1-\|p^*\|)}{I}$. Thus, $x_i^* \in \widetilde{\gamma}_i(p^*)$.

Suppose now by contradiction that there exists $\widetilde{x}_i \in \widetilde{\delta}_i(p^*)$ such that $u_i(\widetilde{x}_i) > u_i(x_i^*)$. Then,

- there is $n_1 \in \mathbb{N}$ such that $\widetilde{x}_i \in \widetilde{\delta}_i(p^n)$ for all $n \geq n_1$,
- since u_i is continuous, there is $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, $u_i(\widetilde{x}_i) > u_i(x_i^n)$.

Hence, there is $\overline{n} \in \mathbb{N}$ such that for all $n \geq \overline{n}$,

$$\widetilde{x}_i \in \widetilde{\delta}_i(p^n) \cap \{x_i' \in C_i : u_i(x_i') > u_i(x_i^n)\}$$

which contradicts the fact that $x_i^n \in \widetilde{\varphi}_i(p^n)$ for all $n \in \mathbb{N}$.

Then $\widetilde{\delta}_i(p^*) \cap \{x_i' \in C_i : u_i(x_i') > u_i(x_i^*)\} = \emptyset$, and so $x_i^* \in \widetilde{\varphi}_i(p^*)$.

As a product of upper semicontinuous correspondences with non-empty convex values, the correspondence $\widetilde{\theta}$ is upper semicontinuous with non-empty convex values from the non-empty compact convex set $P \times C$ to $P \times C$.

By **Kakutani's Fixed Point Theorem**, the correspondence $\widetilde{\theta}$ has a fixed point, i.e.,

$$\exists\ (p^*, x^*) \in \psi(x^*) \times \prod_{i \in \mathcal{I}} \widetilde{\varphi}_i(p^*)$$

That is, there exist $p^* \in P$ and $x^* = (x_1^*, ..., x_i^*, ..., x_I^*) \in C$ such that:

- $p^* \in \psi(x^*)$, i.e., p^* solves the following maximization problem:

$$\begin{array}{ll} \max & p \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) \\ \text{subject to} & p \in P \end{array}$$

- and $x_i^* \in \widetilde{\varphi}_i(p^*)$ for every individual i.

Next, we show that $\|p^*\| = 1$**.** Consequently, $(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ is a fixed point of the correspondence θ, with $p^* \neq 0$.

(a). First, notice that $p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) > 0$ entails $\|p^*\| = 1$.

Indeed, if $p_\ell^* \cdot \sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell}) > 0$ for some commodity ℓ, then we have two cases, either $p_\ell^* > 0$ and $\sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell}) > 0$, or $p_\ell^* < 0$ and $\sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell}) < 0$. In the first case, $p_\ell^* \cdot \sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell})$ increases, by increasing p_ℓ^*. In the second case, $p_\ell^* \cdot \sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell})$ increases, by decreasing p_ℓ^*. In both cases, the norm of p^* increases.

If $\|p^*\| < 1$, one can then obtain some price system $p \in P$ such that $p \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) > p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i)$, that is a contradiction since $p^* \in \psi(x^*)$.

(b). It must be that $p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) = 0$.

Indeed, since $0 \in P$ and $p^* \in \psi(x^*)$, we have that

$$p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) \geq 0.$$

If $p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) > 0$, then $\|p^*\| = 1$, because of **(a)**.

On the other hand, $p^* \cdot x_i^* \leq p^* \cdot e_i + \dfrac{(1 - \|p^*\|)}{I}$ for all i, since $x_i^* \in \widetilde{\varphi}_i(p^*)$ for all i. Then, summing over i, one gets

$$0 < p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) \leq 1 - \|p^*\| = 0,$$

that is a contradiction.

(c). We show that **(b)** implies that $\sum_{i \in \mathcal{I}} (x_i^* - e_i) = 0$.

Indeed, since $p^* \in \psi(x^*)$ and $p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) = 0$, we get

$$p \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) \leq 0, \ \forall \, p \in P.$$

For any commodity ℓ, take all the prices equal to 0, except for $p_\ell = 1$. Then, $\sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell}) \leq 0$.

Take now all the prices equal to 0, except for $p_\ell = -1$. Then, $\sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell}) \geq 0$.

Hence, one deduces that $\sum_{i \in \mathcal{I}} (x_{i\ell}^* - e_{i\ell}) = 0$ for every commodity ℓ.

By **(c)**, we get $x^* \in F \subseteq \mathrm{Int}(K^I)$. Then,

$$x_i^* \in \mathrm{Int}\, K, \ \forall\, i \in \mathcal{I}.$$

Consequently, by **local non-satiation** of u_i, we get

$$p^* \cdot x_i^* = p^* \cdot e_i + \frac{(1 - \|p^*\|)}{I}, \ \forall\, i \in \mathcal{I},$$

because $x_i^* \in \widetilde{\varphi}_i(p^*)$ for all i. Summing over i, one deduces that

$$p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) = 1 - \|p^*\|.$$

Therefore, from **(c)** one obtains $\|p^*\| = 1$, that ends the proof of Step 3.

Step 4. We prove the following lemma.

Lemma (From a fixed point to a quasi-equilibrium)

If $(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ *is a fixed point of* θ *with* $p^* \neq 0$*, then it is a quasi-equilibrium of the economy* $\mathcal{E}$*.*

Proof. By definition of θ, $p^* \in \psi(x^*)$ and $x_i^* \in \varphi(p^*)$ for every i.

- Following the same strategy as in **(b)** and **(c)**, one proves that

$$\sum_{i \in \mathcal{I}} (x_i^* - e_i) = 0.$$

  Consequently, $x_i^* \in \operatorname{Int} K$ for all $i \in \mathcal{I}$.

- Since $x_i^* \in \varphi_i(p^*)$, we know that $x_i^* \in \gamma_i(p^*)$ and

$$x_i \in C_i \text{ and } u_i(x_i) > u_i(x_i^*) \implies p^* \cdot x_i \geq p^* \cdot e_i$$

Claim. $x_i \in X_i$ and $u_i(x_i) \geq u_i(x_i^*) \implies p^* \cdot x_i \geq p^* \cdot e_i$.

Let $x_i \in X_i$ be such that $u_i(x_i) \geq u_i(x_i^*)$. Suppose by contradiction that

$$p^* \cdot x_i < p^* \cdot e_i.$$

Since $x_i^* \in \operatorname{Int} K$, there exists $t \in]0, 1[$ such that

$$x_i(t) = tx_i + (1 - t)x_i^* \in \operatorname{Int} K.$$

Since $x_i^* \in \gamma_i(p^*)$, by the strict inequality above, one obviously gets

$$p^* \cdot x_i(t) < p^* \cdot e_i.$$

Since u_i is **quasi-concave**, we have that

$$u_i(x_i(t)) \geq u_i(x_i^*)$$

Notice now that there exists an open neighborhood N of $x_i(t)$ such that

$$\forall\, x'_i \in N,\ p^* \cdot x'_i < p^* \cdot e_i$$

Without loosing of generality we can suppose that $N \subseteq \operatorname{Int} K$. By **local non-satiation** of u_i, for some $x'_i \in N \cap X_i$ one gets

$$u_i(x'_i) > u_i(x_i(t))$$

Then, there exists some $x'_i \in C_i$ such that $p^* \cdot x'_i < p^* \cdot e_i$ and $u_i(x'_i) > u_i(x^*_i)$. But, $x^*_i \in \varphi_i(p^*)$, a contradiction that completes the proof of the Claim.

Finally, from the Claim and $x^*_i \in \gamma_i(p^*)$, one gets $p^* \cdot x^*_i = p^* \cdot e_i$.

Therefore, $(p^*, (x^*_1, ..., x^*_i, ..., x^*_I))$ is a quasi-equilibrium. ■

Proposition (From EMP to UMP)

Assume that X_i *is convex and* u_i *is continuous on* X_i*. Let* $p^* \in \mathbb{R}^L$ *be a price system with* $p^* \neq 0$*. If* x_i^* *solves the expenditure minimization problem below:*

$$\begin{array}{ll} \min\limits_{x_i \in X_i} & p^* \cdot x_i \\ \textit{subject to} & u_i(x_i) \geq u_i(x_i^*) \end{array}$$

and there exists $\widetilde{x}_i \in X_i$ *such that* $p^* \cdot \widetilde{x}_i < p^* \cdot x_i^*$*, then* x_i^* *solves the utility maximization problem below*

$$\begin{array}{ll} \max\limits_{x_i \in X_i} & u_i(x_i) \\ \textit{subject to} & p^* \cdot x_i \leq p^* \cdot x_i^* \end{array}$$

Proof. This is an exercise of the Homework. ■

Proposition

1. *For for all i, assume* **Assumptions (B.1), (B.2)**, *and $e_i \gg 0$* **(strong survival condition)**. *Then, a quasi-equilibrium is a competitive equilibrium.*

2. *For for all i, assume* **Assumptions (B.1), (B.2)**. *If u_i is* **strictly increasing** *on X_i for all i, and $\sum_{i=1}^{I} e_i \gg 0$* **(survival condition)**, *then a quasi-equilibrium is a competitive equilibrium. Furthermore, the equilibrium price of all commodities is strictly positive.*

Proof. This is an exercise of the Homework. ∎

Pareto optimality in exchange economies

$(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I) \in X$ is a Pareto optimal allocation of the economy $\mathcal{E}$ if:

1. $\displaystyle\sum_{i \in \mathcal{I}} \overline{x}_i = \sum_{i \in \mathcal{I}} e_i,$

2. there is no other allocation $(x_1, ..., x_i, ..., x_I) \in X$ such that:

   $$\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i,$$

   $u_i(x_i) \geq u_i(\overline{x}_i)\ \forall\ i \in \mathcal{I}$, and $u_k(x_k) > u_k(\overline{x}_k)$ for some $k \in \mathcal{I}$.

Optimality properties of equilibria

Theorem (First Welfare Theorem)

For all $i \in \mathcal{I}$, assume **Assumption (B.4)**. *If $(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ is a competitive equilibrium of the economy $\mathcal{E}$, then the equilibrium allocation $(x_1^*, ..., x_i^*, ..., x_I^*)$ is a Pareto optimal allocation of the economy $\mathcal{E}$.*

Sketch of the proof. By **local non-satiation** of u_i, one verifies that for every $i \in \mathcal{I}$, $x_i \in X_i$ and $u_i(x_i) \geq u_i(x_i^*)$ entails $p^* \cdot x_i \geq p^* \cdot e_i$. Assume now that $(x_1^*, ..., x_i^*, ..., x_I^*)$ is not Pareto optimal. Using the definitions of Pareto optimality, competitive equilibrium, and the property above, one deduces that there exists $(x_1, ..., x_i, ..., x_I) \in X$ such that $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i$,

$p^* \cdot x_i \geq p^* \cdot e_i$, $\forall\, i \in \mathcal{I}$ and $p^* \cdot x_k > p^* \cdot e_k$ for some $k \in \mathcal{I}$.

Summing over i, we obviously get a contradiction. ■

Theorem (Second Welfare Theorem)

For all $i \in \mathcal{I}$, assume that X_i is a non-empty convex subset of $\mathbb{R}^L$, and **Assumptions (B.3), (B.4)**. *If $(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I)$ is a Pareto optimal allocation of the economy $\mathcal{E}$, then there exists $\overline{p} \neq 0$ such that $(\overline{p}, (\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I))$ is a quasi-equilibrium of the 'rearranged' economy where the initial endowment of individual i is $\overline{x}_i$ for every $i \in \mathcal{I}$.*

Sketch of the proof.

- For all $i \in \mathcal{I}$, define the set $V_i = \{x_i \in X_i \colon u_i(x_i) > u_i(\overline{x}_i)\}$.

  V_i is non-empty by **local non-satiation** of u_i, V_i is convex because u_i is **quasi-concave**. Then, the set $V = \sum_{i \in \mathcal{I}} V_i$ is a non-empty convex subset of $\mathbb{R}^L$.

- Consider the total resources of the economy $\mathcal{E}$, i.e., $r = \sum_{i \in \mathcal{I}} e_i$.

  Notice that $r \notin V$, because $(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I)$ is a Pareto optimal allocation of $\mathcal{E}$.

  Using the finite dimensional Separation Theorem for two non-empty disjoint convex sets, there exists $\overline{p} \neq 0$ such that

$$\overline{p} \cdot v \geq \overline{p} \cdot r, \; \forall \; v \in V.$$

- We now claim that:

$$u_i(x_i) \geq u_i(\overline{x}_i), \; \forall \; i \in \mathcal{I} \implies \overline{p} \cdot \sum_{i \in \mathcal{I}} x_i \geq \overline{p} \cdot r.$$

  Indeed, since $u_i(x_i) \geq u_i(\overline{x}_i)$, then by **local non-satiation** of u_i, for every i there exists a sequence $(x_i^n)_{n \in \mathbb{N}}$ such that $x_i^n \longrightarrow x_i$ and $x_i^n \in V_i$ for every n.

Then, $\overline{p} \cdot \sum_{i \in \mathcal{I}} x_i^n \geq \overline{p} \cdot r$, for every n. Taking the limit over n, we complete the proof of the claim.

- Also notice that $\overline{p} \cdot r = \overline{p} \cdot \sum_{i \in \mathcal{I}} \overline{x}_i$, because $\sum_{i \in \mathcal{I}} \overline{x}_i = \sum_{i \in \mathcal{I}} e_i$.
- Consider now any $x_i \in X_i$ such that $u_i(x_i) \geq u_i(\overline{x}_i)$.

  From the latter two properties, one gets:

$$\overline{p} \cdot (x_i + \sum_{h \neq i} \overline{x}_h) \geq \overline{p} \cdot \sum_{i \in \mathcal{I}} \overline{x}_i.$$

  Then, $\overline{p} \cdot x_i \geq \overline{p} \cdot \overline{x}_i$.

  Consequently, $(\overline{p}, (\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I))$ is a quasi-equilibrium of the economy where the initial endowment of individual i is $\overline{x}_i$ for all i. ■

Corollary

For all $i \in \mathcal{I}$*, assume* **Assumptions (B.1), (B.2), (B.3), (B.4)**, *and* $\sum_{i=1}^{I} e_i \gg 0$*. Let* $(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I)$ *be a Pareto optimal allocation of the economy* $\mathcal{E}$ *with* $\overline{x}_i \gg 0$*, for all* $i \in \mathcal{I}$*.*

Then there is $\overline{p} \neq 0$ *such that* $(\overline{p}, (\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I))$ *is a competitive equilibrium of the 'rearranged' economy where the initial endowment of individual* i *is* $\overline{x}_i$ *for every* $i \in \mathcal{I}$*.*

Proof. This is an easy consequence of the previous theorem. ■

First order conditions (FOC) for Pareto optimality
$\longrightarrow$ Using more demanding assumptions.

Assumptions. For all $i \in \mathcal{I} = \{1, ..., I\}$,

(C.1) $X_i = \mathbb{R}^L_{++}$ and $\sum_{i=1}^{I} e_i \gg 0$.

(C.2) **(Differentiable utilities)** u_i is differentiable on X_i.

(C.3) **(Convexity of preferences)** u_i is quasi-concave on X_i.

(C.4) $Du_i(x_i) \gg 0$ for every $x_i \in X_i$ **(stronger than u_i is strictly increasing)**.

Proposition (FOC for Pareto optimality)

For all $i \in \mathcal{I}$, *assume* **Assumptions (C.1), (C.2), (C.3), (C.4)**. $(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I) \in \mathbb{R}^{LI}_{++}$ *is a Pareto optimal allocation of the economy* $\mathcal{E}$ *if and only if there exists* $(\theta_2, ..., \theta_i, ..., \theta_I) \in \mathbb{R}^{I-1}_{++}$ *such that*

$$Du_1(\overline{x}_1) = \theta_i Du_i(\overline{x}_i), \ \forall\, i = 2, \dots, I$$

and $\sum_{i \in \mathcal{I}} \overline{x}_i = \sum_{i \in \mathcal{I}} e_i$.

Proof.

Step 1. Consider the problem of maximizing the utility of one individual, for instance individual 1, subject to maintain the utilities of all the others at least as high as they get at $(\overline{x}_2, ..., \overline{x}_i, ..., \overline{x}_I)$, and feasibility conditions. That is:

$$\max_{(x_1,...,x_i,...,x_I)\in\mathbb{R}^{LI}_{++}} \quad u_1(x_1)$$

$$\text{subject to} \quad \begin{cases} u_i(x_i) \geq u_i(\overline{x}_i), \ \forall\, i = 2, \ldots, I \\ \displaystyle\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i \end{cases}$$

One verifies that $(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I) \in \mathbb{R}^{LI}_{++}$ is a Pareto optimal allocation of the economy $\mathcal{E}$ if and only if it solves this maximization problem.

Step 2. Consider the Karush-Kuhn-Tucker conditions associated with the problem above, and verify that they are **necessary and sufficient conditions** to solve that problem. Then, one gets:

$$\begin{cases} (1)\ Du_1(x_1) - \gamma = 0 \\ (2)\ \forall\ i \neq 1,\ \theta_i Du_i(x_i) - \gamma = 0 \\ (3)\ \forall\ i \neq 1, \theta_i \geq 0,\ u_i(x_i) - u_i(\overline{x}_i) \geq 0 \text{ and } \theta_i[u_i(x_i) - u_i(\overline{x}_i)] = 0 \\ (4)\ \sum_{i \in \mathcal{I}} e_i - \sum_{i \in \mathcal{I}} x_i = 0 \end{cases}$$

- For every individual $i \neq 1$, θ_i is the Lagrange multiplier associated with the constraint $u_i(x_i) - u_i(\overline{x}_i) \geq 0$.
- $\gamma = (\gamma_1, ..., \gamma_\ell, ..., \gamma_L)$ and for every commodity ℓ, γ_ℓ is the Lagrange multiplier associated with the feasibility constraint $\sum_{i \in \mathcal{I}} e_{i\ell} - \sum_{i \in \mathcal{I}} x_{i\ell} = 0$.

By (1) and **(C.4)**, $\gamma \gg 0$. Then, $\theta_i > 0$ because of (2) and **(C.4)**.

■

- One easily recognizes the First and Second Welfare Theorems in the previous proposition. Indeed, it is enough to write the Karush-Kuhn-Tucker conditions associated with every individual i's utility maximization problem, and Market Clearing Conditions. Then, one simply remarks that $Du_1(\overline{x}_1)$ is obviously a supporting price of the allocation $(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I)$.

- Also notice that from the above proposition, one deduces the characterization of Pareto optimality in terms of **individual** marginal rates of substitution (MRS), i.e.,

  $\forall$ individuals $i \neq h$ and $\forall$ commodities $\ell \neq k$,

$$MRS_{\ell,k}u_i(\overline{x}_i) = \frac{D_{x_{i\ell}}u_i(\overline{x}_i)}{D_{x_{ik}}u_i(\overline{x}_i)} = \frac{D_{x_{h\ell}}u_h(\overline{x}_h)}{D_{x_{hk}}u_h(\overline{x}_h)} = MRS_{\ell,k}u_h(\overline{x}_h),$$

and $\sum_{i \in \mathcal{I}} \overline{x}_i = \sum_{i \in \mathcal{I}} e_i$.

The economy with consumption externalities

The consumption of individual i is $x_i \in X_i = \mathbb{R}^L_+$, and the consumption of individuals other than i is denoted by:

$$x_{-i} = (x_j)_{j \neq i} \in \mathbb{R}^{L(I-1)}_+$$

The bundle $x = (x_i)_{i \in \mathcal{I}} \in X = \mathbb{R}^{LI}_+$ is an allocation. With innocuous abuse of notation, an allocation x is also denoted by (x_i, x_{-i}).

The preferences of individual i are represented by a utility function:

$$u_i : x \in \mathbb{R}^{LI}_+ \mapsto u_i(x) \in \mathbb{R}$$

where $u_i(x)$ is the utility level of individual i associated with the allocation $x = (x_i, x_{-i}) \in \mathbb{R}^{LI}_+$. That is, u_i also describes how individual i's preferences are affected by the consumption x_{-i} of the other individuals.

1. Economies where the externality is generated by the presence of a public good, also known as a **collective** consumption good, see Samuelson (1954, *Rev. Econ. Stat.*). Without loss of generality, this is commodity L, and for all i the utility function is:

$$u_i(x_i, \sum_{j \in \mathcal{I}} x_{jL})$$

2. Bergson-Samuelson utility functions, that is:

$$u_i(x) = V_i(m_1(x_1), ..., m_i(x_i), ..., m_I(x_I)),$$

where V_i is increasing in component i, and that m_k is continuous, monotone, and quasi-concave for all $k \in \mathcal{I}$.

For instance, altruistic utility functions as in Bourlès, Bramoullé, and Perez-Richet (2017, *Econometrica*), where:

$$u_i(x) = m_i(x_i) + \sum_{j \neq i} \alpha_{ij} m_j(x_j), \text{with } 0 \leq \alpha_{ij} < 1.$$

3. Negative externalities: pollution, envy, etc...

For instance, the following examples in del Mercato and Nguyen (2023, *J. Econ. Theory*).

a) There are three individuals and two commodities, e.g., two drivers $i = 1, 2$, and one non-driver $i = 3$. Commodity 1 is food and it is desirable for all the individuals. Every driver consumes gas (commodity 2) and pollutes. The non-driver does not care about consuming gas, i.e., $x_{32} = 0$. All of them suffer from the global gas emission $(x_{12} + x_{22})$ in the economy.

The utility functions are of the CARA form:

$u_i(x) = -e^{-x_{i1}} - e^{-x_{i2}+\varepsilon(x_{12}+x_{22})}$ for all $i = 1, 2$,
$u_3(x) = -e^{-x_{31}} - e^{\delta(x_{12}+x_{22})}$

The parameters ε and δ measure the dissatisfaction about the global gas emission of drivers and non-driver, respectively.

b) One commodity and two individuals. The total endowment is $r = 1$, and the utility functions are:

$$u_1(x) = x_1 - \lambda \max\{x_2 - \delta, 0\},\ u_2(x) = x_2 - \lambda \max\{x_1 - \delta, 0\},$$

with $\lambda > 0$ and $\frac{1}{2} \leq \delta \leq 1$. If the consumption of an individual is above a certain threshold δ, then the other suffers because he is envious. The parameter λ can be interpreted as a measure of envy about the excess consumption of the others.

Competitive equilibrium à la Nash

Let $\mathcal{E}$ be an economy with consumption externalities.

$(p^*, (x_1^*, ..., x_i^*, ..., x_I^*))$ is a competitive equilibrium (à la Nash) of the economy $\mathcal{E}$ if:

1. for every individual i, x_i^* solves the utility maximization problem at the equilibrium price p^* and the externality x_{-i}^*:

$$\max_{x_i \in X_i} \quad u_i(x_i, x_{-i}^*)$$
$$\text{subject to} \quad p^* \cdot x_i \leq p^* \cdot e_i$$

2. $(x_1^*, ..., x_i^*, ..., x_I^*)$ satisfies Market Clearing Conditions, i.e.,

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} e_i.$$

Existence of a competitive equilibrium à la Nash

There is a large literature on the existence of a competitive equilibrium with various sorts of externalities, pioneered by McKenzie (1955, *The Second Symposium of Linear Programming*).

McKenzie has focused on a production economy with linear technologies where individual preferences are affected by the price system, the consumption of the others and the inputs of the firms.

As for consumption and production externalities in preferences, one finds in the textbook of Arrow and Hahn (1971, *General Competitive Analysis*) the natural adaptation of standard assumptions for establishing the existence of a competitive equilibrium (see Section 2 of Chapter 6).

Recent contributions can be found in del Mercato (2006, *J. Math. Econ.*), Mandel (2008, *J. Math. Econ.*), del Mercato and Platino (2017, *J. Econ.*).

The regularity of competitive equilibria à la Nash has been recently studied in economies with various kinds of externalities.

- Villanacci and Zenginobuz (2005, *J. Math. Econ.*): Public goods.
- Bonnisseau and del Mercato (2010, *Econ. Theory*): Consumption externalities.
- Balasko (2015, *J. Math. Econ.*): Wealth externalities.
- del Mercato and Platino (2017, *Econ. Theory*): Production and consumption externalities.
- Nguyen (2020, *J. Math. Econ.*): Endowment externalities.

Pareto optimality with consumption externalities

Let $\mathcal{E}$ be an economy with consumption externalities.

$(\overline{x}_1, ..., \overline{x}_i, ..., \overline{x}_I) \in X$ is a Pareto optimal allocation of the economy $\mathcal{E}$ if:

1. $\displaystyle\sum_{i \in \mathcal{I}} \overline{x}_i = \sum_{i \in \mathcal{I}} e_i$,
2. there is no other allocation $(x_1, ..., x_i, ..., x_I) \in X$ such that:

   $$\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i,$$

   $u_i(x) \geq u_i(\overline{x})\ \forall\ i \in \mathcal{I}$, and $u_k(x) > u_k(\overline{x})$ for some $k \in \mathcal{I}$.

Market failure

As is well known, under standard assumptions, the First and Second Welfare Theorems may fail.

This is shown below by means of a simple example.

Example. There are two individuals and two commodities.

Commodity 1 is an externality good for everybody, and there is no externality for commodity 2. $X_1 = X_2 = \mathbb{R}^2_{++}$ and the utility functions are $u_1(x_1, x_{21})$ and $u_2(x_2, x_{11})$.

For all i, u_i is differentiable, quasi-concave, and for every externality $x_{j1} \in \mathbb{R}_{++}$, $D_{x_i} u_i(x_i, x_{j1}) \gg 0$ for all $x_i \in \mathbb{R}^2_{++}$.

The competitive equilibrium price p^* must be positively proportional to $D_{x_i} u_i(x_i^*, x_{j1}^*)$. Hence, at a competitive equilibrium à la Nash, one gets $MRS_{1,2} u_1 = MRS_{1,2} u_2 = \frac{p_1^*}{p_2^*}$.

On the other hand, using the Karush-Kuhn-Tucker conditions, one gets the first order conditions associated with Pareto optimality, that entail (this is an exercise!):

$$\frac{D_{x_{11}} u_1(x_1, x_{21})}{D_{x_{12}} u_1(x_1, x_{21})} + \frac{D_{x_{11}} u_2(x_2, x_{11})}{D_{x_{22}} u_2(x_2, x_{11})} =$$

$$\frac{D_{x_{21}} u_1(x_1, x_{21})}{D_{x_{12}} u_1(x_1, x_{21})} + \frac{D_{x_{21}} u_2(x_2, x_{11})}{D_{x_{22}} u_2(x_2, x_{11})}$$

These conditions reflect **social** marginal rates of substitutions that do not necessarily match with the competitive equilibrium price.

- Notice that if there are no externalities, i.e., $D_{x_{11}} u_2(x_2, x_{11}) = D_{x_{21}} u_1(x_1, x_{21}) = 0$, one then has the classical conditions associated with Pareto optimality, i.e., $MRS_{1,2} u_1 = MRS_{1,2} u_2$.

Additional conditions

Additional conditions to retrieve the First or the Second Welfare Theorem have been studied in:

- Winter (1969, *J. Econ. Theory*), Parks (1991, *J. Econ. Theory*): Non-Malevolence, Non-Benevolence.
- Dufwenberg et al. (2011, *Rev. Econ. Stud.*): Social Monotonicity.
- Osana (1972, *J. Econ. Theory*), del Mercato and Nguyen (2023, *J. Econ. Theory*): Social/Strong Redistribution.

Pareto improving policies

The gouvernement or a social planner regulates the possibilities of redistribution (i.e., corrective taxes or subsidies) to Pareto improve competitive outcomes, rather than achieving a full Pareto optimality.

- Shafer and Sonnenschein (1976, *Int. Econ. Rev.*) consider a very general model with consumption, production and price externalities in preferences. They have studied the existence of competitive equilibria in economies with a taxing authority.

  Individual i faces a personalized price $\phi_i(x, y, p)$, to be interpreted as including the commodity taxes and subsidies, and receives a scalar lump transfer $\mu_i(x, y, p)$. Hence, at the state (x, y, p), the individual i's wealth is:

$$\phi_i(x, y, p) \cdot e_i + \mu_i(x, y, p)$$

- Greenwald and Stiglitz (1986, *Q. J. Econ.*) mainly focus on economies with incomplete markets and imperfect information.

  However, in Section I, the authors make a static comparative analysis of Pareto improving policies in a general equilibrium model with consumption and production externalities. The issue of the existence of such Pareto improving policies is not addressed for the general model provided in Section I.

- In Geanakoplos and Polemarchakis (2008, *J. Math. Econ.*), the Pareto improving analysis deals with economies with consumption externalities only.

  There are commodity tax rates, i.e., $t^\ell > 0$ is the tax rate on commodity ℓ, or $t^\ell < 0$ if it is a subsidy. The total tax revenue is collected by the government and it is equally redistributed to all the individuals in terms of a lump-sum transfer τ. Taxes and transfers are anonymous.

The individual i's budget constraint is:

$$(p+t)\cdot(x_i-e_i)_+ - p\cdot(x_i-e_i)_- \leq \tau$$

where, at equilibrium, the balance condition of the gouvernement requires:

$$\tau = \frac{1}{I}\sum_{i\in\mathcal{I}} t\cdot(x_i^*-e_i)_+$$

The generic existence of Pareto improving policies is shown under strong assumptions of separability. Indeed, the utility functions have the following specific form:

$$u_i(x_i, x_{-i}) = \widetilde{u}_i(x_i) + \sum_{j\neq i}\sum_{\ell\in\mathcal{L}} \lambda^i_{j\ell} x_{j\ell}$$

If $\lambda^i_{j\ell} > 0$, the consumption of commodity ℓ by individual j has a positive external effect on individual i. If $\lambda^i_{j\ell} < 0$, the external effect is negative. If $\lambda^i_{j\ell} = 0$, there is no external effect.

Perfect internalization of externalities

On the other hand, from a normative point of view, markets can be extended in order to obtain the First and Second Welfare Theorems, through a **perfect internalization** of the externalities.

This is the idea of the seminal contributions by Samuelson (1954, *Rev. Econ. Stat.*), Arrow (1969, *The P.P.B. System*), and Laffont, (1976, *Eur. Econ. Rev.*), who have introduced additional markets where economic agents (i.e., individuals and firms) face personalized prices à Lindahl and choose allocations.

Markets for consumption externalities

Market structure and equilibrium definitions follow Arrow (1969, *The P.P.B. System*), and Laffont (1976, *Eur. Econ. Rev.*).

An *extended* consumption vector of individual i is

$$\widetilde{x}_i = (x_{ih})_{h \in \mathcal{I}} \in \mathbb{R}_+^{LI}$$

where:

- $x_{ii} \in \mathbb{R}_+^L$ is the effective consumption of individual i,
- $x_{ih} \in \mathbb{R}_+^L$ is the external effect of individual h's consumption as perceived by individual i. Equivalently, x_{ih} is the demand of individual i for the consumption of h.

A price system is

$$\widetilde{p} = (p, ((p_{ih})_{h \neq i})_{i \in \mathcal{I}}) \in \mathbb{R}^L \times \mathbb{R}^{L(I-1)I}$$

where:

- $p \in \mathbb{R}^L$ is the price of the L physical commodities,
- $p_{ih} \in \mathbb{R}^L$ is the price paid by individual i to h for the consumption externality created by individual h on individual i.

The individual i's budget constraint is:

$$B_i(\widetilde{p}) = \{\widetilde{x}_i \in \mathbb{R}^{LI}_+ : (p - \sum_{h \neq i} p_{hi}) \cdot x_{ii} + \sum_{h \neq i} p_{ih} \cdot x_{ih} \leq p \cdot e_i\}$$

Hence, individual i faces a *personalized* price for her own consumption, i.e.,

$$p_{ii} = p - \sum_{h \neq i} p_{hi}$$

By definition of p_{ii}, the price system $\widetilde{p}$ satisfies the classical compatibility condition on prices à la Lindahl, i.e.,

$$p = \sum_{h \in \mathcal{I}} p_{hi}, \forall i \in \mathcal{I}.$$

At equilibrium, the extended consumption vectors are coherent, that is:

$$x_{ih} = x_{hh} = x_h \in \mathbb{R}^L_+, \ \forall \ (i, h)$$

An **equilibrium with markets for externalities** is a pair of an allocation $x^* = (x_i^*)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$ and a price system $\tilde{p}^* \neq 0$ such that:

1. for all i, x^* solves the following individual i's utility maximization problem at the equilibrium price $\tilde{p}^*$:

$$\begin{array}{ll} \max\limits_{x \in \mathbb{R}_+^{LI}} & u_i(x) \\ \text{subject to} & x \in B_i(\tilde{p}^*) \end{array}$$

2. x^* is feasible, i.e., $\sum_{i \in \mathcal{I}} x_i^* \leq \sum_{i \in \mathcal{I}} e_i$.

Related literature on the existence of equilibria in economies with markets for externalities, also called also called Arrow or Lindahl equilibria:

- Foley(1970, *Econometrica*): Public goods.
- Bergstrom (1976, *Theory and Measurement of Economic Externalities*): *Communal* goods.
- Crès (1996, *J. Econ. Theory*): Consumption externalities.
- Bonnisseau, del Mercato and Siconolfi (2023, *J. Econ. Theory*): Consumption externalities.