Master MMMEF, 2022-2023
Final exam:
General Equilibrium Theory:
Economic analysis of financial markets
January 2023

January 16, 2023

1) Why the contingent commodity equilibrium is a benchmark in economies with time and uncertainty?

Because the contingent commodity equilibrium allocations are Pareto optimal, which is a minimal efficiency requirement. Thus, they constitutes a reference point to be reach with other financial market organisations.

2) Are the allocation of an Arrow financial equilibrium Pareto optimal?

Yes because the equilibrium allocation of an Arrow equilibrium are the same as the one of a contingent commodity equilibrium.

3) In a two-period financial economy, how do we build the full payoff matrix $W(p,q)$ from the payoff matrix $V(p)$?

The full payoff matrix $W(p,q)$ has an additional first row than the payoff matrix $V(p)$, which is the opposite of the transpose of the asset price q.

4) In a two-period financial economy, with the payoff matrix $V(p)$ and an arbitrage free asset price q, what is the pricing by arbitrage?

The pricing by arbitrage of a payoff vector $r \in \mathbb{R}^{\mathbb{D}_1}$ which is in the range of the matrix $V(p)$ provides the only price q_r of r such that if we add r as a new asset in the financial structure, then the new asset price with q for the existing assets completed by q_r for the additional asset is still arbitrage free. If λ is a present value vector for the price q, that is, if $q = V(p)^t \lambda$ with $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, then $q_r = \lambda \cdot r$.

5) Let a complete financial structure represented by the payoff matrix $V(p)$ with J assets. Give a simple payoff matrix $\bar{V}$ such that the two financial structures are equivalent.

It suffices to take the identity matrix of dimension the number of states of nature at the second period for $\bar{V}$ since it is the payoff matrix of a complete set of Arrow securities, which is a complete financial structure, thus, equivalent to the financial structure associated to $V(p)$.

Exercise 1 *We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1 = \{\xi_1, \xi_2, \ldots, \xi_k\}$ is the set of states of nature at date 1. The financial structure is composed of J nominal assets with the payoff matrix denoted V. q is an arbitrage free asset price for this financial structure.*

We now consider that an additional state denoted ξ^+ is revealed so the new set of states of nature at date 1 is $\mathbb{D}_1^+ = \{\xi_1, \xi_2, \ldots, \xi_k, \xi^+\}$ and the payoff of the existing assets in this node are denoted $v_j(\xi^+)$ for $j = 1, \ldots, J$. The new payoff matrix is denoted V^+.

1) *Show that if q is still an arbitrage free price for the new payoff matrix V^+, then the last row of V^+, $(v_j(\xi^+))_{j=1}^J$ is a linear combination of the k first rows.*

Since q is an arbitrage free asset price for this financial structure, there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V^t\lambda$. If q is still an arbitrage free price for the new payoff matrix V^+, there exists $\mu \in \mathbb{R}^{\mathbb{D}_1 \cup \{\xi_+\}}_{++}$ such that $q = (V^+)^t\mu$. So,

$$q = \sum_{j=1}^{k} \lambda_{\xi_j} V_{\xi_j} = \sum_{j=1}^{k} \mu_{\xi_j} V_{\xi_j} + \mu_{\xi_+} V_{\xi_+}$$

So, $V_{\xi_+} = \frac{1}{\mu_{\xi_+}} \sum_{j=1}^k (\lambda_{\xi_j} - \mu_{\xi_j}) V_{\xi_j}$ is a linear combination of the k first rows of V.

2) *Show that if the last row of V^+, $(v_j(\xi^+))_{j=1}^J$ is a linear combination of the k first rows, then q is an arbitrage free price for the new payoff matrix V^+.*

Let us assume that $V_{\xi_+} = \sum_{j=1}^k \nu_{\xi_j} V_{\xi_j}$. Again, as q is an arbitrage free asset price for this financial structure, there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V^t\lambda$. Then, for all $\alpha > 0$, $q = V^t\lambda + \alpha V_{\xi_+} - \alpha V_{\xi_+}$, which means that

$$q = \sum_{j=1}^{k} \lambda_{\xi_j} V_{\xi_j} + \alpha V_{\xi_+} - \alpha \left(\sum_{j=1}^{k} \nu_{\xi_j} V_{\xi_j} \right) = \sum_{j=1}^{k} (\lambda_{\xi_j} - \alpha \nu_{\xi_j}) V_{\xi_j} + \alpha V_{\xi_+}$$

Since $\lambda_{\xi_j} > 0$ for all j, for $\alpha > 0$ small enough, $\lambda_{\xi_j} - \alpha \nu_{\xi_j} > 0$ for all j. Hence q is a positive linear combination of the rows of the matrix V^+, so, from the characterisation of arbitrage free asset price in the course, q is an arbitrage free price for the payoff matrix V^+.

3) *Show that if V has no redundant asset, then q is an arbitrage free price for the new payoff matrix V^+ whatever are the payoffs $v_j(\xi^+)$ for $j = 1, \ldots, J$.*

If V has no redundant asset, the matrix V is one-to-one, so V^t is onto, which means that the rows of V is a spanning family of $\mathbb{R}^J$. Hence, all payoffs $V_{\xi+}$ in $\mathbb{R}^J$ is a linear combination of the rows of V. From the previous question, this implies that q is an arbitrage free price for the new payoff matrix V^+ whatever are the payoffs at the new state of nature ξ_+.

4) *We consider the case where $k = 2$, $J = 2$, $V = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $v_1(\xi^+) = 1$, $v_2(\xi^+) = -1$.*

a) Show that $q = (1, 2)$ is an arbitrage free asset price for V;

From the characterization of arbitrage free asset price, $q = (1, 2)$ *is an arbitrage free asset price for* V *since*

$$q = V^t \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

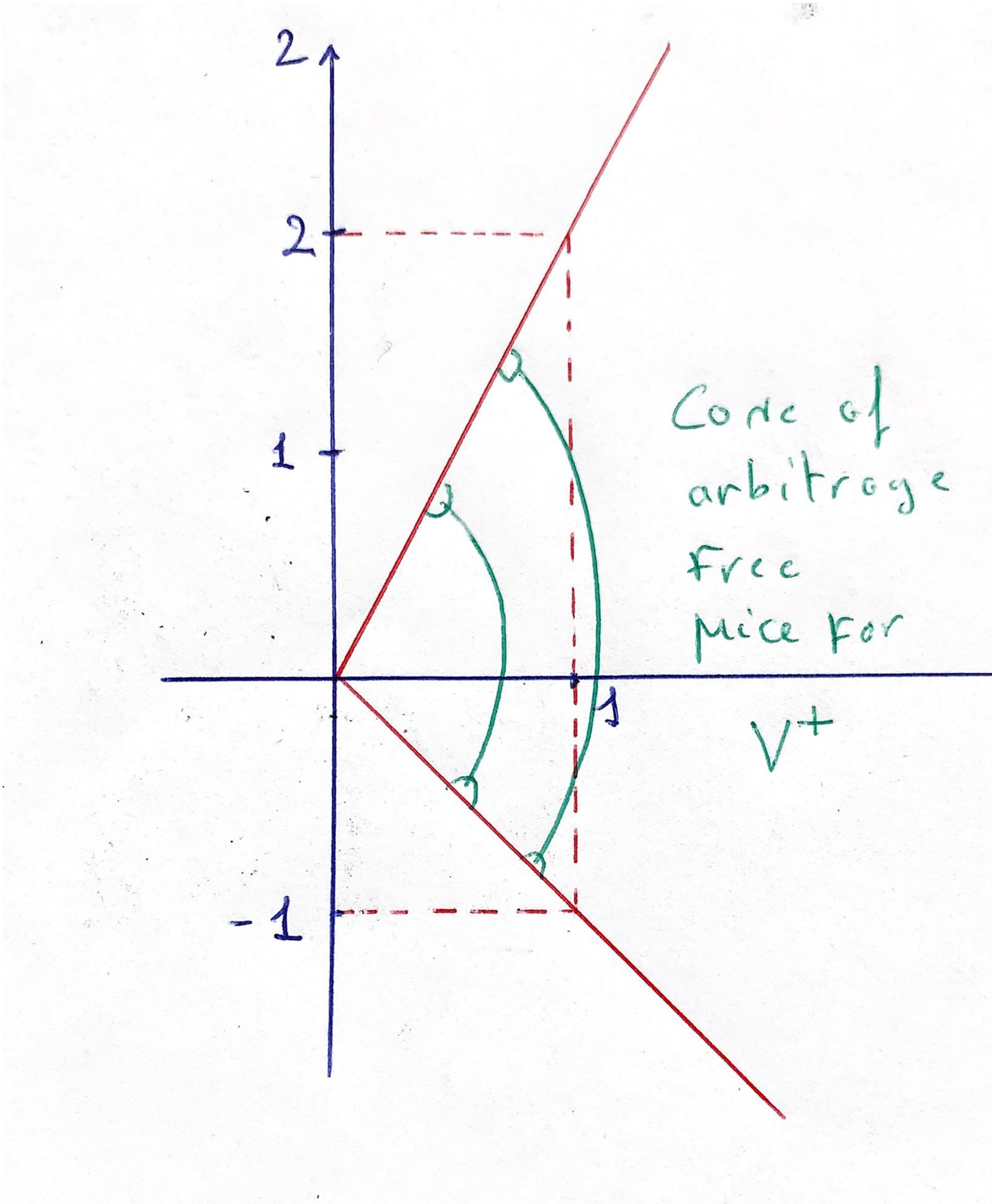
b) Show that q *is not an arbitrage free asset price for* V^+*;*

From the two first question of the exercise, q *is not an arbitrage free asset price for* V^+ *since* $V_{\xi^+} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *is not a linear combination of* V_{ξ_1} *and* V_{ξ_2} *which are both equal to* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

c) Find an arbitrage portfolio for q *and* V^+*.*

We are looking for (z_1, z_2) *such that* $\begin{pmatrix} -1 & -2 \\ 1 & 2 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_1 - 2z_2 \\ z_1 + 2z_2 \\ z_1 + 2z_2 \\ z_1 - z_2 \end{pmatrix} \geq 0_4$ *with at least one strict inequality. So,* $z_1 = -2z_2$ *and* $-3z_2 > 0$*. For example, it suffices to take* $z_1 = 2$ *and* $z_2 = -1$*.*

d) Draw on a picture in $\mathbb{R}^2$ *the set of arbitrage free asset prices for* V^+*.*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V+