Master MMMEF, 2022-2023
Correction of the Homeword on:
General Equilibrium Theory:
Economic analysis of financial markets
December 2022

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1) Show that if q is arbitrage free for the pair (V, Σ) with $\Sigma \neq \emptyset$, then there exists $\lambda \in \mathbb{R}^{\Sigma}_{++}$ such that $q = \sum_{s \in \Sigma} \lambda_s V_s$.

It suffices to duplicate the proof of Proposition 7 in Chapter 5 of the lecture notes replacing the space $\mathbb{R}^{\mathbb{D}}$ by the space $\mathbb{R}^{\{\xi_0\} \cup \Sigma}$.

An asset price $q \in \mathbb{R}^{\mathcal{J}}$ is said arbitrage free for the payoff matrix V and the information structure (S_i) if q is arbitrage free for all pairs (V, S_i), $i \in \mathcal{I}$.

2) Show that if the information structure $(S_i)_{i \in \mathcal{I}}$ is symmetric, that is $S_i = S_j$ for all $(i, j) \in \mathcal{I} \times \mathcal{I}$, then there exists at least one arbitrage free price for $(V, (S_i)_{i \in \mathcal{I}})$.

Let $\lambda \in \mathbb{R}^{S_1}_{++}$ and $q = \sum_{s \in S_1} \lambda_s V_s$. Using the proof of Proposition 7 in Chapter 5, one shows that q is arbitrage free for the pair (V, S_1) and since all sets S_i are equal to S_1, then q is arbitrage free for all pair (V, S_i), then arbitrage free for V and the structure $(S_i)_{i \in \mathcal{I}}$.

Let us consider an economy with two agents $\mathcal{I} = \{1, 2\}$, five states at date 1, $\mathbb{D}_1 = \{1, 2, 3, 4, 5\}$, an information structure $S_1 = \{1, 2, 3\}$ and $S_2 = \{1, 4, 5\}$ and the payoff matrix V:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3)

1. Show that (V, S_1, S_2) has no arbitrage free price.
2. Show that (V, Σ_1, Σ_2), with $\Sigma_1 = \{1\} = \Sigma_2$ has an arbitrage free price.

3. Show that $(V, \bar{\Sigma}_1, \bar{\Sigma}_2)$, with $\bar{\Sigma}_1 = \{1\}$ and $\bar{\Sigma}_2 = \{1, 5\}$ = has an arbitrage free price.

4. Show that $(V, \tilde{\Sigma}_1, \tilde{\Sigma}_2)$, with $\tilde{\Sigma}_1 = \{1, 2\}$ and $\bar{\Sigma}_2 = \{1, 4, 5\}$ has an arbitrage free price.

a) Let q be a no arbitrage free price of (V, S_1, S_2). Then, there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_{++}$ and $(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3_{++}$ such that

$$\begin{cases} q_1 = -\lambda_1 + \lambda_2 = -\mu_1 \\ q_2 = \lambda_2 = \mu_2 \\ q_3 = \lambda_3 = 0 \end{cases}$$

which is impossible since $\lambda_3 > 0$.
b) The structure is symmetric, then it has an arbitrage free price, $(-1, 0, 0)$ for example.
c) $(-1, 0, 0)$ is an arbitrage free price with the multipliers 1 and $(1, 1)$.
d) $(-1, 1, 0)$ is an arbitrage free price with the multipliers $(2, 1)$ and $(1, 1, 1)$.

The purpose of the end of this homework is to show that the structure $(V, (S_i)_{i\in\mathcal{I}})$ has at least one arbitrage free price if and only if it satisfies the following condition:

(AFAO) there is no $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $\sum_{i\in\mathcal{I}} z_i = 0_{\mathcal{J}}$ and $V_{s_i} \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$, with at least one strict inequality.

4) Let us assume that Condition (AFAO) holds true. Let F be the linear mapping from $(\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ to $\mathbb{R}^{\mathcal{J}} \times \mathbb{R}^{\mathcal{J}} \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}$ defined by:

$$F((z_i)_{i\in\mathcal{I}}) = \left(\sum_{i\in\mathcal{I}} z_i, -\sum_{i\in\mathcal{I}} z_i, ((V_{s_i} \cdot z_i)_{s_i\in S_i})_{i\in\mathcal{I}} \right)$$

a) Show that $\text{Im}F \cap [\mathbb{R}^{\mathcal{J}}_+ \times \mathbb{R}^{\mathcal{J}}_+ \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}_+] = \{0\}$.

Let $(a, b, (\zeta_i)_{i\in\mathcal{I}}) \in [\mathbb{R}^{\mathcal{J}}_+ \times \mathbb{R}^{\mathcal{J}}_+ \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}_+]$ in the image of F. Then, there exists $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that:

$$\begin{cases} a = \sum_{i\in\mathcal{I}} z_i \\ b = -\sum_{i\in\mathcal{I}} z_i \\ \zeta_i = (V_{s_i} \cdot z_i)_{s_i\in S_i} \text{ for all } i \in \mathcal{I} \end{cases}$$

Since $a = -b$ and $a \geq 0$ and $b \geq 0$, one concludes that $a = b = 0 = \sum_{i\in\mathcal{I}} z_i$ and since $\zeta_i \in \mathbb{R}^{\mathcal{J}}_+$, $V_{s_i} \cdot z_i = \zeta_{is_i} \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$. If for some $i \in \mathcal{I}$, $\zeta_i \neq 0$, then the AFAO is not satisfied since there is one strict inequality among $V_{s_i} \cdot z_i = \zeta_{is_i} \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$. Then $a = b = 0$ and $\zeta_i = 0$ for all i, so $\text{Im}F \cap [\mathbb{R}^{\mathcal{J}}_+ \times \mathbb{R}^{\mathcal{J}}_+ \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}_+] = \{0\}$.

b) Using the same argument as the one in the proof of the characterisation of arbitrage free price, show that there exists $(\alpha, \beta, (\lambda_i)_{i\in\mathcal{I}}) \in [\mathbb{R}^{\mathcal{J}}_{++} \times \mathbb{R}^{\mathcal{J}}_{++} \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}_{++}]$, such that for every $i \in \mathcal{I}$,

$$0 = \alpha - \beta + \sum_{s_i \in S_i} \lambda_{i,s_i} V_{s_i}$$

Since $\mathrm{Im}F \cap [\mathbb{R}^{\mathcal{J}}_{+} \times \mathbb{R}^{\mathcal{J}}_{+} \times \prod_{i \in \mathcal{I}} \mathbb{R}^{S_i}_{+}] = \{0\}$, using again the proof of Proposition 7 of Chapter 5 of the lecture notes, one deduces that there exists $(\alpha, \beta, (\lambda_i)_{i \in \mathcal{I}}) \in [\mathbb{R}^{\mathcal{J}}_{++} \times \mathbb{R}^{\mathcal{J}}_{++} \times \prod_{i \in \mathcal{I}} \mathbb{R}^{S_i}_{++}]$ such that $0 = F^t(\alpha, \beta, (\lambda_i)_{i \in \mathcal{I}})$.

To compute $F^t(\alpha, \beta, (\lambda_i)_{i \in \mathcal{I}})$, we remark that for all $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$, $F((z_i)_{i \in \mathcal{I}}) \cdot (\alpha, \beta, (\lambda_i)_{i \in \mathcal{I}}) = ((z_i)_{i \in \mathcal{I}}) \cdot F^t(\alpha, \beta, (\lambda_i)_{i \in \mathcal{I}})$. So, the first inner product is equal to:

$$\alpha \cdot \left(\sum_{i \in \mathcal{I}} z_i \right) + \beta \cdot \left(- \sum_{i \in \mathcal{I}} z_i \right) + \sum_{i \in \mathcal{I}} \sum_{s_i \in S_i} \lambda_{is_i} (V_{s_i} \cdot z_i)$$

Reordering the terms, we get that this is equal to:

$$\sum_{i \in \mathcal{I}} z_i \left(\alpha - \beta + \sum_{s_i \in S_i} \lambda_{is_i} V_{s_i} \right)$$

So, $F^t(\alpha, \beta, (\lambda_i)_{i \in \mathcal{I}}) = \left(\alpha - \beta + \sum_{s_i \in S_i} \lambda_{is_i} V_{s_i} \right)_{i \in \mathcal{I}}$, which leads to the desired result.

c) Conclude by showing that $q = \beta - \alpha$ is an arbitrage free price for V and the information structure (S_i).

From the above result, for all i, $q = \beta - \alpha = \sum_{s_i \in S_i} \lambda_{is_i} V_{s_i}$, so q is an arbitrage free price for the structure $(V, (S_i)_{i \in \mathcal{I}})$ by the very definition.

5) Let us assume the structure $(V, (S_i)_{i \in \mathcal{I}})$ has at least one arbitrage free price q.

a) Show that if $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ satisfies $V_{s_i} \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$, with at least one strict inequality, then $q \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ with at least one strict inequality.

For all $i \in \mathcal{I}$, there exists $\lambda_i \in \mathbb{R}^{S_i}_{++}$ such that $q = \sum_{s_i \in S_i} \lambda_{is_i} V_{s_i}$. So, $q \cdot z_i = \left(\sum_{s_i \in S_i} \lambda_{is_i} V_{s_i} \right) \cdot z_i = \sum_{s_i \in S_i} \lambda_{is_i} V_{s_i} \cdot z_i$. Since all terms of the previous sum are nonnegative and at least one is positive, one concludes that the sum is positive.

b) Using an argument by contraposition, conclude that Condition (AFAO) holds true.

If the AFAO does not hold, then there exists $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $\sum_{i \in \mathcal{I}} z_i = 0_{\mathcal{J}}$ and $V_{s_i} \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$, with at least one strict inequality. Hence, for all i, $q \cdot z_i \geq 0$ and there exists one i_0 such that $q \cdot z_{i_0} > 0$. Consequently, $q \cdot \sum_{i \in \mathcal{I}} z_i > 0$, which is in contradiction with $\sum_{i \in \mathcal{I}} z_i = 0_{\mathcal{J}}$.