Master MMMEF, 2022-2023
Homeword on:
General Equilibrium Theory:
Economic analysis of financial markets
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We consider a standard two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_0 = \{\xi_0\}$, $\mathbb{D}_1 = \{\xi_1, \xi_2, \ldots, \xi_S\}$ is the set of states of nature at date 1. We consider a financial structure with a finite collection of $\mathcal{J}$ nominal assets and the payoff matrix is denoted V, which is a $\sharp\mathbb{D}_1 \times \mathcal{J}$ matrix.

We consider a finite set $\mathcal{I}$ of consumers. At date 0, Agent i has a private information represented by a nonempty subset S_i of $\mathbb{D}_1$, which means that she knows that the state that will occur at date 1 belongs to S_i. We assume that the information is reliable in the sense that the true state occurring at date 1 belongs indeed to S_i. The collection $(S_i)_{i\in\mathcal{I}}$ is called an information structure if $\cap_{i\in\mathcal{I}} S_i \neq \emptyset$ and a collection $(\Sigma_i)_{i\in\mathcal{I}}$ such that $\Sigma_i \subset S_i$ and $\cap_{i\in\mathcal{I}}\Sigma_i \neq \emptyset$ is called a refinement of (S_i).

Given a subset $\Sigma \subset \mathbb{D}_1$, an asset price $q \in \mathbb{R}^{\mathcal{J}}$ is said arbitrage free for the pair (V, Σ) if there is no portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $q \cdot z \leq 0$, $V_s \cdot z \geq 0$ for all $s \in \Sigma$ with at least one strict inequality. If Σ is empty, we say by convention that all prices in $\mathbb{R}^{\mathcal{J}}$ are arbitrage free.

1) Show that if q is arbitrage free for the pair (V, Σ) with $\Sigma \neq \emptyset$, then there exists $\lambda \in \mathbb{R}^{\Sigma}_{++}$ such that $q = \sum_{s\in\Sigma} \lambda_s V_S$.

An asset price $q \in \mathbb{R}^{\mathcal{J}}$ is said arbitrage free for the payoff matrix V and the information structure (S_i) if q is arbitrage free for all pairs (V, S_i), $i \in \mathcal{I}$.

2) Show that if the information structure $(S_i)_{i\in\mathcal{I}}$ is symmetric, that is $S_i = S_j$ for all $(i, j) \in \mathcal{I}\times\mathcal{I}$, then there exists at least one arbitrage free price for $(V, (S_i)_{i\in\mathcal{I}})$.

Let us consider an economy with two agents $\mathcal{I} = \{1, 2\}$, five states at date 1, $\mathbb{D}_1 = \{1, 2, 3, 4, 5\}$, an information structure $S_1 = \{1, 2, 3\}$ and $S_2 = \{1, 4, 5\}$ and the payoff matrix V:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3)

1. Show that (V, S_1, S_2) has no arbitrage free price.
2. Show that (V, Σ_1, Σ_2), with $\Sigma_1 = \{1\} = \Sigma_2$ has an arbitrage free price.
3. Show that $(V, \bar{\Sigma}_1, \bar{\Sigma}_2)$, with $\bar{\Sigma}_1 = \{1\}$ and $\bar{\Sigma}_2 = \{1, 5\} =$ has an arbitrage free price.
4. Show that $(V, \tilde{\Sigma}_1, \tilde{\Sigma}_2)$, with $\tilde{\Sigma}_1 = \{1, 2\}$ and $\bar{\Sigma}_2 = \{1, 4, 5\}$ has an arbitrage free price.

The purpose of the end of this homework is to show that the structure $(V, (S_i)_{i\in\mathcal{I}})$ has at least one arbitrage free price if and only if it satisfies the following condition:

(AFAO) there is no $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $\sum_{i\in\mathcal{I}} z_i = 0_{\mathcal{J}}$ and $V_{s_i} \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$, with at least one strict inequality.

4) Let us assume that Condition (AFAO) holds true. Let F be the linear mapping from $(\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ to $\mathbb{R}^{\mathcal{J}} \times \mathbb{R}^{\mathcal{J}} \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}$ defined by:

$$F((z_i)_{i\in\mathcal{I}}) = \left(\sum_{i\in\mathcal{I}} z_i, -\sum_{i\in\mathcal{I}} z_i, ((V_{s_i} \cdot z_i)_{i\in S_i})_{i\in\mathcal{I}} \right)$$

a) Show that $\mathrm{Im} F \cap [\mathbb{R}^{\mathcal{J}}_{+} \times \mathbb{R}^{\mathcal{J}}_{+} \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}_{+}] = \{0\}$.
b) Using the same argument as the one in the proof of the characterisation of arbitrage free price, show that there exists $(\alpha, \beta, (\lambda_i)_{i\in\mathcal{I}}) \in [\mathbb{R}^{\mathcal{J}}_{++} \times \mathbb{R}^{\mathcal{J}}_{++} \times \prod_{i\in\mathcal{I}} \mathbb{R}^{S_i}_{++}]$, such that for every $i \in \mathcal{I}$,

$$0 = \alpha - \beta + \sum_{s_i\in S_i} \lambda_{i,s_i} V_{s_i}$$

c) Conclude by showing that $q = \beta - \alpha$ is an arbitrage free price for V and the information structure (S_i).

5) Let us assume the structure $(V, (S_i)_{i\in\mathcal{I}})$ has at least one arbitrage free price q.
a) Show that if $(z_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ satisfies $V_{s_i} \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ and all $s_i \in S_i$, with at least one strict inequality, then $q \cdot z_i \geq 0$ for all $i \in \mathcal{I}$ with at least one strict inequality.
b) Using an argument by contraposition, conclude that Condition (AFAO) holds true.