Master MMMEF, 2021-2022
Homeword on:
General Equilibrium Theory: Economic analysis of financial markets
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We consider a three-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$ is the set of states of nature at date 1 and $\mathbb{D}_2$ the set of states of nature at date 2. We assume that we have a unique commodity at each state. We consider a financial structure with a finite collection of $\mathcal{J}$ nominal assets. The main difference with the two-period model is the fact that the assets can be issued at date 0 but also at the nodes at date 1. For an asset $j \in \mathcal{J}$, we denote by $\xi(j)$ its node of issuance. An asset is bought or sold only at its node of issuance at a price q_j. An asset yields payoffs only at the successor nodes of its issuance node. So, if an asset is issued at date 0, at the unique node ξ_0, then it yields payoff at the nodes in $\mathbb{D}_1 \cup \mathbb{D}_2$, and if an asset j is issued at date 1, it yields payoffs only at the immediate successors of its issuance node $\xi(j)^+ \subset \mathbb{D}_2$. We assume that we have no trivial asset with a zero payoff for all nodes. To simplify the notation, we consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and we assume that it is equal to zero if ξ is not a successor of the issuance node $\xi(j)$. So we consider the payoff matrix V, which is a $\sharp\mathbb{D} \times \mathcal{J}$ matrix. Note that the first row of the matrix V has only 0 entries. A portfolio z is an element in $\mathbb{R}^{\mathcal{J}}$.

For asset prices $q \in \mathbb{R}^{\mathcal{J}}$, the full payoff matrix $W(q)$ is the $\sharp\mathbb{D} \times \mathcal{J}$ matrix defined by:

$$W_\xi^j(q) \ := V_\xi^j - \delta_{\xi,\xi(j)} q_j,$$

where $\delta_{\xi,\xi'} = 1$ if $\xi = \xi'$ and 0 otherwise.

The asset price $q \in \mathbb{R}^{\mathcal{J}}$ is an arbitrage free price if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(q)z > 0$.

Let $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}\}$ with $\xi_0^+ = \{\xi_1, \xi_2\}$, $\xi_1^+ = \{\xi_{11}\}$ and $\xi_2^+ = \{\xi_{21}\}$.

1) Draw the tree $\mathbb{D}$.

We consider the financial structure with three assets $\mathcal{J} = \{j^1, j^2, j^3\}$, $\xi(j^1) =$

ξ_0, $\xi(j^2) = \xi_1$ and $\xi(j^3) = \xi_2$. The payoff matrix is

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & 0 & 0 \\ 1 & \mathbf{0} & 0 \\ -1 & 0 & \mathbf{0} \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{21} \end{matrix}$$

2) Write the full payoff matrix $W(q)$ with $\bar{q} = (0, -1, 1)$.
3) Show that the kernel of V is included and different to the kernel of $W(\bar{q})$.

We are now coming back to the general model.
4) By mimicking the proof given in the course, show that q is an arbitrage free price if and only if it exists a state price vector $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p, q)\lambda = 0$ or equivalently

$$\forall j \in \mathcal{J}, \lambda_{\xi(j)} q_j = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi V^j_\xi$$

5) Show that the kernel of V is equal to the kernel of $W(q)$ for all arbitrage free asset price q when the assets are all issued at date 0 (resp. at date 1).
6) Show that the kernel of V is equal to the kernel of $W(q)$ for all arbitrage free asset price q when the assets issued at date 0 are short term assets with a non-zero payoff only for the nodes in $\mathbb{D}_1$ of the period 1.
7) Let $\mathbb{D}^e_1$ be the nodes at date 1 such that there is at least one asset issued at this node. For $\xi \in \mathbb{D}^e_1$, we denote by E_ξ the linear subspace of $\mathbb{R}^{\xi^+}$ generated by the vectors of payoffs of the assets issued at node ξ. For $\xi \in \mathbb{D}^e_1$, we denote by E^0_ξ the linear subspace of $\mathbb{R}^{\xi^+}$ generated by the vector of payoffs of the assets issued at node ξ_0 if any, or $E^0_\xi = \{0_{\mathbb{R}^{\xi+}}\}$.

Show that the kernel of V is equal to the kernel of $W(q)$ for all arbitrage free asset price q when the following condition is satisfied: for all nodes $\xi \in \mathbb{D}^e_1$, $E_\xi \cap E^0_\xi = \{0_{\mathbb{R}^{\xi+}}\}$.

We are considering again the numerical example presented above.
8) Show that $\bar{q} = (0, -1, 1)$ is an arbitrage free asset.
9) For $\xi = \xi_1$ and ξ_2, determine the spaces E_ξ and E^0_ξ and show that the condition of Question 7 is not satisfied.
10) Determine all arbitrage free asset prices q such that the kernel of V is not equal to the kernel of $W(q)$.