Master MMMEF, 2021-2022
Solution of the Final Exam on: General Equilibrium Theory: Economic analysis of financial markets
December 2021
2 hours

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Q1) In a two-period model with uncertainty, explain the non satiation state by state assumption for the utility function of a consumer.

A utility function satisfies the non satiation state by state assumption if, for all given state of nature, it is possible to strictly increase the utility level of a given consumption by modifying the consumption at the given state of nature, keeping constant the consumptions at the other states.

Q2) What is the relationship between an equilibrium with a full set of Arrow securities and a contingent commodity equilibrium?

The equilibrium allocations are the same and the CC equilibrium price of a contingent commodity is the product of the equilibrium Arrow spot price of this commodity by the equilibrium Arrow security price associated to the state of the contingent commodity.

Q3) In a two-period model, given a financial structure represented by the payoff matrix function $p \to V(p)$, what is the definition of the full payoff matrix ?

The full payoff matrix has a number of rows equals to the number of states of nature and a number of column equals to the number of assets. It depends on the spot price p and the asset price q. It is obtained from the payoff matrix $V(p)$ by adding an additional first row which is the transpose of the column vector q.

Q4) Give a necessary condition on the financial structure to get the same equilibrium allocations for the financial equilibrium and for the contingent commodity equilibrium.

The financial structure must be complete that is the rank of the payoff matrix $V(p)$ must be the number of states of nature at date 1.

Q5) With a nominal financial structure represented by the matrix V and an arbitrage free asset price q, what is a present value vector associated to q?

A present value vector is a vector λ in $\mathbb{R}^{\mathbb{D}_1}_{++}$ satisfying $q = V^t\lambda$.

Exercise 1 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$ and a financial structure with J assets represented by the constant $\sharp\mathbb{D}_1 \times J$ payoff matrix V. We assume that the financial structure has no useless portfolio. Show that the vector $q \in \mathbb{R}^J$ is arbitrage free for the financial structure if and only if $q \cdot z > 0$ for all $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(z) \geq 0$.

By definition, q is arbitrage free if there is no $z \in \mathbb{R}^J$ such that $V(z) \geq 0$, $-q \cdot z \geq 0$ with at least one strict inequality.

If q is arbitrage free and $z \in \mathbb{R}^J \setminus \{0\}$ satisfies $V(z) \geq 0$, then we remark that $V(z) \neq 0$ since, otherwise, z would be a useless portfolio. Furthermore, there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V^t\lambda$. So, $q \cdot z = V^t\lambda \cdot z = \lambda \cdot V(z) > 0$. The last inequality comes from the fact that $V(z) \in \mathbb{R}^{\mathbb{D}_1}_{+} \setminus \{0\}$ and $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$. So $q \in \mathbb{R}^J$ is arbitrage free for the financial structure implies that $q \cdot z > 0$ for all $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(z) \geq 0$.

Conversely, by contraposition, let us assume that $q \cdot z > 0$ for all $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(z) \geq 0$ and there exists a vector $\zeta \in \mathbb{R}^J$ such that $V(\zeta) \geq 0$, $-q \cdot \zeta \geq 0$ with at least one strict inequality. Then, the strict inequality implies that $\zeta \neq 0$, so, since there is no useless portfolio, $V(\zeta) \neq 0$, and from the first part of our assumption $q \cdot \zeta > 0$, which leads to a contradiction with $-q \cdot \zeta \geq 0$. So, if $q \cdot z > 0$ for all $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(z) \geq 0$, then q is arbitrage free.

Exercise 2 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D} = \{\xi_0, \xi_1, \xi_2\}$ where ξ_1 and ξ_2 are the two successors of ξ_0. There is a unique commodity at each state and the price of the commodity on the spot market is normalized to 1. There are two consumers with the same utility function:

$$u(x_0, x_1, x_2) = x_0 x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

The initial endowments are: $e^1 = (3, 1, 1)$ and $e^2 = (\frac{1}{3}, 3, 2)$.

We first assume that there is a unique asset (the riskless bond) on the financial market with the payoffs $(1, 1)$. We denote by $q > 0$ the price of this asset.

1) Write explicitely the utility maximisation problem over the financial budget set for both consumers.

$$\left\{\begin{array}{l} \text{Maximise } x_0^1 (x_1^1)^{\frac{1}{2}} (x_2^1)^{\frac{1}{2}} \\ x_0^1 + q z^1 \leq 3 \\ x_1^1 \leq 1 + z^1 \\ x_2^1 \leq 1 + z^1 \end{array}\right. \qquad \left\{\begin{array}{l} \text{Maximise } x_0^2 (x_1^2)^{\frac{1}{2}} (x_2^2)^{\frac{1}{2}} \\ x_0^2 + q z^2 \leq \frac{1}{3} \\ x_1^2 \leq 3 + z^2 \\ x_2^2 \leq 2 + z^2 \end{array}\right.$$

2) Show that the above problem for the first consumer can be reduced to the following one where z^1 is the unique unknown:

$$\max\{(3 - q z^1)(1 + z^1)^{\frac{1}{2}} (1 + z^1)^{\frac{1}{2}} \mid z^1 \in [-1, \frac{3}{q}]\}$$

and write the equivalent problem for the second consumer with the quantity of asset as unique unkown.

Since the utility function is strictly increasing, the budget constraints are binding at the optimal solution, that is $x_0^1 + qz^1 = 3$, $x_1^1 = 1 + z^1$ and $x_2^1 = 1 + z^1$. So, the only unknown is the quantity z^1. This quantity is bounded by the fact that the consumptions must be non negative, so, $1 + z^1 \geq 0$ and $3 - qz^1 \geq 0$. Hence, z^1 must belong to the interval $[-1, \frac{3}{q}]$ and the objective function is $(3 - qz^1)(1 + z^1)^{\frac{1}{2}}(1 + z^1)^{\frac{1}{2}}$.

With the same reasoning for the second consumer, z^2 must belong to the interval $[-2, \frac{1}{3q}]$ and the objective function is $(\frac{1}{3} - qz^2)(3 + z^2)^{\frac{1}{2}}(2 + z^2)^{\frac{1}{2}}$.

3) Show that for $q = 1$, $z^1 = 1$ and $z^2 = -1$ are solutions of the two above problem. Deduce a financial equilibrium of this economy. Is the equilibrium allocation Pareto optimal?

The equilibrium consumptions associated to $q = 1$, $z^1 = 1$ and $z^2 = -1$ are $x^1 = (2, 2, 2)$ and $x^2 = (\frac{4}{3}, 2, 1)$ and we immediately check that the market clearing conditions are satisfied.

For $q = 1$, the derivative of the logarithm of the objective function is for the first consumer:

$$\frac{-1}{3 - z^1} + \frac{1}{2(1 + z^1)} + \frac{1}{2(1 + z^1)}$$

and for the second consumer:

$$\frac{-1}{\frac{1}{3} - z^2} + \frac{1}{2(3 + z^2)} + \frac{1}{2(2 + z^2)}$$

and one easily checks that these derivatives vanish for $z^1 = 1$ and $z^2 = -1$, which means that the objective functions reach their maximum for these portfolios since the logarithms are concave functions.

This equilibrium allocation is Pareto optimal if and only if the two gradients of the utility functions are colinear.

$$\nabla u(2, 2, 2) = (2, 1, 1) \quad \nabla u\left(\frac{4}{3}, 2, 1\right) = \left(\sqrt{2}, \frac{\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right)$$

so, they are not colinear and the equilibrium allocation is not Pareto optimal.

We now assume that there is a second asset (an Arrow security) on the financial market with the payoffs $(1, 0)$.

4) Show that the financial structure is complete with these two assets.

The financial structure is complete since the payoff matrix $V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is of rank 2, the number of states of nature at the second period.

5) Give the definition of a contingent commodity equilibrium in this economy.

A contingent commodity equilibrium is a price vector $\pi \in \mathbb{R}^3$ and two consumption x^{*1} and x^{*2} such that for $i = 1, 2$, x^{*i} maximise $u(x^i)$ under the budget constraint $\pi \cdot x^i \leq \pi \cdot e^i$ and $x^{*1} + x^{*2} = e^1 + e^2$.

6) Show that $(\pi = (1, \pi_1, \pi_2), x^{*1} = (x_0^{*1}, x_1^{*1}, x_2^{*1}), x^{*2} = (x_0^{*2}, x_1^{*2}, x_2^{*2}))$ is a contingent commodity equilibrium if:

$$\left\{\begin{array}{l} \frac{x_0^{*1}}{2x_1^{*1}} = \frac{x_0^{*2}}{2x_1^{*2}} = \pi_1 \\ \frac{x_0^{*1}}{2x_2^{*1}} = \frac{x_0^{*2}}{2x_2^{*2}} = \pi_2 \\ x_0^{*1} + \pi_1 x_1^{*1} + \pi_2 x_2^{*1} = 3 + \pi_1 + \pi_2 \\ x_0^{*2} + \pi_1 x_1^{*2} + \pi_2 x_2^{*2} = \frac{1}{3} + 3\pi_1 + 2\pi_2 \\ x_1^{*1} + x_1^{*2} = 4 \\ x_2^{*1} + x_2^{*2} = 3 \end{array}\right.$$

The two last equations are the market clearing conditions at states 1 and 2. Together with the two equalities for the budget constraints, they imply the market clearing condition at state 0. The two first equation shows that the marginal rates of substitution for both agents between commodities at state 1 and 0 and 2 and 0 are equal to the relative price. These conditions together with the equality for the budget constraints are necessary and sufficient conditions for the optimality of the consumption x^{*1} and x^{*2} since the common utility function is quasi-concave and increasing. So (π, x^{*1}, x^{*2}) is a contingent commodity equilibrium.

7) Check that $\big((1, \frac{5}{12}, \frac{5}{9}), \frac{143}{240}(\frac{10}{3}, 4, 3), \frac{97}{240}(\frac{10}{3}, 4, 3)\big)$ is the contingent commodity equilibrium. Is this allocation Pareto optimal?

It suffices to check that the 6 above equalities are satisfied by the given datas.

This allocation is Pareto optimal since an equilibrium allocation of a contingent commodity equilibrium is always Pareto optimal.

8) Give a financial equilibrium of the economy with the two assets.

Since the financial structure is complete, we know that the equilibrium allocations of the financial equilibrium are the same as the ones of the contingent commodity equilibrium given in the previous question. Since the spot prices are normalized to 1, we just have to compute the portfolio (z_1^1, z_2^1) for the first consumer and the asset prices (q_1, q_2). Then the portfolio (z_1^2, z_2^2) for the second consumer is $-(z_1^1, z_2^1)$. To find (z_1^1, z_2^1), we have to solve the following system:

$$\left\{\begin{array}{l} 4\frac{143}{240} = 1 + z_1^1 + z_2^1 \\ 3\frac{143}{240} = 1 + z_1^1 \end{array}\right.$$

So, $z^1 = (\frac{189}{240}, \frac{143}{240})$.

For the asset price q, we have an infinite number of solutions. It must satisfy $0 < q_2 < q_1$ to be an arbitrage free asset price and the first budget constraint, that is,

$$\frac{143}{240}\frac{10}{3} + q_1\frac{189}{240} + q_2\frac{143}{240} = 3$$

or equivalently

$$\frac{189q_1 + 143q_2}{240} = \frac{73}{72}$$