Master MMMEF, 2020-2021
Homeword on:
General Equilibrium Theory:
Economic analysis of financial markets
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We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$ is the set of states of nature at date 1. We assume that we have a unique commodity at each state. We consider a financial structure with a finite collection of $\mathcal{J}$ assets, represented by the payoff mapping $p \to V(p)$ from $\mathbb{R}^{\mathbb{D}}$ to the set of $\sharp\mathbb{D}_1 \times \mathcal{J}$ matrices.

1) Show that the rank of the full payoff matrix $W(p, q)$ for a commodity price p and an asset price $q \in \mathbb{R}^{\mathcal{J}}$ has the same rank than the payoff matrix V if q is arbitrage free.

2) We assume that $V(p)$ is one-to-one. Show that the set $Q(p)$ is an open convex cone of $\mathbb{R}^{\mathcal{J}}$.

3) We assume that the bond is among the collection of assets $\mathcal{J}$, that is the asset whose payoffs are equal to 1 at each state $\xi \in \mathbb{D}_1$. The bond is the asset j_0. We consider an arbitrage free asset price q such that $q_{j_0} = 1$. Show that for all asset j,

$$q_j \in]\min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}, \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}[$$

4) We assume now that $\sharp\mathbb{D}_1 = 2$ and that the collection $\mathcal{J}$ contains only nominal assets. The payoff matrix is

$$V = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Let q be an arbitrage free price such that $q_1 = 1$. Show that $q_2 \in]-1, 2[$. Compute the no arbitrage price of the asset k with payoffs $(0, 2)$ as a function of q_2. Compute the portfolio $z \in \mathbb{R}^2$ of the two initial assets which duplicates Asset k.

5) We consider a partition $\mathcal{P} = (\Xi_j)_{j \in \mathcal{J}}$ of $\mathbb{D}_1$ where all subsets Ξ_j is nonempty. For all $j \in \mathcal{J}$, we associates the nominal asset j defined by its payoffs: 1 for $\xi \in \Xi_j$ and 0 otherwise.

a) Show that the payoff matrix V of this financial structure is one-to-one.
b) Show that the set of no arbitrage asset prices is $\mathbb{R}^{\mathcal{J}}_{++}$.
c) Show that the asset price q defined by $q_j = \frac{\sharp \Xi_j}{\sharp \mathbb{D}_1}$ is arbitrage free and give one present value vector associated to q_j.
d) Let π be a probability on $\mathbb{D}_1$ such that $\pi(\xi) > 0$ for all $\xi \in \mathbb{D}_1$. Show that q defined by $q_j = \pi(\Xi_j)$ is an arbitrage free price and give one present value vector associated to q_j.
e) Show that the financial structure V is complete if and only if the partition $\mathcal{P}$ contains all singletons $\{\xi\}_{\xi \in \mathbb{D}_1}$.