Master MMMEF, 2020-2021
Solution of the homework on: General Equilibrium Theory: Economic analysis of financial markets November 2020

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We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$ is the set of states of nature at date 1. We assume that we have a unique commodity at each state. We consider a financial structure with a finite collection of $\mathcal{J}$ assets, represented by the payoff mapping $p \to V(p)$ from $\mathbb{R}^{\mathbb{D}}$ to the set of $\sharp\mathbb{D}_1 \times \mathcal{J}$ matrices.

1) Show that the rank of the full payoff matrix $W(p,q)$ for a commodity price p and an asset price $q \in \mathbb{R}^{\mathcal{J}}$ has the same rank than the payoff matrix V if q is arbitrage free.

Let q be an arbitrage free asset price. Then, from the characterisation proved in the course, there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V(p)^t\lambda$. We know that the rank of V plus the dimension of the kernel of $V(p)$ is equal to $\sharp J$ the dimension of the space $\mathbb{R}^{\mathcal{J}}$. The same formula holds true for $W(p,q)$, which is defined on the space $\mathbb{R}^{\mathcal{J}}$. So, to prove the equality of the rank, it suffices to prove that the kernel are identical.

Since

$$W(p,q) = \begin{pmatrix} -q \\ V(p) \end{pmatrix}$$

one clearly has $\mathrm{Ker}W(p,q) \subset \mathrm{Ker}V(p)$. If $z \in \mathrm{Ker}V(p)$, then $V(p)z = 0$ and $\lambda \cdot V(p)z = V(p)^t\lambda \cdot z = q \cdot z = 0$, so $W(p,q)z = 0$ and the kernels coincide.

2) We assume that $V(p)$ is one-to-one. Show that the set $Q(p)$ is an open convex cone of $\mathbb{R}^{\mathcal{J}}$.

$Q(p)$ is the image by a linear mapping $V(p)^t$ of the convex cone $\mathbb{R}^{\mathbb{D}_1}_{++}$, so it is a convex cone.

If $V(p)$ is one-to-one, then $V(p)^t$ is onto and the image of an open set by a onto linear mapping is open. So $Q(p) = V(p)^t\mathbb{R}^{\mathbb{D}_1}_{++}$ is open.

3) We assume that the bond is among the collection of assets $\mathcal{J}$, that is the asset whose payoffs are equal to 1 at each state $\xi \in \mathbb{D}_1$. The bond is the asset j_0. We consider an arbitrage free asset price q such that $q_{j_0} = 1$. Show that for all asset j,

$$q_j \in]\min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}, \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}[$$

Let $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ be the present value vector associated to q. We remark that $q_{j_0} = 1 = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi$. For all asset $j \in \mathcal{J}$, $q_j = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V_\xi(p)$. For all $\xi \in \mathbb{D}_1$, $V_\xi(p) \geq \min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$ and $V_\xi(p) \leq \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$. So multiplying these inequalities by λ_ξ and summing over ξ, we get $q_j \geq \min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$ and $q_j \leq \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$.

If the asset j has not a constant payoff across the state of nature, that is if $\min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\} < \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$, then the inequalities are strict since for at least one $\bar{\xi}$, $V_{\bar{\xi}}(p) > \min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$ and for at least one ξ', $V_{\xi'}(p) < \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}$, so

$$q_j \in]\min\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}, \max\{v_j(p,\xi) \mid \xi \in \mathbb{D}_1\}[$$

in this case.

4) We assume now that $\sharp\mathbb{D}_1 = 2$ and that the collection $\mathcal{J}$ contains only nominal assets. The payoff matrix is

$$V = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Let q be an arbitrage free price such that $q_1 = 1$. Show that $q_2 \in]-1, 2[$. Compute the no arbitrage price of the asset k with payoffs $(0,2)$ as a function of q_2. Compute the portfolio $z \in \mathbb{R}^2$ of the two initial assets which duplicates Asset k.

Let $\lambda \in \mathbb{R}^2_{++}$ the present value vector associated to q. Then $q_1 = 1 = \lambda_1 + \lambda_2$. Using the same argument as in the previous question, $q_2 = -\lambda_1 + 2\lambda_2 \in]-1, 2[$.

From the pricing by arbitrage, the price q_k of the asset k is equal to $2\lambda_2$ and $q_2 = -(1-\lambda_2) + 2\lambda_2$. So, $\lambda_2 = \frac{q_2+1}{3}$. So, $q_k = 2\frac{q_2+1}{3}$.

We remark that

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

so the portfolio $z = (\frac{2}{3}, \frac{2}{3})$ duplicates Asset k.

5) We consider a partition $\mathcal{P} = (\Xi_j)_{j \in \mathcal{J}}$ of $\mathbb{D}_1$ where all subsets Ξ_j is nonempty. For all $j \in \mathcal{J}$, we associates the nominal asset j defined by its payoffs: 1 for $\xi \in \Xi_j$ and 0 otherwise.

a) Show that the payoff matrix V of this financial structure is one-to-one.

Since $\mathcal{P}$ is a partition, for each j, Ξ_j is nonempty, so there exists a state of nature $\xi \in \mathbb{D}_1$ such that $\xi \in \Xi_j$ and $\xi \notin \Xi_{j'}$ for all $j' \neq j$. Hence, on the row ξ of the payoff matrix V of this financial structure, there is only one entries equals to 1 on the jth column. So, if $Vz = 0$, this implies that $z_j = 0$ by considering the row ξ. Since this is true for all $j \in \mathcal{J}$, then $z = 0$, which shows that V is one-to-one.

b) Show that the set of no arbitrage asset prices is $\mathbb{R}^{\mathcal{J}}_{++}$.

Let $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$. Then, the jth component of $V^t\lambda$ is equal to $\sum_{\xi \in \Xi_j} \lambda_\xi$. So, one deduces that $V^t\lambda \in \mathbb{R}^{\mathcal{J}}_{++}$, so the set of no arbitrage asset prices is included in $\mathbb{R}^{\mathcal{J}}_{++}$.

Let $q \in \mathbb{R}^{\mathcal{J}}_{++}$. Let $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ defined as follows: for all $\xi \in \mathbb{D}_1$, there exists a unique $j \in \mathcal{J}$ such that $\xi \in \Xi_j$. So, we define $\lambda_\xi = \frac{q_j}{\sharp \Xi_j}$. From the formula of the above paragraph, we deduces that q_j is equal to the jth component of $V^t\lambda$. So q is an arbitrage free asset price. Hence, the set of no arbitrage asset prices is equal to $\mathbb{R}^{\mathcal{J}}_{++}$.

c) Show that the asset price q defined by $q_j = \frac{\sharp \Xi_j}{\sharp \mathbb{D}_1}$ is arbitrage free and give one present value vector associated to q_j.

Using the computation of the previous question, we check that $q = V^t\lambda$ for $\lambda_\xi = \frac{1}{\sharp \mathbb{D}_1}$ for all ξ. So, q is an arbitrage free asset price and λ is one present value vector associated to q.

d) Let π be a probability on $\mathbb{D}_1$ such that $\pi(\xi) > 0$ for all $\xi \in \mathbb{D}_1$. Show that q defined by $q_j = \pi(\Xi_j)$ is an arbitrage free price and give one present value vector associated to q_j.

Once again, using the computation of (b), we check that $q = V^t\pi$. So, q is an arbitrage free asset price and π is one present value vector associated to q.

e) Show that the financial structure V is complete if and only if the partition $\mathcal{P}$ contains all singletons $\{\xi\}_{\xi \in \mathbb{D}_1}$.

If the partition $\mathcal{P}$ contains all singletons $\{\xi\}_{\xi \in \mathbb{D}_1}$, then $\sharp J = \sharp \mathbb{D}_1$ and from Question (a), V is a one-to-one square matrix, so, it is regular, hence onto, and so the financial structure is complete.

Conversely, if the financial structure is complete, then V is of rank $\sharp \mathbb{D}_1$, which means that the number of columns of V, $\sharp \mathcal{J}$ is greater or equal to $\sharp \mathbb{D}_1$. The only partition of $\mathbb{D}_1$ with more than $\sharp \mathcal{J}$ elements is the partition of all singletons of $\mathbb{D}_1$.