Master MMMEF, 2020-2021
Crrection of the Final Exam on: General Equilibrium Theory: Economic analysis of financial markets December 2020

December 13, 2021

Q1) What is the definition of an Arrow Security?

A security which delivers one unit of account or the unit price of one commodity in one state and nothing in all other states.

Q2) What is the definition of a redundant asset?

An assets, the payoff vector of which is a linear combination of the payoff vectors of the other assets.

Q3) Under the assumption that $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$, provide a necessary and sufficient condition under which $V(p)$ is complete.

$V(p)$ is complete if and only if $V(p)$ is onto.

Q4) Let (p, q) be a spot - asset price pair such that q is arbitrage free. How can we choose a price $\pi \in \mathbb{R}^{\mathbb{L}}$ such that the financial budget set $B^{\mathcal{F}}(p, q)$ is included in the Walrasian budget set $B^W(\pi, \pi \cdot e_i)$?

Let $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ a present value vector associated to q. The price π is then defined by $\pi(\xi_0) = p(\xi_0)$ and $\pi(\xi) = \lambda_\xi p(\xi)$ for all $\xi \in \mathbb{D}_1$.

Q5) What is the definition of the over hedging price of an asset for a given financial structure?

Let $V(p)$ be the payoff matrix and q be the arbitrage free asset price. Then, the over hedging price of an asset with payoff vector $w \in \mathbb{R}^{\mathbb{D}_1}$ is the value of the following optimisation problem:

$$\begin{cases} \text{Minimise } q \cdot z \\ V(p)z \geq w \\ z \in \mathbb{R}^J \end{cases}$$

Exercise 1 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$, the set of states of nature at date 1, is equal to $\{\xi_1, \xi_2, \xi_3, \xi_4\}$. The financial structure is composed of two nominal assets with the following payoff matrix:

$$V = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}$$

1) Represent graphically in $\mathbb{R}^2$ the set Z^+ of portfolios z such that $Vz \geq 0$.

See the associated picture.

2) Show that q is an arbitrage free portfolio if and only if $q \cdot z > 0$ for all $z \in Z^+ \setminus \{0\}$.

If $q \cdot z > 0$ for all $z \in Z^+ \setminus \{0\}$, then q is arbitrage free since there is no portolio z such that $Vz \geq 0$, $q \cdot z \leq 0$ with at least one strict inequality.

Conversely, we remark that for all $z \in Z^+ \setminus \{0\}$, $Vz \geq 0$ and $Vz \neq 0$ since V is one-to-one (injective. So, if there exists $\bar{z} \in Z^+$ such that $q \cdot \bar{z} \leq 0$, then $\bar{z}$ is an arbitrage opportunity. Hence, if q is arbitrage free, for all $z \in Z^+$, $q \cdot z > 0$.

3) Represent graphically the set of arbitrage free portfolios.

See the associated picture where the cone of arbitrage free price is the interior of the red cone.

Exercise 2 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$ is the set of states of nature at date 1. We assume that we have a unique commodity at each state. We consider a financial structure $\mathcal{F}$ with a nonempty finite collection of $\mathcal{J}$ assets, represented by the payoff mapping $p \to V(p)$ from $\mathbb{R}^{\mathbb{D}}$ to the set of $\sharp\mathbb{D}_1 \times \mathcal{J}$ matrices. We assume that $p(\xi) > 0$ for all $\xi \in \mathbb{D}$.

Let k be an asset whose payoffs are $(v_k(p,\xi))_{\xi \in \mathbb{D}_1}$. We consider the financial structure $\tilde{\mathcal{F}}$ obtained by adding this new asset to the structure $\mathcal{F}$: the collection of assets of $\tilde{\mathcal{F}}$ is $\mathcal{J} \cup \{k\}$ and the $\sharp\mathcal{J}$ first columns of the payoff matrix $\tilde{V}(p)$ are the columns of the matrix $V(p)$ and the last column is the column of the payoffs of the asset k:

$$\tilde{V}(p) = \begin{pmatrix} V(p) & \vdots & (v_k(p,\xi))_{\xi \in \mathbb{D}_1} \end{pmatrix}$$

1) Show that if $\tilde{q} = ((q_j)_{j \in \mathcal{J}}, q_k)$ is arbitrage free for the structure $\tilde{\mathcal{F}}$ at p, then the asset price $(q_j)_{j \in \mathcal{J}}$ is arbitrage free for the structure $\mathcal{F}$ at p.

If $\tilde{q} = ((q_j)_{j \in \mathcal{J}}, q_k)$ is arbitrage free for the structure $\tilde{\mathcal{F}}$ at p, then there exists a present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $\tilde{V}(p)^t\lambda = \tilde{q}$. So, one deduces from the structure of the matrix $\tilde{V}(p)$ that $V(p)^t\lambda = q$, which implies that q is arbitrage free for the structure $\mathcal{F}$.

2) Show that the financial structures $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p if and only if the payoff vector $(v_k(p,\xi))_{\xi \in \mathbb{D}_1}$ belongs to the range of $V(p)$.

Since $p(\xi) > 0$ for all $\xi \in \mathbb{D}$, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent if and only if the range of $V(p)$ is equal to the range of $\tilde{V}(p)$. From the structure of $\tilde{V}(p)$, its range is equal to $\mathrm{Im}V(p) + \mathbb{R}v_k(p)$. So, $\mathrm{Im}\tilde{V}(p) = \mathrm{Im}V(p)$ if and only if $\mathrm{Im}V(p) = \mathrm{Im}V(p) + \mathbb{R}v_k(p)$, which holds true if and only if $v_k(p) \in \mathrm{Im}V(p)$.

3) Show that if q is an arbitrage free asset price for the structure V at p and the structures V and $\tilde{V}$ are equivalent at p, then there exists a unique asset price q_k for the asset k such that $\tilde{q} = (q, q_k)$ is arbitrage free for the structure $\tilde{V}$ at p.

Let q is an arbitrage free asset price for the structure V at p. Since V and $\tilde{V}$ are equivalent at p, from the previous question $v_k(p) \in \mathrm{Im}V(p)$, so there exists $\bar{z} \in \mathbb{R}^J$ such that $v_k(p) = V(p)\bar{z}$. If q_k is a price for the asset k such that $\tilde{q} = (q, q_k)$ is arbitrage free for the structure $\tilde{V}$ at p, then q_k is the value of the portfolio $\bar{z}$, which replicates the payoffs of the asset k. Otherwise, an arbitrage opportunity exists by selling one unit of the asset k and buying the portfolio $\bar{z}$ or the converse. So, the unique possible price $q_k = \bar{z} \cdot q$.

4) Show that if the financial structure $\mathcal{F}$ is complete at p, then $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p.

If the financial structure $\mathcal{F}$ is complete at p, $\mathrm{Im}V(p) = \mathbb{R}^{\mathbb{D}_1}$ so necessarily $v_k(p) \in \mathrm{Im}V(p)$, which implies that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p from question 2.

We assume now that there is no redundant asset for the financial structure $\mathcal{F}$ at the price p.

5) Show that the financial structures $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p if and only if the financial structure $\tilde{\mathcal{F}}$ has a useless portfolio.

If $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p, from Question 2, $v_k(p) \in \mathrm{Im}V(p)$, so there exists $\bar{z} \in \mathbb{R}^J$ such that $v_k(p) = V(p)\bar{z}$. We remark that $(z, -1)$ is then a useless portfolio of $\tilde{\mathcal{F}}$.

If the financial structure $\tilde{\mathcal{F}}$ has a useless portfolio, there exists $\tilde{\zeta} \in \mathbb{R}^{J+1} \setminus \{0\}$ such that $\tilde{V}(p)\tilde{\zeta} = 0$. $\tilde{\zeta}_k \neq 0$. Indeed, if $\tilde{\zeta}_k = 0$, then $\tilde{V}(p)\tilde{\zeta} = V(p)\zeta = 0$ where ζ is obtained from $\tilde{\zeta}$ by removing the kth component. Furthermore $\zeta \neq 0$. So ζ is a useless portfolio of $\mathcal{F}$. But this is in contradiction with the fact that the financial structure $\mathcal{F}$ at the price p has no redundant asset, which is equivalent to the absence of useless portfolio. Then, one deduces that $\tilde{\zeta}_k v_k(p) = -V(p)\zeta$, so $v_k(p) = -V(p)\frac{1}{\tilde{\zeta}_k}\zeta$. Hence, $v_k(p)$ belongs to the range of $V(p)$ which implies that the financial structures $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p from Question 2.

Exercise 3 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1 = \{\xi_1, \ldots, \xi_K\}$ is the set of states of nature at date 1. We assume that we have a unique commodity at each state. We consider a financial structure $\mathcal{F}$ with a nonempty finite collection $\mathcal{J} = \{1, \ldots, J\}$ of nominal assets defined as follows. Asset 1 has a positive payoff $v_k > 0$ for all $\xi_k \in \mathbb{D}_1$. Then there exists $0 < k_1 < k_2 < \ldots < k_{J-1} < K$ and the payoffs of Asset j at node ξ_k is 0 if $k \leq k_{j-1}$ and v_{ξ_k} otherwise. So the payoff matrix is as follows:

$$V = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_1} & 0 & 0 & \dots & 0 \\ v_{(k_1+1)} & v_{(k_1+1)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_2} & v_{k_2} & 0 & \dots & 0 \\ v_{(k_2+1)} & v_{(k_2+1)} & v_{(k_2+1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_3} & v_{k_3} & v_{k_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{(k_{J-1}+1)} & v_{(k_{J-1}+1)} & v_{(k_{J-1}+1)} & \dots & v_{(k_{J-1}+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_K & v_K & v_K & \dots & v_K \end{pmatrix}$$

1) Show that the payoff matrix V of this financial structure is one-to-one.

Let $z \in \mathbb{R}^J$ such that $Vz = 0$. Then $v_1 z_1 = 0$ which implies $z_1 = 0$ since $v_1 > 0$. One then deduces that $v_{(k_1+1)}(z_1 + z_2) = v_{(k_1+1)} z_2 = 0$ which implies $z_2 = 0$ since $v_{(k_1+1)} > 0$. Repeating the same argument for $k_2 + 1$ until $k_{J-1} + 1$, one deduces that $z_3 = 0, \dots, z_j = 0$, so $z = 0$ which shows that V is one-to-one.

2) Show that the set of no arbitrage asset prices is

$$Q = \{q \in \mathbb{R}^{\mathcal{J}}_{++} \mid q_1 > q_2 > \dots > q_J\}$$

Hint: you can start by showing that $Q \subset \{q \in \mathbb{R}^{\mathcal{J}}_{++} \mid q_1 > q_2 > \dots > q_J\}$ and then show the converse inclusion.

If q is an arbitrage free asset price, then there exists $\lambda \in \mathbb{R}^K_{++}$ such that $V^t \lambda = q$. Then, since all $v_k > 0$, $q_J = \sum_{k=k_{J-1}+1}^K \lambda_k v_k < q_{J-1} = \sum_{k=k_{J-2}+1}^K \lambda_k v_k < \dots < q_2 = \sum_{k=k_1+1}^K \lambda_k v_k < q_1 = \sum_{k=1}^K \lambda_k v_k$.

Conversely, let q such that $q_1 > q_2 > \dots > q_J > 0$. Then, let $\lambda \in \mathbb{R}^K_{++}$ defined as follows: for $k_{J-1} + 1 \leq k \leq K$, $\lambda_k = \frac{1}{K - k_{J-1}} q_J$, for $k_{J-2} + 1 \leq k \leq k_{J-1}$, $\lambda_k = \frac{1}{k_{J-1} - k_{J-2}}(q_{J-1} - q_J)$, and so on, for $k_1 + 1 \leq k \leq k_2$, $\lambda_k = \frac{1}{k_2 - k_1}(q_3 - q_2)$, and finally, for $1 \leq k \leq k_1$, $\lambda_k = \frac{1}{k_1}(q_2 - q_1)$. Clearly λ has only positive components and one checks from q_J to q_1 that $V^t \lambda = q$, so q is arbitrage free.

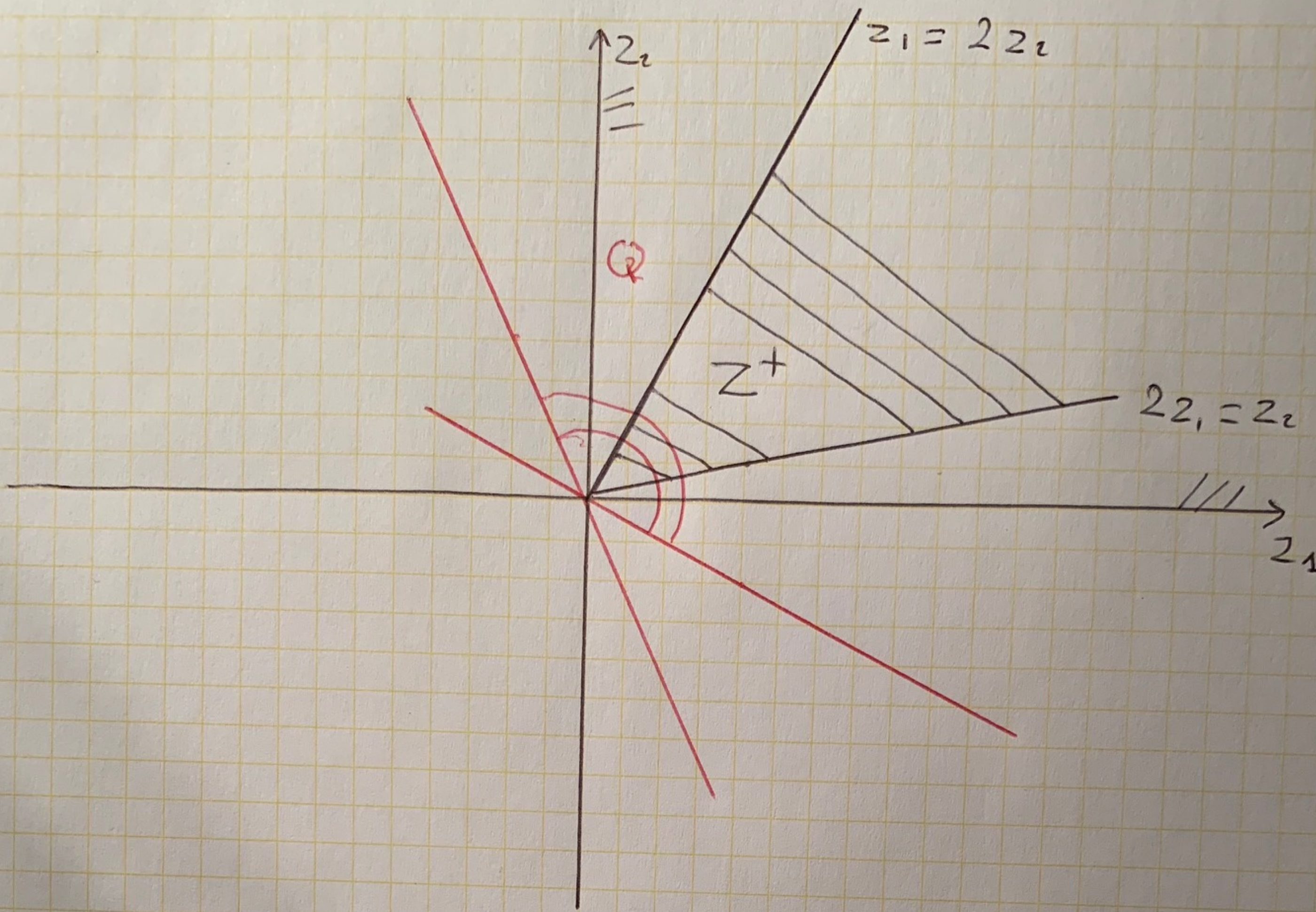
3) Show that the financial structure $\mathcal{F}$ is complete if and only if $K = J$ and $k_1 = 1$, $k_2 = 2$, ..., $k_{J-1} = J - 1$.

The financial structure $\mathcal{F}$ is complete if and only if the range of V is $\mathbb{R}^K$. So, V must have at least K columns. From the structure of V, since $0 < k_1 < k_2 < \dots < k_{J-1} < K$, V has K columns if and only if there are K assets, V is a square matrix and $k_1 = 1, k_2 = 2, \dots, k_K = K$.

4) Show that the financial structure $\mathcal{F}$ is equivalent to the financial structure $\tilde{\mathcal{F}}$ associated to the following payoff matrix:

$$\tilde{V} = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_1} & 0 & 0 & \dots & 0 \\ 0 & v_{(k_1+1)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & v_{k_2} & 0 & \dots & 0 \\ 0 & 0 & v_{(k_2+1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & v_{k_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v_{(k_{J-1}+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v_K \end{pmatrix}$$

Let $(v^1, \dots, v^J)$ be the column vectors of V and $(w^1, \dots, w^J)$ be the column vectors of $\tilde{V}$. We remark that $v^J = w^J$, $v^{J-1} = W^J + W^{J-1}, \dots, V^2 = \sum_{k=2}^{J} W^k$ and $V^1 = \sum_{k=1}^{J} W^k$. So, the range of V spanned by the column vectors $(v^1, \dots, v^J)$ is a subset of the range of W spanned by the column vectors $(w^1, \dots, w^J)$. But, since V is one-to-one, the range of V is of dimension J and the range of W is of dimension at most J. So, one conclude that the dimension of the two spaces is equal to J and that they are equal. So, $\mathcal{F}$ is equivalent to the financial structure $\tilde{\mathcal{F}}$.



z_2
$z_1 = 2z_2$
Q
Z^+
$2z_1 = z_2$
z_1