Economic analysis of financial market
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J.-M. Bonnisseau

Arrow securities and general financial structures

Outline

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A new market organization

Spot markets *at each node* ξ *where the current commodities are traded at a spot price* $p(\xi) \in \mathbb{R}^\ell$. *This means that the paiements are made at the node* ξ *and not at date* 0 *as for the contingent commodities.*

Financial market *at the initial node* ξ_0*, a financial market is open where Arrow securities are traded.*

Definition of the Arrow securities

An Arrow security j^{ξ} *is associated to a node* $\xi \in \mathbb{D}^{+}(\xi_0) = \mathbb{D} \setminus \{\xi_0\}$*. It is a contract signed at date* ξ_0 *with a paiement at this date which promised to deliver one unit of unit of account at the node* ξ *if this node prevails and nothing otherwise.*

Agent's behaviour

In this model, the agent i chooses a consumption $x_i \in X_i$ as previously but also a portfolio $z_i \in \mathbb{R}^{\mathbb{D}^+(\xi_0)}$. $|z_i(\xi)|$ is the quantity of the asset j^ξ sold ($z_i(\xi) < 0$) or bought ($z_i(\xi) > 0$) on the financial market at date 0.

Budget constraints

At node ξ_0, the budget constraint is :

$$p(\xi_0) \cdot x_i(\xi_0) + q \cdot z_i \leq p(\xi_0) \cdot e_i(\xi_0)$$

At each node $\xi \in \mathbb{D}^+(\xi_0)$, the budget constraint is

$$p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi) + z_i(\xi)$$

So, at the global level, the budget set of the agent is $B_i^A(p, q, e_i)$ defined by :

$$\left\{ x_i \in X_i \;\middle|\; \exists z_i \in \mathbb{R}^{\mathbb{D}^+(\xi_0)} \begin{array}{c} p(\xi_0) \cdot x_i(\xi_0) + q \cdot z_i \leq p(\xi_0) \cdot e_i(\xi_0) \\ p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi) + z_i(\xi) \\ \forall \xi \in \mathbb{D}^+(\xi_0) \end{array} \right\}$$

Arrow financial equilibrium

Definition

A Arrow financial equilibrium

$$((x_i^*, z_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathbb{D}^+(\xi_0)})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathbb{D}^+(\xi_0)}$$

such that
(a) [Preference maximization] for every $i \in \mathcal{I}$,
(x_i^*, z_i^*) is a "maximal" element of u_i in the budget set $B_i^A(p^*, q^*, e_i)$ in the sense that

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot z_i^* \leq p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \leq p^*(\xi) \cdot e_i(\xi) + z_i^*(\xi), \quad \forall \xi \in \mathbb{D}^+(\xi_0) \end{cases}$$

and

$$B_i^A(p^*, q^*, e_i) \cap \{x_i \in X_i \mid u_i(x_i) > u_i(x_i^*)\} = \emptyset;$$

Definition continued

Definition

(b) [Market clearing conditions on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i;$$

(c) [Market clearing conditions on the financial market]

$$\sum_{i \in \mathcal{I}} z_i^* = 0.$$

No-arbitrage with Arrow securities

Proposition

Under Assumptions C and NSS, if $((x_i^*, z_i^*), p^*, q^*)$ *is a Arrow financial equilibrium, then* $q_{j\xi}^* > 0$ *for all* $\xi \in \mathbb{D}^+(\xi_0)$.

Arrow financial equilibrium and CC equilibrium

Proposition

We consider an exchange economy satisfying Assumptions C and NSS.
Let $((x_i^*, z_i^*), p^*, q^*)$ *be a Arrow financial equilibrium. Let* $\tilde{p}^*$ *defined by* $\tilde{p}^*(\xi_0) = p^*(\xi_0)$ *and for all* $\xi \in \mathbb{D}^+(\xi_0)$,
$\tilde{p}^*(\xi) = q^*_{j\xi} p^*(\xi)$. *Then, for all* $i \in \mathcal{I}$,
$B_i^A(p^*, q^*, e_i) \subset B_i^W(\tilde{p}^*, \tilde{p}^* \cdot e_i)$.

Consequently, $((x_i^*),\ \tilde{p}^*)$ *is a contingent commodity equilibrium.*

Proposition continued

Conversely, let $((x_i^*),\ \bar{p}^*)$ *be a contingent commodity equilibrium, let* $\bar{q}^*$ *be the asset price such that* $\bar{q}^*(\xi) = 1$ *for all* $\xi \in \mathbb{D}^+(\xi_0)$ *and for all* $i \in \mathcal{I}$, z_i^* *be the portfolio defined by* $z_i^*(\xi) = \bar{p}^*(\xi) \cdot (x_i^*(\xi) - e_i(\xi))$ *for all* $\xi \in \mathbb{D}^+(\xi_0)$. *Then, for all* $i \in \mathcal{I}$, $B_i^W(\tilde{p}^*, \tilde{p}^* \cdot e_i) \subset B_i^A(p^*, q^*, e_i)$.

Consequently, $((x_i^*, z_i^*), \bar{p}^*, \bar{q}^*)$ *is a Arrow financial equilibrium.*

Necessity of a complete set of Arrow securities

For example, let us consider the simplest tree $\mathbb{D}$ *with* $T = 1$ *and just one node* ξ_0 *at date* 0 *and one,* ξ_1 *at date* 1*. We also assume that there is just one commodity per date,* $\ell = 1$*. Then, we have two agents* $\mathcal{I} = \{1, 2\}$ *having the identical preferences on* $\mathbb{R}^2_+$ *defined by* $u(x_0, x_1) = x_0 x_1$ *and initial endowments* $e_1 = (2, 1)$ *and* $e_2 = (1, 2)$*. Then if the unique Arrow security is missing, we have only two spot markets.*

Payoff expressed in terms of a numéraire

The return of the Arrow securities can be expressed in real terms of the value or a numéraire commodity or a numéraire commodity basket. For example, if a commodity h is chosen as numéraire or if a numéraire commodity basket $\nu \in \mathbb{R}^{\ell}_{++}$ *is chosen, then the return of one unit of the Arrow security* j^{ξ} *at node* ξ *is equal to* $p_h(\xi)$ *or* $p(\xi) \cdot \nu$*. Then the budget constraints become* $p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi) + p_h(\xi) z_i(\xi)$ *or* $p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi) + (p(\xi) \cdot \nu) z_i(\xi)$.

The equivalence results holds true if the value of the numéraire basket $p^*(\xi) \cdot \nu$ *are positive at every node* $\xi \in \mathbb{D}^+(\xi_0)$.

Link with the financial literature

$\ell = 1$ at each node, Spot price $p(\xi) = 1$ on each spot market.

Let us consider a risk neutral consumer i with a discounted expected utility u defined by :

$$u_i(x_i) = \sum_{t=0}^{T} \beta^t \sum_{\xi \in \mathbb{D}_t} \pi_t(\xi) x_i(\xi)$$

Interpretation of the asset price

Assume that the Arrow financial equilibrium allocation x_i^* of this consumer is an interior condition. Then the first order optimality conditions tell us that there exists multipliers such that :

$$\text{(i) } \lambda_\xi = \beta^t \pi_t(\xi) \text{ item(ii) } \lambda_\xi = \lambda_{\xi_0} q_{j\xi}^* \text{ for all } \xi \in \mathbb{D}_t.$$

So $\beta^t \pi_t(\xi) = \lambda_{\xi_0} q_{j\xi}^*$. Note that $q_{j\xi}^*$ is cost paid at ξ_0 to have one additional unit of wealth at node ξ or in other words is the price at date 0 of a unit of wealth at node ξ.

Interest rate

To have one additional unit of wealth at all nodes of date t, the cost is $\sum_{\xi \in \mathbb{D}_t} q^*_{j\xi}$. Since $(\pi_t(\xi))$ is a probability on $\mathbb{D}_t$, the total price is β^t at date 0. So in terms of interest rate r, we note that the return at date t of a paiement of β^t at date 0 is $\beta^t(1+r)^t = 1$, or, in other words, $\beta = \frac{1}{1+r}$.

Risk neutral probability

Now we remark that the discounted price process on the final states $\mathbb{D}_T$, $(\frac{1}{\beta^T} q^*_{j\xi} = (1+r)^T q^*_{j\xi})_{\xi \in \mathbb{D}_T}$ defined a “risk-neutral” probability measure on the final states and $(\frac{1}{\beta^t} q^*_{j\xi} = (1+r)^t q^*_{j\xi})_{\xi \in \mathbb{D}_t}$ is the conditional probability on the states at date t. These are the usual assumptions on a price process in a standard financial model.

Outline

1. Arrow securities
2. Pure spot market equilibrium

Pure spot market economy

No financial market, only pure spot markets

Budget constraints

$$p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi), \quad \forall \xi \in \mathbb{D}$$

Remark

If $\ell = 1$ and $p^*(\xi) > 0$ for all ξ, then the unique pure spot equilibrium is the autarky equilibrium $x_i^* = e_i$ for all i.

Existence of pure spot market equilibrium

Remark

Note that the above assumptions C, S and NSS are sufficient to guarantee the existence of a pure spot market equilibrium. It suffices to adapt the proof of a standard Competitive equilibrium checking that the budget sets have a closed graph and are lower semicontinuous which implies that the quasi-demands are upper semicontinuous if we truncate in a suitable way the consumption sets. Then, the step from a quasi-equilibrium to an equilibrium is obtained thanks to the survival assumption and the non satiation at each state.