Exercises Logic and Set

Master Mathematical Models in Economics and Finance (MMEF).
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Contents

1 Logic

1.1 Symbolic logic

Exercise 1.1. Let p be "it is cold" and q "it is raining". Give a simple verbal sentence which describes each of the following propositions:

1. $\neg p$
2. $p \wedge q$
3. $p \vee q$
4. $p \leftrightarrow q$
5. $p \rightarrow \neg q$
6. $q \ \vee \ \neg p$
7. $\neg p \ \wedge \ \neg q$
8. $p \leftrightarrow \neg q$
9. $(p \ \wedge \ \neg q) \rightarrow p$

Exercise 1.2. Let p be "He is tall" and let q be "He is bright", write each of the following statements in symbolic form using p and q.

1. He is tall and bright
2. He is bright and short
3. It is false that he is bright or short
4. He is tall, or he is short and bright

Exercise 1.3. Let p be "it is raining", q be "it is going to rain" and r be "one can see the heaven". Give a simple verbal statement which describe the following proposition:

$$(r \rightarrow q) \wedge (\neg r \rightarrow p).$$

Exercise 1.4. Determine the truth value of each of the following propositions:

1. If $3 + 2 = 7$ then $4 + 4 = 8$
2. It is not true that $2 + 2 = 5$ if and only if $4 + 4 = 10$
3. Paris is in England or Venezia is not in Italy.
4. It is not true that, $1 + 1 = 3$ or $2 + 1 = 3$
5. It is false that (if Paris is in England then London is in France).

Exercise 1.5. Determine the negation of each of the following propositions:

1. He is tall and handsome
2. He is not rich and not happy (He is neither rich nor happy).
3. If she comes, she will talk to you.
4. Mark is rich or Eric is poor.
5. If Marc is sad, then both Marie and Jean are happy
6. Eric is handsome if and only if Marie is intelligent.

Exercise 1.6. 1) Find the truth table of each of the following proposition.
2) Determine their negation.

1. $\neg(p) \wedge q$
2. $\neg(q) \rightarrow \neg(p)$
3. $(p \wedge q) \rightarrow (p \vee q)$
4. $\neg(p \wedge q) \vee \neg(p \leftrightarrow q)$
5. $(\neg(p \wedge q) \rightarrow r) \rightarrow (q \wedge r)$

1.2 First-order logic

Exercise 1.7. Translate in English (or in Mathematics) the following formula

1. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ x \geq y$
2. $\exists x \in \mathbb{R}, \exists y \in \mathbb{R},\ x \geq y$
3. For all integer, there exists a real number whose square is smaller than itself.
4. There exists no integer that is smaller than 20.

Exercise 1.8. Determine the truth value of each of the following statements :

1. $\forall x \in \{1, 2, 4, 5\},\ x + 2 \in \{3, 4, 7, 8\}$,
2. $\exists x \in \{1, 2, 4, 5\},\ x + 2 \in \{3, 4, 7, 8\}$,
3. $\exists x \in \mathbb{R},\ x^2 - 2x + 1 = 0$
4. $\forall x \in \mathbb{R},\ |x| = x$
5. $\forall x \in \mathbb{R}_+,\ |x| = x$
6. $\forall x \in \mathbb{R}, \exists n \in \mathbb{N},\ n > x$ and same question for $\forall x \in \mathbb{R}_+^*, \exists n \in \mathbb{N},\ \frac{1}{n} < x$
7. $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y = x^2$,
8. $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x^2 = y$,
9. $\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ x^2 = y$,
10. $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ (x < y) \rightarrow (\exists z \in \mathbb{R}, x < z < y)$,
11. $\exists t \in \mathbb{R},\ \forall x \in \mathbb{R}_-,\ x < t$.

Exercise 1.9. Determine the negation of each of the following proposition:

1. Every human being is smart or tall.
2. If there exists a human being smart and tall then there exists a tree blue or red.
3. If a blue dog exists then the world has a beginning or an end.
4. Everything that has a beginning has an end

Exercise 1.10. Determine the negation of the following propositions

1. $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ \forall z \in \mathbb{R},\ (x \in [y, z]\ or\ z \in [x, y]),$
2. $\exists x \in \mathbb{R},\ \exists n \in \mathbb{N},\ \forall z \in \mathbb{R},\ (z \geq n \to x \leq z),$
3. For every real number x, there exists y a real number such that $x \leq y$ or there exists a natural number z such that $x \geq z$.
4. $\forall x \in \mathbb{R},\ \exists z \in \mathbb{N}$ such that $z < x$ implies that for all $y \in Z$, $z < y$.
5. $\forall n \in \{1, 2, 3\},\ \forall m \in \{3, 4, 5\},\ n^m < 3$

Exercise 1.11. Let $p(x, y, z)$ be a predicate on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, determine the logical relations between the following propositions:

1. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists z \in \mathbb{R}\ p(x, y, z),$
2. $\exists z \in \mathbb{R}\ \forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ p(x, y, z),$
3. $\forall y \in \mathbb{R}\ \forall x \in \mathbb{R}\ \exists z \in \mathbb{R}\ p(x, y, z),$
4. $\forall x \in \mathbb{R}\ \exists z \in \mathbb{R}\ \forall y \in \mathbb{R}\ p(x, y, z).$

Exercise 1.12. A function f on an interval I is uniformly continuous on I if there exists

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, \forall y \in I \cap]x - \delta, x + \delta[,\ |f(x) - f(y)| \leq \varepsilon.$$

1. Recall the definition of f being continuous on I.
2. Explain the difference between the two definitions.
3. Does one notion imply the other?

2 Reasoning in mathematics

2.1 Basic proof

Exercise 2.1. Write for each of the following statements:

1. A conditional/universal proof
   (a) for all real numbers $(x+1)^3 = x^3 + 3x^2 + 3x + 1$,
   (b) for all natural numbers x, $x-1$ divides $x^3 - 1$,
2. An existential proof
   (a) there exists an integer such that $x^2 - 2x + 1 = 0$,
   (b) there exists a real number y such that $y^2 = 4$.
3. A counterexample (that shows the statement is false)
   (a) Every integer is divisible by 4
   (b) For every real number x, there exists a real number y such that
   $$y^2 \leq x.$$
4. A proof by contrapositive
   (a) Let $a, b, n \in \mathbb{N}$, if n does not divide (ab) then n does not divide a and does not divide b.
   (b) Let $x \in \mathbb{N}$, if $x^2 - 6x + 5$ is even then x is odd.
5. A proof by contradiction that
   (a) If n^2 is even, then n is even.
   (b) $\sqrt{2}$ is not a rational number[1].

Exercise 2.2. Proof the following statements:

1. Let n be an integer. n^2 is even if and only if n is even.
2. Let n be an integer, then n^3 is either divisible by 9, 1 more or 1 less than an integer divisible by 9.
3. For every positive real number, $|x+2| - |x-2| > 0$.

[1]We admit that x is a positive rational number if there exists $p, q \in \mathbb{N}^*$ such that $x = \frac{p}{q}$ while p and q have only 1 as a common divisor

Exercise 2.3. We will now make some proof on sequence and convergence: recall that $(u_n)_{n\geq 1}$ converges to $l \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ |u_n - l| \leq \varepsilon,$$

1. Prove that $(2u_n)_{n\geq 1}$ converges to $2l$,

2. Let $(v_n)_{n\geq 1}$ that converges to k. Define for every $n \geq 1$, $w_m = v_n + u_n$. Prove that the sequence $(w_n)_{n\geq 1}$ converges to $k + l$ by using that for every $n \geq 1$,

$$|w_n - (l + k)| = |v_n - k + u_n - l| \leq |v_n - k| + |u_n - l|.$$

2.2 Induction

Exercise 2.4.
Show by induction that for all $n \in \mathbb{N}$,

1. $\sum_{t=1}^{n} t^2 = 1 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.
2. $\sum_{t=1}^{n}(2t - 1) = 1 + 3 + 5 + ... + (2n - 1) = n^2$,
3. $n(n^2 + 5)$ is a multiple of 6.

Exercise 2.5.
Define the sequence $(u_n)_{n\in\mathbb{N}}$ by $u_0 = 0$ and $\forall n \in \mathbb{N}$, $u_{n+1} = \sqrt{u_n + 2}$.

1. Show that the sequence is bounded by above by 2.

2. Deduce that the sequence is increasing.

3. Conclude.

Exercise 2.6.

- Recall that a natural number n is a prime number if it has two divisors 1 and n. (1 is not a prime number).

- Using the strong principle of induction, show that every number n has a prime divisor.

3 Set theory

3.1 Sets

Exercise 3.1. For each of the following statement say if it is True or False.

1. $John \in \{John, Marc\}$,
2. $Julie \in \{John, Marc\}$,
3. $\{1,2\} \subset \{1,2\}$,
4. The function $\theta \to \cos(\theta)$ belongs to the set of functions $\{\theta \to a\cos(\theta) + b\sin(\theta) : a, b \in \mathbb{R}\}$,
5. $\{1,3\} \subset \{x, (x-1)(x-2)(x-4) = 0\}$,
6. $7 \in \{x \in \mathbb{R}, x^2 - 5x - 14 = 0\}$.
7. $\{1\} \subset \{1,2,3\}$

Exercise 3.2. 1. Let $A = \{1,2,3,4\}$, $B = \{2,4,6,8\}$, and $C = \{3,4,5,6\}$. Compute the following sets

   (a) $A \cup B$,
   (b) $B \cup C$,
   (c) $A \cap B$,
   (d) $A \cap B \cap C$.
   (e) $A - B$,
   (f) $B - C$.

2. Same question with $A = \{Victor, Pierre, Luc\}$, $B = \{Victor, Jean, Eric\}$, and $C = \{Marc, Eric, Alain\}$.

Exercise 3.3. Compute the following sets.

1. $\{1,2,3,4\} \cap \{x \in \mathbb{N} \mid x \textit{ is a multiple of } 2\} = ?$
2. $\{x \in \mathbb{N} \mid x \leq 40\} \cap (\{x \in \mathbb{N} \mid x \textit{ is a multiple of } 3\} \cap \{x \in \mathbb{N} \mid x \textit{ is a multiple of } 4\}) = ?$

Exercise 3.4.

1. What is the complement of $\{a,b,c\}$ in the set of letters of the Latin alphabet?
2. What is the complement of the set of positive odd numbers in the set of positive integers?
3. What is the complement of the set of positive odd numbers in the set of all integers?

Exercise 3.5. Let A, B, C be three sets. Suppose that $A \subset B$, $B \subset C$ and $C \subset A$. Show that $A = B = C$.

Exercise 3.6. Let A, B be two sets. Prove the following results

- $(A \cup B)^c = A^c \cap B^c$,
- $(A \cap B)^c = A^c \cup B^c$.

3.2 Operations on sets

Exercise 3.7. Let A and B be two sets. Prove the following results

1. $A \subset A \cup B$.
2. $B \subset A \cup B$.
3. If $A \subset B$ then $A \cup B = B$.
4. $A \cup B = B \cup A$.
5. $\emptyset \cup A = A$.
6. $A \cup B = \emptyset$ implies $A = \emptyset$ and $B = \emptyset$.

Exercise 3.8. Let A and B be two sets. Prove the following results

1. $A \cap B \subset A$.
2. $A \cap B \subset B$.
3. $A \cap \emptyset = \emptyset$.
4. $A \cap B = B \cap A$.
5. If $A \subset B$ then $A \cap B = A$.

Exercise 3.9. Let A and B be two sets. Prove the following results

1. $(A - B) \cap B = \emptyset$.
2. $(A - B) \cap (B - A) = \emptyset$.
3. $(A - B) \cap (A \cap B) = \emptyset$.
4. $(A - B) = (B - A)$ if and only if $A = B$.

Exercise 3.10. Let A and B be two sets. Prove that $A^c - B^c = B - A$.

3.3 Family of sets

Exercise 3.11. Tell if the following statements are true or false

1. $\{1, 2\} \in \{1, 2\}$,
2. $\{1\} \in \{1, \{1\}\}$,
3. $\{1\} \subset \{1, \{1\}\}$,
4. $\{\emptyset\}$ is empty,
5. $\{1\} \in \mathbb{N}$,
6. $\emptyset \in \emptyset$,
7. $\emptyset \subset \emptyset$.

Exercise 3.12. Let $\Omega = \{1, 2\}$. Tell if the following statements are true or false:

1. $\{\{1\}\} \in \mathcal{P}(\Omega)$,
2. $\{\{1\}\} \subset \mathcal{P}(\Omega)$,
3. $\{1\} \in \mathcal{P}(\Omega)$,
4. $\{1\} \subset \mathcal{P}(\Omega)$,
5. $\{1\} \in \Omega$,
6. $\{1\} \subset \Omega$,

Exercise 3.13. Let $\Omega = \{a, b, c, d\}$. We consider the following family of $\mathcal{P}(\Omega)$:

$$\mathcal{F} = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}.$$

1. Is $\mathcal{F}$ stable by complements?
2. Does it contain the empty set?
3. Is it stable by union [2]?

Exercise 3.14. Let $(A_i)_{i\in I}$ be a family of sets. Prove the following results

1. $(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c$,
2. $(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c$,

3.4 Cartesian product

Exercise 3.15. We want to study how the Cartesian product and the union behave together.

1. Let $A = [1, 3]$, $A' = [2, 4]$, $B = [0, 2]$ and $B' = [1, 3]$.
   (a) Draw $A \times B$, $A' \times B'$.
   (b) Draw $(A \cup A') \times (B \cup B')$.
   (c) Compare $(A \times B) \cup (A' \times B')$ and $(A \cup A') \times (B \cup B')$.
2. Let A, A', B, B' be 4 sets. Prove that

$$(A \times B) \cup (A' \times B') \subset (A \cup A') \times (B \cup B').$$

3. Find A,A', B and B' such that the previous inclusion is strict (no equality).

Exercise 3.16. We want to study how the Cartesian product and the intersection behave together.

1. Let $A = [1, 3]$, $A' = [2, 4]$, $B = [0, 2]$ and $B' = [1, 3]$.
   (a) Draw $(A \cap A') \times (B \cap B')$.
   (b) Compare $(A \times B) \cap (A' \times B')$ and $(A \cap A') \times (B \cap B')$.
2. Let A, A', B, B' be 4 sets. Prove that

$$(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B').$$

[2] considering a union of elements in the family is still in the set

4 Functions

- In this section, we consider two sets X and Y and a mapping $f : X \to Y$.
- The sets A_1, A_2 and A are subsets of X and the sets B_1, B_2 and B are subsets of Y.

Exercise 4.1. 1. If $A_1 \subset A_2$ show that $f(A_1) \subset f(A_2)$.

2. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
3. Show that $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. and give an example showing that the equality may not hold.

Exercise 4.2. 1. If $B_1 \subset B_2$ show that $f^{-1}(B_1) \subset f^{-1}(B_2)$.

2. Show that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
3. Show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Exercise 4.3. 1. Show that $f^{-1}(Y \setminus B) = X \setminus (f^{-1}(B))$.

2. Is it true that $f(X \setminus A) = Y \setminus (f(A))$?

Exercise 4.4. 1. Show that $f(f^{-1}(B)) \subseteq B$ and that the equality may not hold.

2. Show that $A \subseteq f^{-1}(f(A))$ and that the equality may not hold.

Exercise 4.5. Show that $(f_{|A})^{-1}(B) = A \cap f^{-1}(B)$.

4.1 Injection, surjection and bijection

Exercise 4.6. Give an example of a mapping which is:

1. injective and surjective.
2. injective but not surjective.
3. Not injective but surjective.
4. Not injective and not surjective.

Exercise 4.7. Let A, B, C be three sets and $f : A \to B$, $g : B \to C$ be two mappings. Show that:

1. $g \circ f$ injective implies f injective.
2. $g \circ f$ surjective implies g surjective.

Exercise 4.8. Let X, Y be two sets and $f : X \to Y$, $g : Y \to X$ be two mappings such that $g \circ f = id_X$ (the identity mapping of X). Show that:

1. f is injective and g is surjective.
2. f may not be surjective and g may not be injective.

Exercise 4.9. Let E, F, G be three sets and $f : E \to F$, $g : F \to G$ be two bijective mappings. Show that $g \circ f : E \to G$ is bijective and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 4.10. Let X_1, X_2 be two sets and $\pi_1 : X_1 \times X_2 \to X_1$ be the projection mapping on the first coordinate defined by $\pi_1(x_1, x_2) = x_1$. Show that:

1. π_1 is surjective.
2. π_1 may not be injective.

Exercise 4.11. Let X and Y be two sets and $f : X \to Y$ be a mapping. For every $y \in Y$, what can we say of the sets $f^{-1}(\{y\})$ when f is surjective, injective, bijective?

Exercise 4.12.

1. If f is surjective, show that $f(f^{-1}(B)) = B$.
2. If f is injective, show that $A = f^{-1}(f(A))$.

5 Relations

5.1 Basic properties

Exercise 5.1. Let $\Omega = \{2, 3, 4, 5, 10\}$ and the following relation

$$a\mathcal{R}b \text{ if and only if } a \leq b < a^2.$$

- Represent this relation with a table and a diagram.
- Is it reflexive? transitive? symmetric? anti-symmetric?

Exercise 5.2. Are the following relations reflexive? transitive? symmetric? anti-symmetric?

1. On $\mathbb{N}_* \times \mathbb{N}_*$ $a\mathcal{R}b$ if and only if a *divides* b
2. On 2^E, the inclusion relation between subsets.
3. On $\mathbb{R} \times \mathbb{R}$ the relation $x\mathcal{R}y$ if and only if $|x| = |y|$
4. On 2^E the relation $A\mathcal{R}B$ if and only if $A \cap B = \emptyset$
5. On $\mathbb{R}^2$ the relation ($x\mathcal{R}y$ iff $u(x) > u(y)$) where $u : \mathbb{R} \to \mathbb{R}$ is a function.
6. On $\mathbb{R}^2$ the relation ($x\mathcal{R}y$ iff $u(x) \geq u(y)$) where $u : \mathbb{R} \to \mathbb{R}$ is a function.
7. On $(\mathbb{Z} \times \mathbb{N} - \{0\})^2$ the relation $(a, b)R(c, d)$ if and only if $ad - bc = 0$

Exercise 5.3.

- Which of the preceding relations are equivalence relations ? Order relations ?
- Give a characterization of an equivalence relation (resp. order relation) in terms of its graph.

5.2 Equivalence relation

Exercise 5.4. Find all partitions of the following sets:

1. $U = \{John, Elsa\}$,
2. $S = \{a, b, c\}$.

Exercise 5.5. Consider the set of words $W = \{sheet, last, sky, wash, winf, sit\}$. Find W/R where R is the following equivalence relation

1. "has the same number of letters",
2. "begins with the same letter".

Exercise 5.6. (Theorem 11) Let E be a set, prove that Π is a partition of E if and only if there exists an equivalence relation such that $\Pi = \{\mathcal{R}(x)\}$ where $\mathcal{R}(x)$ is the equivalence class of x for $\mathcal{R}$.

5.3 Order relation

Exercise 5.7.

1. Let $u : X \to \mathbb{R}$ be a utility function and $\preceq_u$ be the preference relation associated with u, that is, $x \preceq_u y$ if and only if $u(x) \leq u(y)$.
   Show that $\preceq_u$ is a complete preorder.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function and define $v : X \to \mathbb{R}$ by $v(x) = f(u(x))$.
   (a) Show that $\preceq_u$ and $\preceq_v$ define the same relation: $x \preceq_u y$ iff $x \preceq_v y$.
   (b) Give a counterexample if f is not increasing.
3. Give an example of an order relation that can not be written as $\preceq_u$ for some u.

Exercise 5.8. We consider the set $\mathbb{R}^2$ and define two different relations on $\mathbb{R}^2$.

1. For every $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$:
   $$x \geq y \text{ if } (x_1 \geq y_1 \text{ and } x_2 \geq y_2).$$
   (a) Show that $\geq$ is an order on $\mathbb{R}^2$.
   (b) Show that the order $\geq$ is not complete.
2. For every $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X^2:
   $$x \preceq_L y \text{ if } (x_1 < y_1) \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2).$$
   (a) Show that $\geq$ is a complete order on X^2.
   (b) It is called the *lexicographic order*. Justify this name.

Exercise 5.9. Do the following subsets have a greatest lower bound ? A least upper bound ? A minimum ? A maximum?

· Subsets of $\mathbb{R}$:

1. $]0, 1[$
2. $\cup_{n\in\mathbb{N}}\{-n\}$
3. $[0, 1]$
4. $\{\frac{1}{n}\}_{n\in\mathbb{N}-\{0\}}$
5. $\{\frac{1}{x}\}_{x\in\mathbb{R}-\{0\}}$

· Subsets of $\mathbb{Q}$

1. $\{x \in \mathbb{Q} \mid x > 0\}$
2. $\{x \in \mathbb{Q} \mid x > \pi\}$

Exercise 5.10. Let A and B be two subsets of $\mathbb{R}$ such that A is bounded by above, B is bounded by above and $A \cap B \neq \emptyset$.

1. Prove that $A \cap B$ and $A \cup B$ are bounded by above.
2. Compare $\sup(A \cup B)$, $\sup(A \cap B)$ and $\max\{\sup(A), \sup(B)\}$.

6 Cardinality

Exercise 6.1. Construct a bijection between the following sets

1. $[0, \pi]$ and $[-1, 1]$,
2. $(0, +\infty)$ and $\mathbb{R}$,
3. $[0, 1)$ and $(0, 1]$,
4. $(0, 1)$ and $(0, 1)$.
5. $\mathbb{N}$ and $\mathbb{Z}^-$,
6. the set of prime number and $\mathbb{N}$.

Exercise 6.2. Prove that $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality.

Exercise 6.3. Given two sets A and B, we denote by $A \approx B$ that A and B have the same cardinality. Prove that

1. if $A \approx B$ and $B \approx C$ then $A \approx C$,
2. if $A \approx B$ and $C \approx D$ then $A \times C \approx C \times D$.

Exercise 6.4. Tell if the following sets are countable or uncountable:

1. $\mathbb{R}^*$,
2. $\mathbb{R}^2$,
3. $\mathbb{N}$,
4. $\mathbb{N}^{10}$,
5. $\mathbb{Q}$,
6. (hard) $\mathbb{N}^{\mathbb{N}}$.

7 Annals

7.1 Midterms (October 2015)

Exercise 7.1. (11 points)
Let A, B, C and D be four sets. Let p, q and r be three propositions. Let $P(.,.)$ be a predicate on $\mathbb{R} \times \mathbb{R}$. For each of the following statements, say if it is TRUE or FALSE. (+0.5 if your answer is correct, -0.5 if your answer is wrong, 0 otherwise.)

1. If (Paris is on mars) then (London is in England).
2. (A unicorn exists) if and only if (2+2=5).
3. The negation of $p \to q$ is $\neg p \wedge q$.
4. The negation of $((p \wedge q) \leftrightarrow r)$ is $(\neg p \vee \neg q) \leftrightarrow \neg r$.
5. The negation of "(Marc and Julie are tall) or Sam is small" is "Sam is tall and (Marc or Julie is small)".
6. The negation of "If it is raining then everybody is sad" is "If it is raining then everybody is happy".
7. The contrapositive of $p \to q$ is $\neg q \to \neg p$.
8. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$.
9. $\forall x \in \mathbb{R}_+,\ \exists y \in \mathbb{R},\ x = y^2$.
10. $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}_+, \quad x = y^2$.
11. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, \quad x = y^2$.
12. The negation of "$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}, P(x, y)$" is "$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R}, \neg P(x, y)$".
13. $\{1, 2, 3\} \subset \emptyset$.
14. $\{2\} \in \{1, \{2\}\}$.
15. $\{2\} \subseteq \{1, \{2\}\}$.
16. For all $f : A \to B$ and all $C, D \subseteq A$, $f(C \cap D) = f(C) \cap f(D)$.
17. For all $f : A \to B$ and all $C, D \subseteq A$, $f(C \cup D) = f(C) \cup f(D)$.
18. $(A \cup B) \subseteq (A \cap B)$
19. $A \cup (B \cap C) = (A \cup B) \cap C$.
20. $\emptyset \in \mathcal{P}(\emptyset)$.
21. $\displaystyle\bigcap_{C \in \{\emptyset, \{1,2\}, \{1\}\}} C = \{1\}$.

22. $\displaystyle\bigcup_{C\in\{\emptyset,\{1,2\},\{\{1\}\}\}} C = \{1,2\}.$

Exercise 7.2. (4 points)
Given two propositions p and q, we define the proposition $p \oplus q$, called exclusive disjunction, with the following truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1. Write the truth table of $(\neg p \wedge q) \vee (\neg q \wedge p)$. What do you observe?

2. Check that;

   (a) $(p \oplus T) \leftrightarrow (\neg p)$ is a tautology.

   (b) $(p \oplus p) \leftrightarrow F$ is a tautology.

3. Use the operator $\oplus$ in order to give a new definition of $A\Delta B$ where A and B are two sets.

4. Find a proposition equivalent to $p \vee q$ using only $\oplus$ and $\wedge$ (try first with $\oplus$,$\wedge$ and $\neg$).

Exercise 7.3. (5 points)

1. Let A and B be two sets. Prove the following propositions from the definitions:

   (a) $B \subset A \cup B$.

   (b) If $A \subset B$ then $f(A) \subset f(B)$ for a function f from C to D such that $A, B \subset C$.

2. Given three sets A, B and C. We define the following set

$$[A, B, C] = \{x \in A \cup B \cup C, \exists y \in \{A, B, C\}, x \notin y\}$$

   (a) Draw on a picture this set for three generic sets A, B and C (such that any intersection is non-trivial).

   (b) Express this set with usual symbol: $\cap$, $\cup$, and $\setminus$ (bonus if proof).

3. Compare $(A \setminus A) \setminus A$ and $A \setminus (A \setminus A)$. Is the $\setminus$ operation commutative?

7.2 Exam (December 2015)

Exercise 7.4. (10 points)
Let A, B and C be three set. Let p, q and r be three propositions. Let f, g and h be three functions. For each of the following statements write on the left if it is TRUE or FALSE.

1. The negation of $(p \vee q) \wedge r$ is $(\neg p \vee \neg r) \wedge (\neg q \vee \neg r)$.
2. The negation of $p \rightarrow q$ is $p \wedge \neg q$.
3. The contrapositive of: "If it is not raining, we will go to the beach" is "If we go to the beach, it is not raining".
4. $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x = e^y$.
5. $\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ x = e^y$.
6. Let $X = \{1, \{2\}\}$ and $Y = \{1, \{1, \{2\}\}\}$, $X \in Y$.
7. $\emptyset \subseteq \{\{\emptyset\}\}$.
8. $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
9. $x \in f^{-1}(A)$ if and only if $f(x) \in A$.
10. $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.
11. If $f \circ g \circ h$ is surjective then f is surjective.
12. $sin(x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
13. There exists f such that f is a bijection and f is not invertible.
14. A relation that is reflexive, antisymmetric and transitive is an order relation.
15. If $\mathcal{R}$ is not an equivalence relation, it is not reflexive.
16. The usual inclusion on $\mathcal{P}(A)$ is a total order.
17. $\mathbb{R} \setminus \{2\}$ and $\mathbb{R} \setminus \{10\}$ have the same cardinality.
18. $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{N},\ \exists z \in \mathbb{N},\ x = \frac{y}{z}$.
19. $\mathbb{Q}$ is uncountably infinite.
20. $\mathbb{R}$ is uncountable infinite.

Exercise 7.5. (2 points)
Answer the following questions and justify your answers. We define the following relations on $\mathbb{R}^2$.

$$x\mathcal{R}y \text{ if and only if } x^2 - y^2 \leq 0$$

1. Is $\mathcal{R}$ transitive?
2. Is $\mathcal{R}$ reflexive?
3. Is $\mathcal{R}$ symmetric?
4. Is $\mathcal{R}$ anti-symmetric?

Exercise 7.6. (2 points)
Prove by induction that for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} k^3 = \frac{1}{4}n^2(n+1)^2.$$

Exercise 7.7. (3 points)

1. Give a bijection between $\mathbb{R}$ and $\mathbb{R}_+^*$.
2. Using the bijection of (1), exhibit a bijection between $\mathbb{R} \times \{0, 1\}$ and $\mathbb{R}^*$.
3. Using (1), prove that there exists a bijection between $\mathbb{R}^*$ and $\mathbb{R}$.
4. Deduce that for every $n \geq 1$, $\mathbb{R} \times \{1, ..., n\}$ and $\mathbb{R}$ have the same cardinality.

Exercise 7.8. (3 points)
Prove the following result. Let A, B and C be three sets and $f : B \to C$ and $g : A \to B$ such that $f \circ g$ is injective.

1. Let us assume that g is surjective. Prove that f is injective.
2. Give an example where g is not surjective and f is not injective.