Logic and Sets
Final exam 2021 (2h)

Name: QEM/MMEF

Exercise 1 (5pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. The least number principle (Fermat infinite descent) is stronger than the weak principle of induction. F
2. $(A\Delta B)\Delta C = A\Delta(B\Delta C)$ T
3. f not injective means that $x = y$ implies $f(x) = f(y)$. F
4. $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$ but not the converse inclusion. F
5. $x \mapsto e^x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. F
6. $x \mapsto x^2$ is an injection from $\mathbb{R}_+$ to $\mathbb{R}$. T
7. $x \mapsto x^2$ is a surjection from $\mathbb{R}$ to $\mathbb{R}_+$. T
8. $f : X \longrightarrow Y$ is invertible if there exists a mapping $f^{-1} : Y \longrightarrow X$ such that $f^{-1} \circ f = Id_X$. F
9. $g \circ f$ injective implies f injective. T
10. The inverse function of $x \mapsto x$ is the function $x \mapsto \frac{1}{x}$. F
11. The binary relation $\subseteq$ is reflexive, asymmetric, transitive and complete. F
12. If $X \subseteq \mathbb{R}$ is bounded from below, it has an infimum. T
13. It is possible for $X \subseteq E$ to have a least element but no mimimal element. F
14. $\mathbb{R}$ is equipotent with the set of irrational numbers. T
15. $\mathbb{R}$ is equipotent with the set of rational numbers. F
16. $[0, 1] \cap \mathbb{Q}$ is countable. T
17. $2^{\mathbb{N}}$ is countable. F
18. $\mathbb{N}^{\mathbb{N}}$ is uncountable. T
19. Countable cartesian products of countable sets are countable. F
20. The set of all finite subsets of $\mathbb{N}$ is countable. T

Exercise 2 (4 pts)

1. (1pt) Give the rank and nullity of

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 4 & 2 & 0 & 2 & 4 \end{bmatrix}$$

2. (3pts) Solve the following linear system where $a \in \mathbb{R}$. Depending on the value of a, determine the set of its solutions, if any.

$$\begin{cases} x - y - z = 1 \\ 2x - 2y + z = 2 \\ 3x + y - 2z = -1 \\ 4x - 4y - z = 4 \\ x + 3y - 3z = a \end{cases}$$

Exercise 3 (3pts)

Consider a finite set E and the set of partitions of E, denoted by $\Pi(E)$. We define the binary relation $\mathcal{R}$ on $\Pi(E)$ by

$$\pi_1 \mathcal{R} \pi_2 \Leftrightarrow \forall P_1 \in \pi_1, \exists P_2 \in \pi_2 \text{ such that } P_1 \subseteq P_2.$$

Show that $\mathcal{R}$ is an order relation. Is it complete? What are the maximal and minimal elements?

2 0,5 0,5

Exercise 4 (3 pts)

Show that for any mapping $f : X \longrightarrow Y$ and any $A, B \subseteq X$, it holds

$$f^{-1}(f(A) \setminus f(B)) \subseteq B^c.$$

Is it true that $A \setminus B \subseteq f^{-1}(f(A) \setminus f(B))$? Give a counterexample if not.

Exercise 5 (4pts)

Determine, if they exist, the set of lower bounds, upper bounds, the minimal and maximal elements, the infimum and supremum of the following subsets of $\mathbb{R}$:

1. $[0, 1[\cup]2, 3]$
2. $\{\log x \, : \, x \in \mathbb{Q}_+\}$

Same question for the following subsets of $\mathbb{Q}$:

1. $\left\{ \frac{1}{n^3} \, : \, n \in \mathbb{Z} \setminus \{0\} \right\}$
2. $\{x \in \mathbb{Q} \, : \, x > \sqrt{3}\}$

Question (1pt)

Explain the meaning of: A has cardinality at least as large as the cardinality of B.

Exam 2021 (Answers)

Ex 2

1. rank of $\begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 4 & 2 & 0 & 2 & 4 \end{bmatrix}$. The determinant of the submatrix $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ is $-6 \neq 0$, hence rank $A = 2$.
Nullity$A = n -$ rank$A = 5 - 2 = 3$.

2. $[A\ b] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & -2 & 1 & 2 \\ 3 & 7 & -2 & -1 \\ 4 & -4 & -1 & 4 \\ 1 & 3 & -3 & a \end{bmatrix}$

Column 1: $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 2 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & -2 & a-1 \end{bmatrix}$ swap $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 4 & 2 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & -2 & a-1 \end{bmatrix}$

Column 2 $\begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 4 & 2 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -4 & a+3 \end{bmatrix}$

Column 3 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a+3 \end{bmatrix}$ swap $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & a+3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank A = rank$[\bar{A}\ b]$ iff $a + 3 = 0$, i.e., $a = -3$.
If $a \neq -3$, the system has no solution
If $a = -3$, the solution is unique as rank $A = 3$, and is given by: $\begin{cases} x = 0 \\ y = -1 \\ z = 0 \end{cases}$

Ex. 3

- R is reflexive: $\forall P_1 \in \pi_1$, $P_1 \subseteq P_1$.
- R is antisymmetric: consider π_1, π_2 s.t. $\pi_1 R \pi_2$ and $\pi_2 R \pi_1$. Take $P_1 \in \pi_1$. Then $\exists P_2 \in \pi_2$, $P_1 \subseteq P_2$. As $\pi_2 R \pi_1$, $\exists P_1' \in \pi_1$, $P_2 \subseteq P_1'$. As $P_1, P_1' \in \pi_1$ and $P_1 \subseteq P_2 \subseteq P_1'$, we have $P_1 = P_1'$, and therefore $P_1 = P_2$. It follows that $\pi_1 = \pi_2$.

- $\mathcal{R}$ transitive. Consider π_1, π_2, π_3 s.t. $\pi_1 \mathcal{R} \pi_2$, $\pi_2 \mathcal{R} \pi_3$. Take $P_1 \in \pi_1$. Then $\exists P_2 \in \pi_2$, $P_1 \subseteq P_2$, $\exists P_3 \in \pi_3$, $P_2 \subseteq P_3$. Therefore $P_1 \subseteq P_3$. Then $\pi_1 \mathcal{R} \pi_3$.
- $\mathcal{R}$ not complete as for example $\{\{1,2\}, 3\}$ is not comparable with $\{\{1,3\}, 2\}$.
- maximal element is $\hat{\pi} = \{E\}$, minimal element is $\check{\pi} = \{\{x_1\}, \dots, \{x_n\}\}$, with $E = \{x_1, \dots, x_n\}$.

Ex. 4

- Take $x \in f^{-1}(f(A) \setminus f(B))$. Then $f(x) \in f(A) \setminus f(B)$, which implies $f(x) \notin f(B)$. Therefore $x \notin B$.
- $A \setminus B \subseteq f^{-1}(f(A) \setminus f(B))$ is not true: take $f : X \to Y$ a constant function, and $A \not\subseteq B$. Then $A \setminus B \neq \emptyset$ but $f(A) \setminus f(B) = \emptyset$ so that $f^{-1}(f(A) \setminus f(B)) = \emptyset$.

Ex. 5

		lower bounds	upper bounds	min	max	inf	sup
in $\mathbb{R}$	$[0,1[\cup]2,3]$	$]-\infty, 0]$	$[3, +\infty[$	0	3	0	3
	$\{\log x : x \in \mathbb{Q}_+\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
in $\mathbb{Q}$	$\{\frac{1}{n^3} : n \in \mathbb{Z} \setminus \{0\}\}$	$]-\infty, -1] \cap \mathbb{Q}$	$\mathbb{Q} \cap [1, +\infty[$	-1	1	-1	1
	$\{x \in \mathbb{Q} : x > \sqrt{3}\}$	$]-\infty, \sqrt{3}] \cap \mathbb{Q}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$