Macroeconomics: Economic Growth (Licence 3)
Lesson 8: Endogenous growth

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Endogenous growth

Lesson 8

- Endogenous growth
- Economics of ideas
- AK model

- **Economics of ideas: Summary**
- Ideas are non-rivalrous goods and they imply **high fixed costs**
- The presence of ideas in the production function means that the production function is characterized by **increasing returns to scale**
- Increasing returns require imperfect competition

- **Economics of ideas: Summary**
- Increasing returns require imperfect competition
- **if labour and capital are paid at their marginal product, then no output will be left to compensate for the accumulation of knowledge**
- Ideas are at the core of sustainable growth
- but: how can we model them? How can we deal with the increasing returns to scale that are required to endogenize the accumulation of knowledge?

- **Basic AK model: Summary**
- Actually, there is an alternative way to have **increasing returns to scale and maintain perfect competition** in the model
- This is the "first approach" to endogenous growth which is based on AK models
- even if individuals are not compensated for accumulating knowledge, **knowledge accumulates embodied in capital and in the capital accumulation**

Economic Growth

- **Basic AK model**
- Assume $\alpha = 1$, so the production function can be

$$Y = AK \tag{1}$$

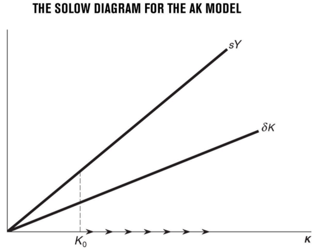
- Assumptions:
- where A is some positive constant and it is assumed that $\frac{\dot{A}}{A} = 0 \rightarrow$ i.e. no technological progress
- There is no population growth
- Notice the linearity between K and Y

- **Basic AK model**
- Accumulation of capital is like in the Solow model:

$$\dot{K} = sY - \delta K \quad (2)$$

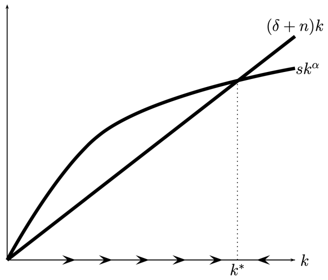
- where s is the investment rate and δ the depreciation rate, both assumed constant

Economic Growth



- sY line represents total investment as a function of capital stock $\rightarrow$ **since Y is linear in K sY curve is a line since $\alpha = 1$**
- δK line represents amount of investment to replace the depreciation of capital
- **Assumption: total investment is larger than depreciation: $sY > \delta K$**
- Capital stock is always growing: Increase in capital accumulation drives economic growth than never stops

The basic Solow diagram



- Plot output per worker against capital per worker
- First curve: is the amount of investment per person $sy = sk^{\alpha}$
- Diminishing returns to capital since $\alpha < 1$ each new unit of capital added was less productive $\rightarrow$ **total I fall to the level of depreciation ending capital accumulation**
- Second curve is the line $(\delta + n)k$: the amount of investment per person required to keep the amount of capital per worker constant.
- The difference between both curves is: the change in the amount of capital per worker. When $sk^{\alpha} = (\delta + n)k$ then $\dot{k} = 0$

- **Basic AK model**
- Accumulation of capital is like in the Solow model:

$$\dot{K} = sY - \delta K \tag{3}$$

- The growth rate of capital is given by diving both sides by K:

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = sA - \delta \tag{4}$$

- notice that A is the marginal product of capital.

- **Basic AK model**

- The growth rate of the economy's income is equal to the growth rate of capital given by:

$$\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \frac{\dot{K}}{K} \tag{5}$$

$$\frac{\dot{Y}}{Y} = sA - \delta \tag{6}$$

- **The growth rate of the economy is an increasing function of the investment rate** s $\rightarrow$ policy implication

- In per capita terms: production is $y = Ak$ and the growth rate of capital is given by $\frac{\dot{k}}{k} = sA - (\delta)$, which is constant and positive (if s and A are sufficiently large)

- **Basic AK model**
- In this model, growth never stops (perpetual growth)
- We have **perpetual growth** because here we have constant returns to capital accumulation since $\alpha = 1$
- **Recall in the Solow model we had diminishing return** because of $\alpha < 1$
- In the AK model the **marginal product of each unit of capital is constant and equal to A:**
- Since $Y = AK \rightarrow dY/dK = A$.

- **Basic AK model**
- Note that in the Solow model, the parameter α measures the curvature of the curve $sy = sk^{\alpha}$ in transition dynamics:
  - If α is large, then the further away the steady state value of k_* **the transition to the steady state is longer**, relative to low value of α.
  - **If $\alpha = 1$ is a limiting case in which the transition dynamics never ends**

- **Differences between Solow and AK model**
  - **In AK model:** $\alpha = 1$, thus $\frac{\dot{k}}{k} = sA - \delta$ and
  - the growth rate of capital and income depends on s and perpetual growth, thus policy changes in s can lead to changes in the long run growth rates;
  - $\rightarrow$ AK model generates endogenous growth because it involves a linearity in the capital accumulation equation from the linear production function

Endogenous Growth: The Romer Model

- Romer (1990), "Endogenous Technological Change"
- We will study the version by Jones (1995) and discuss the differences with Romer (1990)

Economic Growth

- **Endogenous Growth: The Romer Model**
- The model describes advanced economies
- **Endogenous Growth: technological change is endogenous**
- Technological progress is the addition of new varieties of goods to those available in the economy
- E.g: Laptop computers are a new type of good compared to desktop computers and smartphones are new compared to laptops
- Product innovation is a specific activity motivated by profits

- **Endogenous Growth: The Romer Model**
- In this model technological change takes the form of a larger variety of intermediate products
- **The model is composed of three sectors:**
    - (1) A sector producing final goods
    - (2) An intermediate goods sector
    - (3) A research sector

- **Endogenous Growth: The Romer Model**
- The Romer model introduces **research sector:** search for new ideas by researchers interested in raising profits from inventions
- Technological progress is driven by $R\&D$ in advanced economies
- There are 2 main elements in the Romer model:
  - (1) **Production function:** describe how K and L are combine to produce output using the stock of ideas in A
  - (2) **A set of equations describing how inputs for the production function evolve over time.**

- **Endogenous Growth: The Romer Model**
- Rely on the neoclassical production function (Solow model) but **technology A is endogenized**

$$Y = K^{\alpha}(AL_Y)^{1-\alpha} \quad (7)$$

- Take this production function as given, we will later discuss in detail the microfoundation of the model
- Y stands for output, K for capital, (L_Y) for labor used in production sector of goods, and A for technology

- **Endogenous Growth: The Romer Model**

- Rely on the neoclassical production function (Solow model) but **technology A is endogenized**

$$Y = K^{\alpha}(AL_Y)^{1-\alpha} \tag{8}$$

- When we recognize that **ideas (A) is also an input in the production** $\rightarrow$ **increasing returns to scale**

- **Endogenous Growth: The Romer Model**
- The production function has CRS with respect to K and L for a given (exogenous) level of A
- Therefore, **it must exhibit increasing return to scale to all three inputs when A is endogenize:**
- If double K, L and the stock of ideas, then you will more than double output

- **Endogenous Growth: The Romer Model**
- **Accumulation of capital is like in the Solow model:**

$$\dot{K} = s_K Y - \delta K \tag{9}$$

- where $s_K Y$ is the investment in physical capital and δ the depreciation rate of the capital stock
- As in Solow, savings are assumed to be a constant fraction of income $S = I = s_K Y$

- **Endogenous Growth: The Romer Model**
- **Labor accumulation: population grows at a constant and exogenous rate n as in Solow model**

$$\frac{\dot{L}}{L} = n \tag{10}$$

- The key equation that is new relative to the Solow model is the equation describing technological progress (endogenous A)

- **Endogenous Growth: The Romer Model**
- **Evolution of ideas: knowledge externality**
- **Knowledge** depends on the rintroduction of new ideas, which in turn depends on the number of people involved in generating ideas (e.g., researchers)
- Denote with L_A the number researchers (workers in the research sector), and $\bar{\theta}$ **the rate at which they discover new ideas**

$$\dot{A} = \bar{\theta} L_A \tag{11}$$

- **Endogenous Growth: The Romer Model**
- **Evolution of ideas: knowledge externality**
- The rate at which new ideas are produced is assumed to depend on the existing stock of ideas A:
- Such dependence could be either positive or negative
  - **Positive,** if new discoveries benefit from the stock of older ideas ("standing on the shoulders of the giants effect")
  - **Negative**, if new ideas are harder to develop
  - **Jones (1995) assumes that the externality is positive: $\phi < 1$,**
  - **where ϕ represents the dependence of the rate at which new ideas are discovered depends on the stock of ideas A :**

$$\bar{\theta} = \theta A^{\phi} \tag{12}$$

- **Endogenous Growth: The Romer Model**
- **Evolution of ideas: knowledge externality**
- The invention of ideas from the past raises productivity in the present
- In that case $\bar{\theta}$ is an increasing function of A
- The discovery of calculus, the invention of the laser are examples of ideas that have increased productivity of later research

- **Endogenous Growth: The Romer Model**
- **Evolution of ideas: duplication externality**
- We also assume that the impact of the number of researchers on the dynamics of new ideas is nonlinear,
- with a **positive but decreasing effect, summarized by** $\lambda < 1$
- Researchers crowd out research of others: **congestion effect**
- This implies that:

$$\dot{A} = \theta L_A^{\lambda} A^{\phi} \tag{13}$$

- **Endogenous Growth: The Romer Model**
- **Duplication externality**
- Individual researchers being small relative to the economy as a whole take $\bar{\theta}$ as given
- As in $\bar{\theta} = \theta A^{\phi}$, an individual engaged in research creates $\bar{\theta}$ new ideas
- While $\bar{\theta}$ might be changed by only a minuscule amount in response to actions on one researcher,
- it varies with aggregate research effort
- $\lambda < 1$ may reflect an externality associated with duplication: **some ideas created by an individual may not be new for the economy.**

- **Endogenous Growth: The Romer Model**
- **Duplication externality**
- The presence of A^{ϕ} is external to the individual
- When $\phi > 0$ implies positive knowledge spillover in research
- Much of the knowledge created spillover for future reserachers

- **Endogenous Growth: The Romer Model**

- **Allocation of labor across sectors**

- At each point in time, people may choose to work as production workers or as researchers

- Initially, for simplicity, we **assume that the share of people involved in the two activities is exogenous and constant** (we'll relax later this assumption)

$$L_Y + L_A = L \tag{14}$$

- where $L_A/L = S_R$ is the share of researchers in total population and ,$1 - S_R$, is the share of production workers

- **Growth in the Romer Model: Balanced growth path**
- Along a balanced growth path (BGP), output per capita, the capital-labor ratio and technology grow at the same constant rate:
- Let us denote such growth rate as $g_y = g_k = g_A$
- All per capita growth is due to technological progress
- If there is no technological change in the model then there is no growth
- **What is the growth rate of technology along the BGP?**

- **Growth in the Romer Model: Balanced growth path**
- **What is the growth rate of technology along the BGP?**
- The answer to this question is found by re-writing the production function of ideas:
- Dividing by A the dynamic equation for new ideas $\dot{A} = \theta L_A^{\lambda} A^{\phi}$, we obtain the growth rate of new ideas:

$$\dot{A}/A = \theta L_A^{\lambda} A^{\phi-1} \tag{15}$$

$$g_A = \theta L_A^{\lambda} / A^{1-\phi} \tag{16}$$

- where g_A is constant along the Balance Growth Path if and only if the numerator and the denominator grow at the same rate
- In the BGP $\dot{A}/A = g_A$ is constant

Long run growth (per worker variables) Solow model

- **Main difference between Romer model and the Solow model with technological progress is**
- in the long run growth of per worker variables in the BGP. In the Solow model with exogenous A we have:
- $\hat{k} = \frac{k}{A} \rightarrow \frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{k}}{k} - \frac{\dot{A}}{A}$
- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{\hat{A}}}{\hat{A}} = \gamma$
- In the long run the growth rate of capital per worker is

$$\frac{\dot{k}}{k} = \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma$$

Long run growth (per worker variables) Solow model

- **Growth of per capita income in BGP in the Solow model with exogenous A**
- Since $\hat{y} = \frac{y}{A} \rightarrow y = \hat{y}A = \hat{k}^{\alpha}A$
- Thus $\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A}$
- Thereby the growth rate of income per worker in the long run is:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma$$

- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{\hat{A}}}{\hat{A}} = \gamma$
- **In the Solow model with exogenous technological progress in the long run we have:**
- $g_{\hat{k}} = g_{\hat{y}} = 0$ and $g_k = g_y = g_A = \gamma$

- **Growth in the Romer Model: Balanced growth path**

- In this model we can solve for the endogenous growth rate of A, by taking logs and derivatives respect to time of $\theta L_A^{\lambda}/A^{1-\phi}$

$$0 = \lambda \frac{\dot{L_A}}{L_A} - (1-\phi)\frac{\dot{A}}{A} \tag{17}$$

- **In the equilibrium of BGP,** g_A is constant and so $\lambda \frac{\dot{L_A}}{L_A} = (1-\phi)\frac{\dot{A}}{A}$
- Along the BGP, the number of researchers grow at the same rate n as the entire population ($\frac{\dot{L_A}}{L_A} = n$), this in turn implies

$$(1-\phi)\frac{\dot{A}}{A} = \lambda n \tag{18}$$

$$g_A = \frac{\dot{A}}{A} = \frac{\lambda n}{(1-\phi)} \tag{19}$$

- The long-run growth rate of the economy is determined by the parameters of the production function for ideas and the growth rate of researchers $(= n)$
- **More researchers mean more ideas sustaining growth**.

- **Comparing the effect of population in the Romer Model relative to the Solow model**
  - (1) **Effects of population growth in the Solow model** high n reduces the level of income along the BGP, since more people means that more K is needed to keep the K/L ratio constant $\rightarrow$ in steady state the level:

$$y = A\left(\frac{s}{\delta + n + \gamma}\right)^{1/(1-\alpha)} \tag{20}$$

  - (2) **Effects of population growth in the Romer model** people are key inputs to the creative process $\rightarrow$ A larger population generates more ideas and since they are nonrivalrous everyone can benefit

- **Growth in the Romer Model: Balanced growth path**
  $g_A = \frac{\dot{A}}{A} = \frac{\lambda n}{(1-\phi)}$
- long run growth continues only as long as population grows
- which allows the number of researchers to grow
- However, if the number of researchers stops growing, the growth rate of per capita output stops as well
- This means that once and for all improvements in research efforts do not lead to persistent increase in growth

- **Romer (1990) and Jones critique**
- The original work by Romer implies that
- a once and for all increase in the share of researchers in total population, lead to permanent changes in growth rates
- In the context of the model just outlined above, this would occur in the special case of $\lambda = 1 = \phi$, which leads to the following dynamic equation for new ideas: $\frac{\dot{A}}{A} = \theta L_A$
- This version of the model generates sustained growth in the presence of a constant research effort.
- Implies the assumption that **productivity of researchers is proportional to the existing stock of ideas**
- Jones critique: the fact that a once and for all increase in L_A would lead to a permanent increase in the rate of growth of per capita income is in contrast with empirical evidence

- **Jones critique**

- The empirical evidence shows:
  - Note that most research activities of the post-WWII period are located in advanced economies, and the **proportion of researchers over production workers has continuously increased in advanced economies**
  - Nevertheless, one cannot find evidence of an increase in growth rates in advanced economies
  - In the US growth rates have been rather stable for over 100 years, whereas for European countries growth rates have been declining over time
  - Jones (1995): **the assumption that $\phi = 1$ cannot be right; almost certain that $\phi < 1$**

Economic Growth

- **Jones critique**

- Similarity between Solow and Romer model on long-run effects:
  - **In the Solow model**: changes in the policy and investment rate have no long-run effect on growth $\rightarrow$ since growth was due to exogenous technical change
  - **In the Romer model**: we have the same result.
  - Due to: $g_A = \frac{\dot{A}}{A} = \frac{\lambda n}{(1-\phi)}$, depends only on parameters and it is not affected by investment rate or share of RD labor.
  - **Main difference: policy on RD affect the growth rate along the transition path to the new steady state altering the level of income in Romer model.**

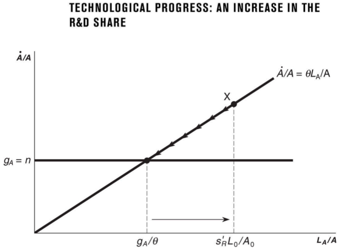
- **Response of the model to shocks: permanent increase in RD share**
  - We now analyze the behavior of the economy along the dynamic path leading to the BGP and the response of the economy to shocks
  - Figure in the next slide describes the dynamics of the model after a permanent increase in the share of researchers for the case in which $\lambda = 1$ a,d $\phi = 0$

$$\frac{\dot{A}}{A} = \theta \frac{L_A}{A} = \theta \frac{S_R L}{A} \tag{21}$$

  - At the steady state the growth rate of new ideas and of output per capita is :

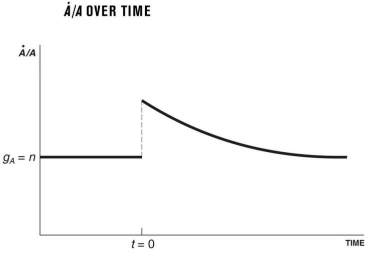
$$g_A = \frac{\dot{A}}{A} = \frac{\lambda n}{(1-\phi)} = n \tag{22}$$

Economic Growth: increase in the share of researchers



- An increase in the share of researchers moves the initial condition to the right of the steady state and the growth rate jumps instantaneously to the point X
- After the initial jump, the denominator of the right-hand side increases as A increases
- As a result the growth rate slows down along the arrows indicated in the figure, until the growth rate equals n again, the condition for balanced growth path

Economic Growth: Temporary boost to growth



- **Output per capita and technology along a BGP**

- Note that after the transition dynamics has been concluded, when the economy is on its balanced growth path, the model becomes equivalent to the Solow model:
  - **Technological change is exogenous, as new ideas grow at the rate of growth of the population, an exogenous rate**
  - Recall, in Solow, along a BGP per capita (with Cobb-Douglas production function) output is

$$y* = \left(\frac{s}{\delta + n + gA}\right)^{\alpha/(1-\alpha)} A \tag{23}$$

- Where $gA = \gamma$ in the Solow model.

- **Output per capita and technology along a BGP**
- Here the model differs from the Solow model, as here only a share of the population works in the production, and we need to adjust for it
  - With S_R the fraction of researchers and $1 - S_R$ the fraction of workers in the production sector, thus on a balanced growth path we have :

$$y* = \left(\frac{s_K}{\delta + n + g_A}\right)^{\alpha/(1-\alpha)} A(1 - S_R) \tag{24}$$

- **Technological level**

- Recall that on a BGP:

$$\frac{\dot{A}}{A} = g_A = \theta \frac{S_R L}{A} \tag{25}$$

- Then, we can rewrite the expression above to highlight the level of technology:

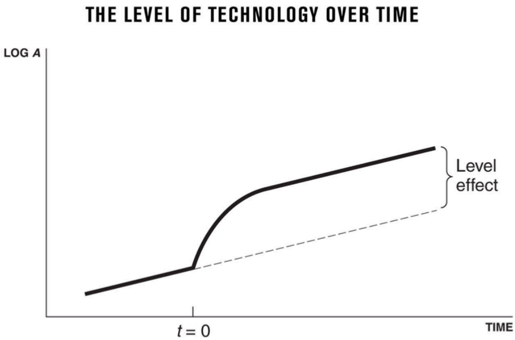
$$A = \theta \frac{S_R L}{g_A} \tag{26}$$

- An increase in S_R permanently increases the level of technology.

- **Technological level**

$$A = \theta \frac{S_R L}{g_A} \tag{27}$$

- If at $t = 0$, S_R jumps to a higher level, then A jumps as well
- However, the growth rate eventually converges to the rate of growth of population,
- as the increase in A reduces the rate of growth of new ideas after the initial jump.
- This process is summarized in the next slide


THE LEVEL OF TECHNOLOGY OVER TIME
LOG A
Level effect
t = 0
TIME

- **Scale effects in the long run**
- Given $A = \frac{S_R L}{g_A}$

$$y* = \left(\frac{s_K}{\delta + n + g_A}\right)^{\alpha/(1-\alpha)} \frac{S_R L}{g_A}(1 - S_R) \qquad (28)$$

- In a world populated by countries of different sizes L, the above equation states that **larger economies will be richer in the long-run**
- Another interpretation of the above equation is that it refers to the world economy, and thus L denotes world population in that case the share of researchers has two opposite effects.

- **Scale effects in the long run**

- Givan $A = \frac{S_R L}{g_A}$

$$y* = \left(\frac{sK}{\delta + n + gA}\right)^{\alpha/(1-\alpha)} \frac{S_R L}{g_A}(1 - SR) \tag{29}$$

- The share of researchers has two opposite effects:

- (1) **A negative effect: a larger share of researchers reduces the number of production workers** (for a given total population)

- (2) **A positive effect: a larger share of researchers implies a larger set of ideas**, which increase productivity.

- $\rightarrow$ we need to see a complete version of the model with an endogenous choice between production and research activities

Economic Growth

- **Complete version of the Romer Model**

- **The model is composed of three sectors:**

  - **A research sector:** creates new ideas (non rivalrous but partially excludable due to patents). Design new variety of machinery and sells the right to produce a capital good to the intermediate good sector;

  - **An intermediate goods sector:** several monopolists (imperfect competition) that gain monopoly power to charge a markup by purchasing the design for specific capital goods and manufacture it and sell it to the final good sector.

  - **A sector producing final goods:** under perfect competition using labor and a number of different capital goods call also intermediate goods

  - In this model technological change takes the form of a larger variety of intermediate products

Final Goods Sector

- Final goods are produced by perfectly competitive firms using this function:

$$Y = L_Y^{1-\alpha} \int_0^A x_j^{\alpha} dj \tag{30}$$

  - L_Y is the number of workers in the final goods sector
  - A indexes the number of intermediate goods used
  - x_j is the amount of capital goods (also call int. good) j used by the final goods sector

Final Goods Sector

- Assume the price of final goods is the numeraire (and we normalize it to 1) $p_Y = 1$

- Firms maximize profits taking the rental price of intermediate goods (pj) and the wage rate (w) as given:

$$max_{L_Y, x_j} L_Y^{1-\alpha} \int_0^A x_j^{\alpha} dj - wL_Y - \int_0^A p_j x_j dj \tag{31}$$

- which gives the first-order conditions (F.O.C.)

- Differentiating with respect to L_Y :

$$w = (1-\alpha)\frac{Y}{L_Y} \tag{32}$$

- which states that firms hire workers until the real wage equals the marginal product of labor due to perfect competition.

Final Goods Sector

- Differentiating with respect to x_j :

$$p_j = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} \qquad (33)$$

- which states that firms demand inputs until their marginal product equals their price

- Final goods firms are just setting marginal cost (wage, price of good j) equal to the marginal product of the input due to perfect competition.

The intermediate-goods sector

- **Market structure: monopolistic competition**
- Several intermediate goods firms producing under imperfect competition
- Each intermediate goods producer is a monopolist that purchase the design for the specific capital good to the research sector and gain monopoly power
- **Due to patent protection** : Only one firm can produce each separate int. good
- They purchase the design for a capital good j and such design is protected by a patent
- Int. good firms produce those int. goods by transforming capital into them one-for-one $\rightarrow$
- **Each firm produces one unit of intermediate good using one unit of capital**$(x_j = k_j)$

The intermediate-goods sector: profit maximizing

- **Each monopolist sets the optimal price for its specific capital good (intermediate good), by maximizing profits.**
- The profits of any given int. good firm j are

$$\pi_j = p_j(x_j)x_j - rx_j \tag{34}$$

- where $p_j(x_j)$ is the demand function of the final goods firm for the intermediate good j.
- Because the firm is a monopolist, they take into account how demand changes as they produce more x_j- that is, they know the price p_j that final good firms will pay for any given output x_j.

The intermediate-goods sector: profit maximizing

$$\pi_j = p_j(x_j)x_j - rx_j. \tag{35}$$

- Firm pick x_j to maximize profits:
- First-order condition (drop j subscripts for simplicity)

$$p'(x)x + p(x) - r = 0 \tag{36}$$

- which says that marginal revenue equals marginal cost. Re-arrange to:

$$p'(x)\frac{x}{p} + 1 = \frac{r}{p} \tag{37}$$

$$p = \frac{1}{1 + \frac{p'(x)x}{p}} r. \tag{38}$$

The intermediate-goods sector: profit maximizing

$$p = \frac{1}{1 + \frac{p'(x)x}{p}} r. \tag{39}$$

- Given the demand from final good firms $p_j = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}$
- we can calculate the inverse of the price elasticity of the demand equal to $\alpha - 1$
- Thus, the optimal price is a constant mark up over the variable marginal cost r (recall $\alpha < 1$)

$$p = \frac{1}{1 + \alpha - 1} r = \frac{r}{\alpha}. \tag{40}$$

- A higher α implies that demand is more elastic and thus monopolistic power is lower.

The intermediate-goods sector: markups

- The int. good firm charges :

$$p = \frac{r}{\alpha} \tag{41}$$

- for its output. Thus price (p) is greater than marginal cost (r), as $\alpha < 1$.
- Intermediate good firms thus earn profits.
- These profits are what will be the reason for innovation. The opportunity to earn profits will incentive people to try and start int. good firms.
- **Note that the profit-maximization problem is identical for all int. good firms.**
- All of them charge the same price. Given that the demand function is identical for all int. goods, this means that the final goods firm buys the same amount of all of them.

$$x_j = x \tag{42}$$

Symmetric equilibrium

- Intermediate goods firms are all identical, and also demand functions for each capital goods are the same, therefore, their prices are the same
- Therefore each capital goods firm earns the same profit $\pi = \alpha(1-\alpha)Y/A$
- The total amount of int. goods produced must be equal to the capital stock, as each int. good firm uses capital to produce ($x_j = k_j$), and because all $x_j = x$:

$$K = \int_0^A x_j dj = Ax \tag{43}$$

- The above equation implies

$$x = \frac{K}{A} \tag{44}$$

Symmetric equilibrium

- Final output is

$$Y = L_Y^{1-\alpha} \int_0^A x_j^\alpha dj = L_Y^{1-\alpha} A x^\alpha \tag{45}$$

- which when we plug in $x = K/A$ becomes

$$Y = K^\alpha (AL_Y)^{1-\alpha}. \tag{46}$$

- Aggregate final output is described just like we did with the Solow model. Only the underlying market structure is different.

Research sector

- Generates new ideas.
- In this model new ideas are design for new capital goods
- New designs are discovered according to :

$$\dot{A} = \theta A^{\phi} L_A^{\lambda} \tag{47}$$

- The accumulation of ideas depends on the rate at which new ideas are discovered (θA^{ϕ}) and the amount of research workers.
- When a new design is discovered the inventor receives a patent from government for the exclusivity to produce this new capital good
- The inventor sells the patent to the intermediate good producer.
- The intermediate good firms charge a markup over MC and earned profits that they used to pay the inventors for the patent.
- Those profits compensate inventors for the time they spent in prospecting new ideas.

Research sector

- We now analyze the market for patents
- How is the price of a new design (patent) determined?
- A new design allows firms to earn a monopolistic profit, for the entire life of the new design (we assume the life of new design is infinite)
- How much the intermediate goods firms are willing to pay?

Research sector

- How much the intermediate goods firms are willing to pay?
- The present discounted value of profits earned by the intermediate good firm
- **If the price of the patent is lower** than the present discounted value of profits earned by the intermediate good firm, there will be another firm ready to pay more to obtain the patent
- **If the price of the patent is higher** than the present discounted value of profits earned by the intermediate good firm, then no one will buy it.

Research sector

- P_A is the price of a new design
- How does P_A evolves over time?
- We can answer this question with an arbitrage condition: the firm has money to invest and so has two options:
- (1) Put the money in the bank and earn an interest rate r, or
- (2) Purchase the patent for one period earn profits and then sell the patent
- In equilibrium the rate of return from both investments is the same.
- **An arbitrage condition is a condition that states that the return from both investments is the same.**

The arbitrage condition

- We use an arbitrage equation to get the value of a patent, P_A (the price of a new design)

$$rP_A = \pi + \dot{P}_A \tag{48}$$

  - rP_A is the amount earned from investing P_A dollars in the bank. This is the alternative to buying the patent
  - π is the profits earned from owning the patent, solved for above
  - $\dot{P}_A$ is the change in the value of the patent if you own it - you could resell it later for a profit (or loss)

- The arbitrage equation says that the return to owning the patent (the right-hand side) must be equal to the return to simply investing the same dollars in the bank (the left-hand side)
- Re-arrange the arbitrage equation to

$$r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}. \tag{49}$$

The arbitrage condition

$$r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}. \quad (50)$$

- Along the BGP, we know that r is constant. So therefore the ratio π/P_A must be constant, and $\dot{P}_A/P_A$ must be constant.
- Along the BGP:

$$\pi = \alpha(1-\alpha)\frac{Y}{A} = \alpha(1-\alpha)\frac{Y}{L}\frac{L}{A}. \quad (51)$$

- Along the BGP, Y/L grows at the rate g. A grows at the rate g. L grows at the rate n. So π must grow at the rate n.
- This implies that $\dot{P}_A/P_A = n$, to keep π/P_A constant. Putting that into the arbitrage equation gives

$$r = \frac{\pi}{P_A} + n \quad (52)$$

We get the price of a Patent in equilibrium along the BGP:

$$P_A = \frac{\pi}{r-n} \quad (53)$$

Solving the model

- We have already solved for the growth rate of the economy in the steady state
- The part of the model that remains to be solved is the allocation or workers between research and final good sector
- We solve for the fraction of workers who do research (so SR is endogenous)

Labour market equilibrium

- We solve for the fraction of workers who do research (so *SR* is endogenous)
- The labor market is frictionless: workers can freely choose to work in the research or in final goods production
- I in equilibrium the wage rate they obtain has to make them indifferent in working in either of the sectors Workers in the production sector are paid their marginal product

$$w_Y = (1-\alpha)\frac{Y}{L_Y} \tag{54}$$

- Workers in the research sector are paid their marginal product $\overline{\theta}$ (productivity of the research sector) times the value of the new created ideas P_A

$$w_R = \overline{\theta} P_A \tag{55}$$

Labour market equilibrium

- Wage equalization across sectors $w_Y = w_R$

$$\bar{\theta} P_A = (1-\alpha)\frac{Y}{L_Y} \tag{56}$$

- Substituting and solving for SR:

$$s_R = \frac{1}{1+\frac{r-n}{\alpha g_A}} \tag{57}$$

- The discount rate $r - n$ has a negative effect. If the discount rate goes up, research effort goes down

- g_A has a positive effect: If the growth rate is high, this is because people are finding new ideas quickly. Therefore doing research is very likely to lead to a patent, so more people want to do research.

Sub-optimal R&D

- Is the share of population that works in research optimal?
- In general, the answer is "no" in the Romer model. There is not enough research. Number of researchers is less than optimal
- Researchers do not internalize their positive knowledge externalities.
- They take $\overline{\theta}$, productivity of the research sector, as given.
- They take A as given and they do not discount the fact that their **ideas increase the productivity of other researchers in the future.**
- The presence of imperfect competition in one of the sectors creates distortions.

Sub-optimal R&D

- **The presence of imperfect competition in one of the sectors creates distortions.**
- Patents on ideas give a monopoly power over the production of each capital good (thus, lower supply to enjoy higher prices).
- **A possible policy intervention** would try to incentivize individuals to engage in more research activities
- e.g., government funds research in universities and research centers