Masters M1 MAEF, M1 IMMAEF & QEM1 –
DU MMEF 2019/2020

Class Notes *

Microeconomics 1A: Individual Decision Making
Microeconomics 1B: Equilibria & Optimality
Teachers: Elena del Mercato and Antoine Mandel
Université Paris 1 Panthéon-Sorbonne, PSE & CES

* Authors: Jean-Marc Bonnisseau and Elena del Mercato, Université Paris 1 Panthéon-Sorbonne, Paris School of Economics & Centre d'Economie de la Sorbonne.

Contents

1. Commodities and prices

1.1 Commodities

We consider a model with a finite number ℓ of commodities. A commodity or a good will be denoted by the letter h, $h = 1 \ldots, \ell$.

A commodity is characterized by its physical properties (color, weight, quality,...), the location where it is available, the date at which it will be delivered and eventually, the state of the world in which it will be available.

We assume that the characteristics of the commodities are known by the economic agent. We have no asymmetry of information.

We assume that the quantity of each commodity can be evaluated and a unit is chosen for each commodity.

The commodity space is $\mathbb{R}^\ell$, the ℓ dimensional Euclidean space. A commodity bundle or commodity basket is a vector x in $\mathbb{R}^\ell$:

$$x = (x_1, x_2, \ldots, x_h, \ldots, x_\ell)$$

The h component x_h represent the quantity of commodity h in the commodity basket x. The quantity x_h may be negative, it means either that there is a debt in commodity h or that x describe a trade or a transaction and the commodity h is given in exchange of another commodities.

We can add two basket of commodities and multiply a basket of commodity by a real number. This correspond to the usual operations in the vector space $\mathbb{R}^\ell$.

Let $x = (x_1, x_2, \ldots, x_h, \ldots, x_\ell)$ and $x' = (x'_1, x'_2, \ldots, x'_h, \ldots, x'_\ell)$, two baskets of commodities and t a real number. Then

$x + x' = (x_1 + x'_1, x_2 + x'_2, \ldots, x_h + x'_h, \ldots, x_\ell + x'_\ell)$

and $tx = (tx_1, tx_2, \ldots, tx_h, \ldots, tx_\ell)$.

We adopt the following notations:

$x \geq x'$ means $x_h \geq x'_h$ for all $h = 1, \ldots, \ell$;

$x \gg x'$ means $x_h > x'_h$ for all $h = 1, \ldots, \ell$;

$\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell \mid x \geq 0\}$;

$\mathbb{R}^\ell_{++} = \{x \in \mathbb{R}^\ell \mid x \gg 0\}$

1.2 Prices

Each commodity h has a price p_h, which is a real number. The price vector $p = (p_1, p_2, \ldots, p_h, \ldots, p_\ell)$ is also a vector of $\mathbb{R}^\ell$. Given the price vector p and a commodity bundle x, the value of x for the price vector p is

$$p \cdot x = \sum_{h=1}^{\ell} p_h x_h$$

This corresponds to the canonical inner product in $\mathbb{R}^\ell$.

In the ordinary life, the price are always quoted with respect to a given currency. Here we do not introduce such mean of exchange in the model. So the prices have a relative value. This means that two positively proportional prices are economically equivalent. Indeed, for two positive prices p_h and p_k, the ratio or relative price $\frac{p_h}{p_k}$ of good h with respect to the good k means that one can exchange $\frac{p_h}{p_k}$ units of good h against one unit of good k.

Often, one considers normalized price vector. Different normalizations are possible. One can choose one good as numéraire, which means that its price is fixed to 1, or one can consider only the price vector such that the value of some reference commodity basket is 1. For example, the reference commodity basket may be $\mathbf{1} = (1, 1, \ldots, 1)$ and a normalized price must satisfy $\mathbf{1} \cdot p = \sum_{h=1}^{\ell} p_h = 1$.

Let $x = (x_1, x_2, \ldots, x_h, \ldots, x_\ell)$ and $x' = (x'_1, x'_2, \ldots, x'_h, \ldots, x'_\ell)$, two baskets of commodities and t a real number. Let p a price vector. One easily checks that:

$p \cdot (x + x') = p \cdot x + p \cdot x'$;

$p \cdot (tx) = t(p \cdot x)$;

If $p \in \mathbb{R}^\ell_{++}$ and $x \in \mathbb{R}^\ell_+ \setminus \{0\}$ or $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $x \in \mathbb{R}^\ell_{++}$, then $p \cdot x > 0$.

From a geometric point of view, in a two dimension plan, for a given price vector p, then the set of commodity baskets with a zero value is the orthogonal line to p passing through the origin $(0, 0)$. A commodity basket has a positive value if it is above this line in the direction of p and has a negative value if it is below this line. A commodity bundle x is more expensive than a commodity bundle x' for the price vector p if the angle between the vector $x - x'$ and the vector p is acute.

2. Consumers

2.1 Economic environment

2.1.1 Consumption set

A consumer is an economic agent, who is buying and selling commodities on the market for her final consumption. Her economic characteristics are described by a consumption set and a preference relation. The consumption set is a subset X of $\mathbb{R}^\ell$, which contains all possible commodity baskets. This means that the consumer can consume a consumption basket x in $\mathbb{R}^\ell$ if and only if $x \in X$. The consumption set summarize the physical constraints on the possible consumption.

In the following, we will assume that the consumption set is the positive orthant $\mathbb{R}^\ell_+$, that is the basket of commodities with nonnegative components. But, to analyze particular economic situations, it is necessary to take into account additional constraints in the consumption set.

Examples: Indivisibility

$$X = \{x \in \mathbb{R}^2_+ \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{N}\}$$

Survival constraint

$$X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \geq 1\}$$

Bounded consumption

$$X = \{x \in \mathbb{R}^2_+ \mid x_1 \in \mathbb{R}, x_2 \in [0, \bar{x}_2]\}$$

2.1.2 Budget set

A this stage, we take as given a price vector $p \in \mathbb{R}^\ell \setminus \{0\}$ and a wealth $w \in \mathbb{R}$.

Definition 2.1.1. *The budget set $B(p, w)$ is the set of commodity bundle x in the consumption set such that the value of x at p is lower or equal to w.*

$$B(p, w) = \{x \in \mathbb{R}^\ell_+ \mid p \cdot x \leq w\}$$

If $p \gg 0$ and $w > 0$, it is easy to draw a picture in $\mathbb{R}^2$ to represent the budget set. It is the triangle $(0,0),(w/p_1,0),(0,w/p_2)$. More generally, with ℓ commodities, the budget set is the convex hull of the $\ell+1$ extreme points $(0,((w/p_h)e^h)_{h=1}^{\ell})$ where e^h is the hth vector of the canonical basis of $\mathbb{R}^\ell$.

The budget line (in $\mathbb{R}^2$) or the budget hyperplane in higher dimension is the set of elements of the budget set, which bind the budget constraint, that is such that $p \cdot x = w$.

Proposition 2.1.1. *Let $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $w \geq 0$.*

(i) $B(p,w)$ is a nonempty, convex and closed subset of $\mathbb{R}^\ell$;
(ii) $B(p,w)$ is bounded if and only if $p \gg 0$;
(iii) for all $t > 0$, $B(tp,tw) = B(p,w)$; Let $p' \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $w' \geq 0$.
(iv) $B(p,w) \subset B(p',w')$ if and only if $p'_h w \leq p_h w'$ for all $h = 1,\dots,\ell$; in particular, $B(p,w) \subset B(p',w')$ if $p = p'$ and $w \leq w'$ or if $p \geq p'$ and $w = w'$.
(v) for all $t \in [0,1]$, $B(tp+(1-t)p',w) \subset B(p,w) \cup B(p',w)$.

Remark. In many cases, it is not possible to conclude that $B(p,w) \subset B(p',w')$ or the converse is true.

Proposition 2.1.2. *Let $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $w \geq 0$. There exists $x \in B(p,w)$ such that $p \cdot x < w$ if and only if $w > 0$. $x = (w/\sum_{h=1}^{\ell} p_h)\mathbf{1}$, where $\mathbf{1} = (1,1,\dots,1)$ is in the budget set and satisfies $p \cdot x = w$. If $x \in B(p,w)$ and satisfies $p \cdot x < w$, then $x' = x + ((w - p\cdot x)//\sum_{h=1}^{\ell} p_h))\mathbf{1}$ satisfies $p \cdot x' = w$ and $x' \gg x$.*

2.2 Consumer's preferences

A consumer is supposed to have some preferences on the possible consumptions, that is the element of her budget set.

Definition 2.2.1. *A preference relation denoted $\preceq$ on $\mathbb{R}^\ell_+$ is a total pre-order on $\mathbb{R}^\ell_+$, which means that it is a binary relation which satisfies the following conditions:*
a) Reflexivity $\forall x \in \mathbb{R}^\ell_+$, $x \preceq x$;
b) Transitivity $\forall (x,\xi,z) \in \mathbb{R}^\ell_+$, $x \preceq \xi$ and $\xi \preceq z$ imply that $x \preceq z$;
c) Completeness $\forall (x,\xi) \in (\mathbb{R}^\ell_+)^2$, either $\xi \preceq x$ or $x \preceq \xi$ holds true.

$x \preceq \xi$ means that the consumer prefers or is indifferent between the consumptions ξ and x. Two consumptions x and ξ are equivalent ($x \sim \xi$) if $x \preceq \xi$ and $\xi \preceq x$. x is strictly preferred to ξ ($\xi \prec x$) if $\xi \preceq x$ and not $x \preceq \xi$. A consumer is supposed to be able to compare any pairs of consumption but she can be indifferent among two consumptions. We remark that the completeness of the preference relation implies that if x is not strictly preferred to ξ, then ξ is preferred or indifferent to x.

Definition 2.2.2. *The preference relation $\preceq$ is :*

- *continuous if for all $(x,\xi) \in (\mathbb{R}^\ell_+)^2$ such that $x \prec \xi$, then, there exists $r > 0$ such that for all $(x',\xi') \in (B(x,r) \cap \mathbb{R}^\ell_+) \times (B(\xi,r) \cap \mathbb{R}^\ell_+)$, $x' \prec \xi'$;*
- *monotonic if for all $(x,\xi) \in (\mathbb{R}^\ell_+)^2$, if $x \ll \xi$ then $x \prec \xi$;*
- *strictly monotonic if for all $(x,\xi) \in (\mathbb{R}^\ell_+)^2$, if $x \leq \xi$ and $x \neq \xi$ then $x \prec \xi$;*
- *convex if for all $x \in \mathbb{R}^\ell_+$, the upper contour set $\{\xi \in \mathbb{R}^\ell_+ \mid x \preceq \xi\}$ is convex;*
- *strictly convex if for all $(x,\xi) \in (\mathbb{R}^\ell_+)^2$ such that $x \neq \xi$ and $x \sim \xi$, then for all $t \in]0,1[$, $x \prec tx + (1-t)\xi$.*

Remark. On can prove that $\preceq$ is continuous if and only if for all $x \in \mathbb{R}^\ell_+$, the lower contour set $\{\xi \in \mathbb{R}^\ell_+ \mid \xi \preceq x\}$ and the upper contour set $\{\xi \in \mathbb{R}^\ell_+ \mid x \preceq \xi\}$ are closed.

Examples.

- Lexicographic preferences on $\mathbb{R}^2_+$:

$x \preceq \xi$ if $x_1 < \xi_1$ or $x_1 = \xi_1$ and $x_2 \leq \xi_2$.
This preference relation is strictly monotonic and strictly convex but is not continuous. One remarks that $x \sim \xi$ if and only if $x = \xi$.

- Leontief preferences :

$x \preceq \xi$ if $\min_h\{x_h/a_h\} \leq \min_h\{\xi_h/a_h\}$, where $a_h > 0$ for all h.
This preference relation is continuous, convexe, monotonic but not strictly convex and not strictly monotonic. This preference relation describes the extreme case where the commodities are complementary commodities.

- Linear preferences :

$x \preceq \xi$ if $\sum_{h=1}^{\ell} a_h x_h \leq \sum_{h=1}^{\ell} a_h \xi_h$, where $a_h > 0$ for all h.
This preference relation is continuous, convex, strictly monotonic, but not strictly convex. It describes the case where the commodities are perfect substitute and the rate of substitution among two commodities is constant.

Graphically, one represents a preference relation with the indifference curves.

Remark. If ϕ is a function from $\mathbb{R}^\ell_+$ to $\mathbb{R}$, then one can define a preference relation $\preceq$ on $\mathbb{R}^\ell_+$ as follows: for all $(x,\xi) \in (\mathbb{R}^\ell_+)^2$, $x \preceq \xi$ if $\phi(x) \leq \phi(\xi)$. We remark that x and ξ are equivalent if $\phi(x) = \phi(\xi)$ and ξ is strictly preferred to x if $\phi(x) < \phi(\xi)$.

One easily checks that one obtains a complete, reflexive and transitive binary relation. One also checks that: the preference relation is continuous if ϕ is continuous; the preference relation is convex if ϕ is quasi-concave ($\forall \alpha \in \mathbb{R}$, $\{x \in \mathbb{R}^\ell_+ \mid \phi(x) \geq \alpha\}$ is convex); the preference relation is monotonic if ϕ is increasing ($\forall (x,\xi) \in (\mathbb{R}^\ell_+)^2$, if $x \ll \xi$ then $\phi(x) < \phi(\xi)$).

A typical example is a preference relation á la Cobb-Douglas, which means that they are define by the function ϕ defined by:

$$\phi(x) = \prod_{h=1}^{\ell} x_h^{\alpha_h}$$

with $\alpha_h > 0$ for all h. One can prove that these preference relations are continuous, convex, monotonic and strictly convex and strictly monotonic on $\mathbb{R}^\ell_{++}$.

We now consider the opposite question: taken a preference relation, does it exist a function u from which one can derive the preference relation.

Definition 2.2.3. *A preference relation $\preceq$ is representable by a utility function if there exists a function u from $\mathbb{R}^\ell_+$ to $\mathbb{R}$ such that :*

$\forall (x,\xi) \in (\mathbb{R}^\ell_+)^2$, $x \preceq \xi \Leftrightarrow u(x) \leq u(\xi)$.

We remark that several utility functions can represent the same preference relation. Indeed, if u represents the preference relation $\preceq$, then for all function φ strictly increasing from $u(\mathbb{R}^\ell_+)$ to $\mathbb{R}$, $\varphi \circ u$ also represents $\preceq$. So one can always obtain a positive utility function or a bounded utility function. The following proposition gives sufficient conditions on the preference relation to be representable.

Proposition 2.2.1. *A preference relation $\preceq$ is representable by a utility function if it is continuous and monotonic. Furthermore, the utility function can be chosen continuous.*

Actually, the result holds true if u is only continuous.

Proof. Let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^\ell_+$. We prove first that for all $x \in \mathbb{R}^\ell_+$, there exists a unique non negative real number denoted $u(x)$ such that $x \sim u(x)\mathbf{1}$. Let $A = \{t \in \mathbb{R}_+ \mid t\mathbf{1} \preceq x\}$ and $B = \{t \in \mathbb{R}_+ \mid x \preceq t\mathbf{1}\}$. B is nonempty since $\preceq$ is monotonic and A also since $0 \in A$. These two sets are closed since the preference relation is continuous. Since $\mathbb{R}_+$ is connected and $\mathbb{R}_+ = A \cup B$,there exists $t \in A \cap B$. Hence $t\mathbf{1} \sim x$. The uniqueness of t is a consequence of the monotonicity of $\preceq$.

We now show that u is continuous. Let $\alpha \in \mathbb{R}_+$. It suffices to prove that $u^{-1}([0,\alpha])$ and $u^{-1}([\alpha, +\infty[)$ are closed. From the construction of u, one deduces that $u^{-1}([0,\alpha]) = \{x \in \mathbb{R}^\ell_+ \mid x \preceq \alpha\mathbf{1}\}$ and $u^{-1}([\alpha, +\infty[) = \{x \in \mathbb{R}^\ell_+ \mid \alpha\mathbf{1} \preceq x\}$ which are closed since $\preceq$ is continuous. Hence u is continuous.

We end the proof by showing that u represents $\preceq$. Let $(x,\xi) \in (\mathbb{R}^\ell_+)^2$. If $\xi \preceq x$, by the transitivity of $\preceq$, $u(\xi)\mathbf{1} \preceq u(x)\mathbf{1}$ and the monotonicity of $\preceq$ implies that $u(\xi) \leq u(x)$. If $u(\xi) \leq u(x)$, again from the monotonicity of $\preceq$, $\xi \sim u(\xi)\mathbf{1} \leq u(x)\mathbf{1} \sim x$ and by the transitivity of $\preceq$, $\xi \preceq x$. □

We can remark that the Leontief preference relation and the linear preference relation are derived from a utility function. On the contrary, using the fact that $\mathbb{R}$ is not countable, one can prove that the lexicographic preference is not representable.

2.3 Demand of a consumer

Given a price vector $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ and a wealth $w \in \mathbb{R}_+$, the consumer chooses a consumption x which is financially affordable and which is the best choice for him with respect to her preference relation $\preceq$. This leads to the formal definition.

Definition 2.3.1. *a) The demand of the consumer for the price $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ and the wealth $w > 0$, denoted $d(p,w)$, is the set of consumptions $x \in B(p,w)$ satisfying one of the three following equivalent conditions:*

(i) there does not exist $\xi \in B(p,w)$ such that $x \prec \xi$;
(ii) for all $\xi \in B(p,w)$, $\xi \preceq x$;
(iii) for all $\xi \in \mathbb{R}^\ell_+$ satisfying $x \prec \xi$, $p \cdot \xi > w$.

b) If the preference relation is represented by a utility function u, $d(p,w)$ is the set of solutions of the following optimization problem:

$$\begin{cases} \text{maximise } u(x) \\ x \in B(p,w) \end{cases}$$

c) The indirect utility function is the value function of the previous problem, that is,

$$v(p,w) = \sup\{u(x) \mid x \in B(p,w)\}.$$

We remark that if x and ξ belong to $d(p,w)$, then $x \sim \xi$. If $x \in d(p,w)$ and $\xi \in B(p,w)$ satisfies $x \sim \xi$, then $\xi \in d(p,w)$. In other words, the elements of $d(p,w)$ are on the same indifference curve. If $x \in d(p,w)$, then $u(x) = v(p,w)$. If $x \in B(p,w)$ satisfies $u(x) = v(p,w)$, then $x \in d(p,w)$.

The equivalence between the four equivalent definitions of the demand is left as an exercise. The next proposition gathers several basic properties of the demand.

Proposition 2.3.1. *Let $\preceq$ be a continuous, convex and monotonic preference relation. Then*

a) $\forall (p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$, $d(p,w)$ is a nonempty, convex, closed and bounded subset. For all $t > 0$, $d(tp,tw) = d(p,w)$.
b) For all $x \in d(p,w)$, $p \cdot x = w$ (Walras Law).
c) If $\preceq$ is strictly convex, then $d(p,w)$ is single valued.
d) Let (p^ν, w^ν, x^ν) a sequence in $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++} \times \mathbb{R}^\ell_+$ converging to $(\bar{p}, \bar{w}, \bar{x}) \in \mathbb{R}^\ell_+ \times \mathbb{R}_+ \times \mathbb{R}^\ell_+$. If, for all ν, $x^\nu \in d(p^\nu, x^\nu)$ and $\bar{w} > 0$, then $\bar{x} \in d(\bar{p}, \bar{w})$.
e) If $\preceq$ is strictly convex, the mapping $(p,w) \to d(p,w)$ is a continuous mapping on $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$.
f) If u is a continuous utility function representing $\preceq$, the indirect utility function v is continuous, decreasing in p, increasing in w, quasi-convex in p and homogeneous of degree 0 in (p,w) on $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$.

Proof. a) The non-emptiness of $d(p,w)$ is a consequence of the continuity of $\preceq$. The convexity comes from the convexity of $\preceq$. The closedness comes from the continuity of $\preceq$. Finally, the boundedness comes from the fact that the budget set is bounded.

b) Let $x \in \mathbb{R}^\ell_+$ such that $p \cdot x < w$. Then, $\xi = x + \frac{w - p\cdot x}{\sum_{h=1}^\ell p_h}\mathbf{1}$ satisfies $x \ll \xi$ and $p \cdot \xi = w$. So, x cannot belong to $d(p,w)$ since the monotonicity of $\preceq$ implies that ξ is strictly preferred to x and ξ belongs to the budget set.

c) If $\preceq$ is strictly convex, let x and ξ in $d(p,w)$ such that $x \neq \xi$. Then, let $x' = \frac{1}{2}(x+\xi)$. From the strict convexity of $\preceq$, $x \prec x'$. Furthermore, x' belongs to $B(p,w)$ since the budget set is convex. Hence, one gets a contradiction between the fact that $x \in d(p,w)$ and $x \prec x'$ and $x' \in B(p,w)$.

d) One easily checks that $\bar{p} \cdot \bar{x} \leq \bar{w}$. If $\bar{x} \notin d(\bar{p},\bar{w})$, then there exists $\xi \in B(\bar{p},\bar{w})$ such that $\bar{x} \prec \xi$. Since $\prec$ is continuous, there exists $t \in [0,1[$ such that $\bar{x} \prec t\xi$. Since $\bar{w} > 0$, $\bar{p} \cdot (t\xi) < \bar{w}$. Then there exist $\underline{\nu} \in \mathbb{N}$, such that for all $\nu \geq \underline{\nu}$, $p^\nu \cdot (t\xi) < w^\nu$. Since $\preceq$ is continuous, there exist $\tilde{\nu} \in \mathbb{N}$, such that for all $\nu \geq \tilde{\nu}$, $x^\nu \prec t\xi$. Finally, for all $\nu \geq \max\{\underline{\nu}, \tilde{\nu}\}$, $t\xi$ belongs to the budget set $B(p^\nu, w^\nu)$ and $x^\nu \prec t\xi$, which contradicts the fact that $x^\nu \in d(p^\nu, w^\nu)$.

e) The continuity of the demand function comes from the previous properties noticing that the demand is a locally bounded function. Indeed, for all $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$, there exists $r > 0$ such that the closed ball of center p and radius r is included in $\mathbb{R}^\ell_{++}$. So for all p' in this ball, for all $h = 1, \ldots, \ell$, p'_h is bounded below by a uniform positive constant. Consequently, for all w' in a bounded neighborhood of w, the budget set $B(p', w')$ is uniformly bounded.

f) The continuity of v is a consequence of (d), the continuity of u and the fact that the demand is uniformly bounded in a neighborhood of $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$. v is decreasing with respect to p since the budget set is smaller when the price increases, v is increasing in w since the budget set is larger when the wealth increases, v is homogeneous since the budget set is constant when the prices and the wealth are multiplied by a positive real number. Finally, the quasi-convexity of v is a consequence of the assertion (v) of Proposition 2.1.1. □

Examples. For the lexicographic preferences in $\mathbb{R}^2_+$, for any price-wealth pair $(p,w) \in \mathbb{R}^2_{++} \times \mathbb{R}_{++}$, the demand is $(\frac{w}{p_1}, 0)$.

For the Leontief preferences, with the utility function $\min_h\{x_h/a_h\}$, the demand for a price-wealth pair $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$ is $(\frac{a_h w}{\sum_{k=1}^\ell p_k a_k})_{h=1}^\ell$.

For the linear preferences, with the utility function $\sum_{h=1}^\ell a_h x_h$, the demand for a price-wealth pair $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$ is:

$$\mathrm{co}\{\frac{w}{p_h}\mathbf{1}^h \mid h \in \mathrm{argmax}\{\frac{a_k}{p_k} \mid k = 1, \ldots, \ell\}\}$$

To find the demand when the utility function is differentiable, we will characterize the demand using the first order condition for optimality. We first recall a property of the gradient of a quasi-concave function.

Proposition 2.3.2. *Let u be a quasi-concave continuously differentiable function on $\mathbb{R}^\ell_{++}$ such that for all $x \in \mathbb{R}^\ell_{++}$, $\nabla u(x) = \left(\frac{\partial u}{\partial x_h}(x)\right)_{h=1}^{\ell} \neq 0$. Then, for all $(x,x') \in (\mathbb{R}^\ell_{++})^2$, if $u(x) \leq u(x')$, then $\nabla u(x) \cdot (x'-x) \geq 0$. If $u(x) < u(x')$, then $\nabla u(x) \cdot (x'-x) > 0$.*

Proposition 2.3.3. *If the continuous, convex, montonic preference relation is representable by a continuous quasi-concave function u on $\mathbb{R}^\ell_+$, which is differentiable on $\mathbb{R}^\ell_{++}$ and satisfies $\nabla u(x) \neq 0$ for all $x \in \mathbb{R}^\ell_{++}$, then, for all $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$, $x \in \mathbb{R}^\ell_{++}$ belongs to $d(p,w)$ if and only if there exists $\lambda > 0$ such that $\nabla u(x) = \lambda p$ and $p \cdot x = w$.*

This proposition can be interpreted in the following term. A point x is in the demand at (p,w) if the budget constraint is binding at x and if the relative price $\frac{p_h}{p_k}$ is equal to the marginal rate of substitution $\frac{\frac{\partial u}{\partial x_h}(x)}{\frac{\partial u}{\partial x_k}(x)}$ for any pair of commodities.

The name "marginal rate of substitution" comes from the following remark. At a consumption $x \in \mathbb{R}^\ell_{++}$, if we want to substitute the commodity h to the commodity k and to keep the same utility level. Let us call $\varphi(t)$ the quantity of commodity k that one can withdraw in the consumption if one increases the quantity of commodity h of t keeping the utility level fixed. This means that $u(\ldots, x_h + t, \ldots, x_k - \varphi(t), \ldots) = u(x)$. The rate of substitution is $\varphi(t)/t$ and the marginal rate of substitution is the limit when t tends to 0, that is the derivative of φ in 0. When $\frac{\partial u}{\partial x_k}(x) \neq 0$, this derivative is equal to $\frac{\frac{\partial u}{\partial x_h}(x)}{\frac{\partial u}{\partial x_k}(x)}$.

One can have an intuition of the necessity of the equality between relative prices and marginal rate of substitution with the following argument. Let us consider the new consumption $(\ldots, x_h + t, \ldots, x_k - \frac{p_h}{p_k}t, \ldots)$ with t in a neighborhood of 0. Clearly, the new allocation is in budget set since the trade $(0, \ldots, t, 0, \ldots, 0, -\frac{p_h}{p_k}t, 0, \ldots, 0)$ has a value 0 at p. Consequently, if x belongs to the demand, $\psi(t) = u((\ldots, x_h+t, \ldots, x_k - \frac{p_h}{p_k}t, \ldots) \leq u(x) = \psi(0)$. Hence, the derivative of ψ at 0 is equal to 0. From the chain rule formula, $\psi'(0) = \frac{\partial u}{\partial x_h}(x) - \frac{p_h}{p_k}\frac{\partial u}{\partial x_k}(x)$. Hence, one gets the equality between the marginal rate of substitution and the relative price.

Proof. If $x \in \mathbb{R}^\ell_{++}$ belongs to $d(p,w)$, we already know that $p \cdot x = w$ by Proposition 2.3.1 (b). We now show that $\nabla u(x) = \lambda p$ for some $\lambda \geq 0$. A standard result of linear algebra shows that it holds true if and only if for all $z \in \mathbb{R}^\ell$ such that $p \cdot z \leq 0$ then $\nabla u(x) \cdot z \leq 0$. Let $z \in \mathbb{R}^\ell$ such that $p \cdot z \leq 0$. There exists $\bar{t} > 0$ such that for all $t \in [0, \bar{t}]$, $x + tz \in \mathbb{R}^\ell_{++}$.

Consequently, $x+tz \in B(p,w)$ and $u(x+tz) \leq u(x)$. Let $\varphi(t) = u(x+tz)$. φ attains its maximum on $[0,\bar{t}]$ at 0, thus $\varphi'(0) \leq 0$. By the chain rule formula, $\varphi'(0) = \nabla u(x) \cdot z$. Hence $\nabla u(x) \cdot z \leq 0$. Hence, there exists $\lambda \geq 0$, such that $\nabla u(x) = \lambda p$. Since $\nabla u(x) \neq 0$, $\lambda > 0$.

If $p \cdot x = w$ and $\nabla u(x) = \lambda p$ for some $\lambda > 0$, one applies the properties of the quasi-concave function to prove that $x' \notin B(p,w)$ if $u(x') > u(x)$. Indeed, if $x' \in \mathbb{R}^\ell_{++}$ satisfies $u(x') > u(x)$ then $\nabla u(x) \cdot (x'-x) > 0$. Consequently, since $\lambda > 0$, $p \cdot (x'-x) > 0$, which implies $p \cdot x' > p \cdot x = w$. Hence $x' \notin B(p,w)$. If $x' \in \mathbb{R}^\ell_{+}$ satisfies $u(x') > u(x)$, then, for all $t \in]0,1[$, close enough to 1, one has $x^t = tx' + (1-t)x \in \mathbb{R}^\ell_{++}$ and $u(x^t) > u(x)$. Hence, from the first part, $p \cdot x^t > p \cdot x$. This implies that $p \cdot t(x'-x) > 0$. Hence, since $t > 0$, one gets $p \cdot x' > p \cdot x = w$, which ends the proof. □

Examples. One can apply the previous proposition to show that the demand function for the Cobb-Douglas preference relation represented by the utility function $u(x) = \prod_{h=1}^{\ell} x_h^{\alpha_h}$, where $\alpha_h > 0$ for all h is defined by

$$d(p,w) = \left(\frac{\alpha_h w}{\left(\sum_{k=1}^{\ell} \alpha_k\right) p_h} \right)_{h=1}^{\ell}$$

If one considers the utility function $u(x_1,x_2) = x_1 + \sqrt{x_2}$, one remarks that the demand is not always in $\mathbb{R}^2_{++}$. The wealth must be large enough with respect to the price. So, the above formula does not always work to compute the demand. Precisely,

$$d(p,w) = \begin{cases} \left(\frac{4p_2 w - p_1^2}{4p_1 p_2}, \frac{p_1^2}{4p_2}\right) & \text{if } w \geq \frac{p_1^2}{4p_2} \\ \left(0, \frac{w}{p_2}\right) & \text{if } w < \frac{p_1^2}{4p_2} \end{cases}$$

2.3.1 Sensitivity of the demand

We now analyze how the demand changes when the price vector is fixed and the wealth varies. The curve $w \to d(p,w)$ is called the *Engel's curve.* Usually, one expects that the demand of a commodity will increase when the wealth increase. In that case, the good is called a normal good. But, when we consider different qualities of the same good or close substitute goods, the demand of low quality commodity can decrease when the wealth increase. In that case, the good is called an inferior commodity.

This case may appear with a very simple utility function when there is some survival constraint limiting the consumption from below. Let us consider the following consumption set: $X = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \geq 1\}$ The preferences relation of the consumer is represented by the utility function $u(x_1,x_2) = x_1 + 4x_2$. Let $p = (1,2)$. Then when $w \in [1,2]$, the demand of the consumer is $d(1,2,w) = (2-w, w-1)$. So the demand of the commodity 1 decreases with w.

When the wealth is kept fixed and the price of commodity h increases, one expects that the demand $d_h(p,w)$ of commodity h decreases since the relative price of the commodity increases and one expects that the consumption of the commodity h will be replaced by the consumption of a substitute which is relatively cheaper. But, like in the case of a wealth change, this is not always the case. An increase in the price of commodity h can lead to an increase in the demand of commodity h. In that case, the commodity is called a giffen goods. Indeed, an increase of the price of commodity h has an indirect wealth effect since the budget set becomes smaller and then, the consumer has less opportunity.

Let us consider the same example as above. Let $w = 1$ and $p_2 = 2$. For $p_1 \in]\frac{1}{2}, 1]$, $d_1(p_1, 2, w) = \frac{1}{2-p_1}$. Hence the demand increases with p_1.

When the price of commodity h increases, one can expect that the demand of the other commodities increase. But this is false when we have a strong complementarity among the commodity. Let us consider a Leontief preference relation represented by the utility function $\min\{x_1, x_2\}$. Let us fix $w = 1$ and $p_2 = 1$. Then, the demand $d(p_1, 1, 1)$ is equal to $(\frac{1}{1+p_1}, \frac{1}{1+p_1})$. One remarks that the demand for the second commodity decreases with respect to p_1.

We now consider a simultaneous change in price and wealth called compensated change. To simplify, we assume that the demand is single valued. Let (p,w) and $x = d(p,w)$. Let p' a different price vector. Then we change the wealth from w to w' in order to compensate the wealth effect due to the price change, which means that we let $w' = p' \cdot x$. Hence, x is financially affordable for the new price-wealth pair (p', w'). Let $x' = d(p', w')$. If $x' \neq x$, since $x \in B(p', w')$, one has $u(x') > u(x)$. Consequently, $p \cdot x' > p \cdot x$. From the Walras law, one has $p' \cdot x' = w' = p' \cdot x$. Consequently, $(p - p') \cdot (x - x') < 0$. One can summarize this by the following formula: for all $(p, p', w) \in (\mathbb{R}^\ell_{++})^2 \times \mathbb{R}_{++}$,

$$(p - p') \cdot (d(p,w) - d(p', p' \cdot d(p,w)) \leq 0$$

with a strict inequality if $d(p,w) \neq d(p', p' \cdot d(p,w))$. This is called the compensated law of demand. Indeed, if p and p' differs only for the commodity h, one obtains $(p_h - p'_h)(d_h(p,w) - d_h(p', p' \cdot d(p,w)) \leq 0$. Hence if the price of commodity h increases, the demand of commodity h decreases.

2.3.2 Axiom of revealed preferences

A crucial question is to know whether it is possible to deduce the preference of a consumer by the observation of her demand. The complete answer to this question goes beyond the scope of these notes but we just give some basic elements.

We consider a consumer with a preference relation $\preceq$. Let d the demand of the consumer. Let $x \in \mathbb{R}^\ell_+$. If we know that there exists $(p,w) \in \mathbb{R}^\ell_+ \setminus$

$\{0\} \times \mathbb{R}_{++}$ such that $x \in d(p,w)$, then for all $x' \in B(p,w)$, one has $x' \preceq x$ from the definition of the demand. In that case, x is revealed preferred to all consumption x' belonging to the budget set $B(p,w)$ by the knowledge of the demand.

If the demand is single valued, we remark that it is not possible x be revealed preferred to x' by the knowledge of the demand and x' be revealed preferred to x by the knowledge of the demand when $x \neq x'$. Indeed, when the demand is single valued and $x' \neq x$, then $x' \prec x$ if $x' \in B(p,w)$. So, if $x' = d(p',w')$, then $x \notin B(p',w')$ and x' is not revealed preferred to x.

The above remark is summarized by saying that the single valued demand deriving from a preference relation satisfies *the weak axiom of revealed preferences*, that is:

For all $(p,w,p',w') \in (\mathbb{R}^\ell_+ \setminus \{0\} \times \mathbb{R}_{++})^2$, if $p \cdot d(p',w') \leq w$ and $d(p,w) \neq d(p',w')$ then $p' \cdot d(p,w) > w'$.

This axiom is then a necessary condition to determine if an observed demand comes from a preference relation. But it is possible to exhibit an homogeneous demand function satisfying the Walras law and the weak axiom but which is not coming from a preference relation.

By a recursive argument, one easily shows that a single valued demand actually satisfies a stronger axiom, called *the strong axiom of revealed preferences*, which appears to be a sufficient condition in order to prove that the demand is coming from a preference relation. The statement of the strong axiom of revealed preferences is the following:

for all finite sequence $(p^\kappa, w^\kappa)_{\kappa=1}^k \in (\mathbb{R}^\ell_+ \setminus \{0\} \times \mathbb{R}_{++})^k$, if for all $\kappa = 1, \ldots, k-1$, $p^\kappa \cdot d(p^{\kappa+1}, w^{\kappa+1}) \leq w^\kappa$ and $d(p^1,w^1) \neq d(p^k,w^k)$, then $p^k \cdot d(p^1,w^1) > w^k$.

2.3.3 Recovering the utility function from the indirect utility function

The next proposition shows that the utility function is the value of a minimization problem where the objective function is the indirect utility function. One then has a dual situation since the indirect utility function is the value function of a maximization problem where the objective function is the utility function.

Proposition 2.3.4. *Let u be a continuous quasi-concave and monotonic utility function on $\mathbb{R}^\ell_+$. Let v be the indirect utility function associated to u. For all $x \in \mathbb{R}^\ell_{++}$, let us consider the following minimization problem:*

$$\begin{cases} \text{Minimize } v(p, p\cdot x) \\ p \in \mathbb{R}^\ell_{++} \end{cases}$$

The value of this problem is $u(x)$ and p is a solution if and only if $d(p, p\cdot x) = x$.

Proof. Let α^0 the value of the above problem, that is $\alpha^0 = \inf\{v(p, p\cdot x) \mid x \in \mathbb{R}^\ell_{++}\}$. For all $p \in \mathbb{R}^\ell_{++}$, $x \in B(p, p\cdot x)$. Thus, $u(x) \leq v(p,w)$ and $u(x) \leq \alpha^0$.

If $u(x) < \alpha^0$, then $x \notin C^0 = \{\xi \in \mathbb{R}^\ell_+ \mid u(\xi) \geq \alpha^0\}$. C^0 is a convex closed subset of $\mathbb{R}^\ell_+$ since u is continuous and quasi-concave. C^0 satisfies $C^0 + \mathbb{R}^\ell_+ = C^0$ since u is monotonic. C^0 is nonempty since $d(p, p\cdot x) \in C^0$ for all $p \in \mathbb{R}^\ell_{++}$. Finally $C^0 \subset \mathbb{R}^\ell_+$. Hence, from a separation theorem for convex subsets, there exists $\bar{p} \in \mathbb{R}^\ell_{++}$, such that $\bar{p}\cdot x < \min \bar{p}\cdot C^0$. Consequently $B(\bar{p}, \bar{p}\cdot x) \cap C^0 = \emptyset$, hence $d(\bar{p}, \bar{p}\cdot x) \notin C^0$, which implies $v(\bar{p}, \bar{p}\cdot x) < \alpha^0$. This contradicts the fact that $\alpha^0 = \inf\{v(p, p\cdot x) \mid x \in \mathbb{R}^\ell_{++}\}$. We can conclude that $u(x) = \alpha^0$.

Let $p \in \mathbb{R}^\ell_{++}$ be a solution of the problem. Then, $u(x) = v(p, p\cdot x)$. Hence, for all $x' \in B(p, p\cdot x)$, $u(x') \leq u(x)$.This implies that $x = d(p, p\cdot x)$. Conversely, if $x = d(p, p\cdot x)$ then $u(x) = v(p, p\cdot x)$. Hence $v(p, p\cdot x)$ is equal to the value of the minimization problem, which implies that p is a solution. □

From the above proposition, one can easily deduce formulas, which give the demand from the partial derivatives of the inverse demand function, at least when these partial derivatives exist.

Proposition 2.3.5. *We consider a continuous convex monotonic preference relation represented by a utility function u. Let us assume that the indirect utility function is continuously differentiable on $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$. Then, for all $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$,*

$$d(p,w) = -\left(\frac{\frac{\partial v}{\partial p_h}(p,w)}{\frac{\partial v}{\partial w}(p,w)}\right)_{h=1}^{\ell}$$

Proof. Let $(\bar{p}, \bar{w}) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$ and $\bar{x} = d(\bar{p}, \bar{w})$. We define the function ϕ from $\mathbb{R}^\ell_{++}$ to $\mathbb{R}$ by $\phi(p) = v(p, p\cdot\bar{x})$. We remark that $\bar{w} = \bar{p}\cdot\bar{x}$ from the Walras law. From the previous proposition, one deduces that $\phi(\bar{p}) \leq \phi(p)$ for all $p \in \mathbb{R}^\ell_{++}$. Consequently, the gradient of ϕ at $\bar{p}$ is equal to 0. From the chain rule formula, one has:

$$\frac{\partial \phi}{\partial p_h}(\bar{p}) = \frac{\partial v}{\partial p_h}(\bar{p}, \bar{p}\cdot\bar{x}) + \bar{x}_h \frac{\partial v}{\partial w}(\bar{p}, \bar{p}\cdot\bar{x})$$

Since $\bar{p}\cdot\bar{x} = \bar{w}$, one deduces the desired formula. □

2.3.4 Expenditure function and compensated demand

We now introduce the expenditure function and the associated compensated demand. We consider a preference relation represented by a continuous utility function u. Let p be a price vector and α^0 a given attainable utility level. Taken the price, the expenditure function gives the minimal wealth above which the consumer is able to buy a consumption having the utility level α^0.

The compensated demand is the set of consumptions, which have a utility level equal to α^0 and a value equals to the expenditure function at the price p. This means that the consumptions in the compensated demand are the cheapest consumptions for the price p having a utility level above α^0.

In the following, we denote by I the interval of attainable utility level, that is $I = \{\alpha \in \mathbb{R} \mid \exists x \in \mathbb{R}^\ell_+, u(x) \geq \alpha\}$.

Definition 2.3.2. *For a given price $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $\alpha^0 \in I$, we consider the following minimization problem:*

$$\begin{cases} \text{minimize } p \cdot x \\ u(x) \geq \alpha^0 \\ x \geq 0 \end{cases}$$

The set of solutions is called the compensated demand and it is denoted $\Delta(p, \alpha^0)$ and the value of this problem is the expenditure function and it is denoted $c(p, \alpha^0)$.

The next proposition gives the basic properties of Δ and c.

Proposition 2.3.6. *Let u be a continuous, quasi-concave and monotonic utility function.*

a) The expenditure function c is finite and non negative for all $(p, \alpha^0) \in \mathbb{R}^\ell_+ \setminus \{0\} \times I$. It is continuous on $\mathbb{R}^\ell_{++} \times I$.
b) For α^0 fixed, $c(\cdot, \alpha^0)$ is increasing, homogeneous of degree 1 and concave on $\mathbb{R}^\ell_{++}$.
c) For $p \in \mathbb{R}^\ell_+ \setminus \{0\}$ fixed, $c(p, \cdot)$ is increasing.
d) For all $(p, \alpha^0) \in \mathbb{R}^\ell_{++} \times I$, $\Delta(p, \alpha^0)$ is nonempty, convex and closed and for all $x \in \Delta(p, \alpha^0) \setminus \{0\}$, $u(x) = \alpha^0$. If u is strictly quasi-concave, then $\Delta(p, \alpha^0)$ is a singleton and continuous.

Proof. a) The finiteness and the non-negativity of c is a direct consequence of the definition. The continuity of c comes from the fact that c is locally bounded and u is continuous and monotonic.

b) The proof is easy by a direct application of the definition of c noticing that the higher are the prices the higher is the value for a given consumption.

c) The proof is easy by a direct application of the definition of c noticing that the set $\{x \in \mathbb{R}^\ell_+ \mid u(x) \geq \alpha\}$ is decreasing with respect to α.

d) The nonemptyness, the convexity and closedness of $\Delta(p, \alpha^0)$ are consequences of the continuity and the quasi-convexity of u. If $x \in \Delta(p, \alpha^0) \setminus \{0\}$ and $u(x) > \alpha^0$, then for $t \in]0, 1[$ close enough to 1, the continuity of u implies that $u(x) \geq \alpha^0$ and $p \cdot tx < p \cdot x$ since $p \in \mathbb{R}^\ell_{++}$. One gets a contradiction with the definition of $\Delta(p, \alpha^0)$. Hence, $u(x) = \alpha^0$. If u is strictly quasi-concave and x and x' belong to $\Delta(p, \alpha^0)$ with $x \neq x'$, one has $p \cdot x = p \cdot x' = c(p, \alpha^0)$. Furthermore, let $\hat{x} = \frac{1}{2}(x + x')$. Then $u(\hat{x}) > \alpha^0$, $p \cdot \hat{x} = c(p, \alpha^0)$ and $\hat{x} \neq 0$ since x and x' belong to $\mathbb{R}^\ell_+$ and are different. Consequently, one gets a contradiction with the previous assertion. The continuity of Δ is a consequence of the continuity and the monotonicity of u and the fact that Δ is locally bounded. □

Remark. The expenditure function is a mean to recover a particular utility function, which is a wealth measurement of the utility level. Let $\bar{p} \in \mathbb{R}^{\ell}_{++}$. Then, if the preference relation $\preceq$ is continuous, convex and monotonic, the mapping

$$\tilde{u}(x) = \inf\{\bar{p} \cdot x' \mid x \preceq x'\}$$

is a utility function representing $\preceq$. For all utility function u representing the preference relation $\preceq$, one has $\tilde{u}(x) = c(\bar{p}, u(x))$. Clearly, different reference prices leads to different utility functions.

Remark. The term compensated demand has a different meaning than the compensated change of a price wealth pair presented in the previous subsection. Indeed, here the utility level is kept fixed instead of the wealth when the price changes. One remark that the compensated law of demand holds also with the compensated demand defined above. Indeed, let us assume to simplify the exposition that the compensated demand $\Delta(p, u)$ is single valued. Let p and p' two price vectors such that $\Delta(p, u) \neq \Delta(p', u)$. Then, from the definition of Δ, one deduces that $p \cdot \Delta(p', u) > p \cdot \Delta(p, u)$ and $p' \cdot \Delta(p, u) > p' \cdot \Delta(p', u)$. Consequently, $(p - p') \cdot (\Delta(p, u) - \Delta(p', u)) < 0$.

As for the computation of the demand from the partial derivatives of the indirect utility function, we can also compute the compensated demand from the derivative of the expenditure function.

Proposition 2.3.7. *If for some utility level α^0, the expenditure function $c(\cdot, \alpha^0)$ is differentiable on $\mathbb{R}^{\ell}_{++}$, then for all $p \in \mathbb{R}^{\ell}_{++}$, $\Delta(p, \alpha^0) = \nabla_p c(p, \alpha^0)$.*

In this proposition, we implicitly show that the compensated demand is single valued.

Proof. Let $\bar{p} \in \mathbb{R}^{\ell}_{++}$ and $\bar{x} \in \Delta(\bar{p}, \alpha^0)$. Then, for all $p \in \mathbb{R}^{\ell}_{++}$, one has $\phi(p) = p \cdot \bar{x} - c(p, \alpha^0) \geq 0$ and $\phi(\bar{p}) = 0$. Hence, $\nabla\phi(\bar{p}) = 0$. Since $\nabla\phi(p) = \bar{x} - \nabla_p c(p, \alpha^0)$, one deduces the result. □

We ends this part by giving some relations between demand, indirect utility function, compensated demand and expenditure function.

Proposition 2.3.8. *Let u be a continuous, quasi-concave and monotonic utility function. Then, for all $(p, w, \alpha^0) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++} \times I$,*

a) $d(p, c(p, \alpha^0)) = \Delta(p, \alpha^0)$;
b)$\Delta(p, v(p, w)) = d(p, w)$;
c)$c(p, v(p, w)) = w$;
d) $v(p, c(p, \alpha^0)) = \alpha^0$.

The proof of this proposition is left as an exercise.

2.3.5 Some properties of a differentiable demand

We will now deduce some properties of the partial derivatives the demand function when it is assumed to be continuously differentiable. We remark that the indirect utility function is then differentiable when the utility function is so.

Proposition 2.3.9. *If the demand function is continuously differentiable on* $\mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++}$, *then for all* $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$,

a) for all $k = 1, \ldots, \ell$, $\sum_{h=1}^\ell p_h \dfrac{\partial d_k}{\partial p_h}(p,w) + w\dfrac{\partial d_k}{\partial w}(p,w) = 0$.

b) $\sum_{h=1}^\ell p_h \dfrac{\partial d_h}{\partial w}(p,w) = 1$.

c) For all $h = 1, \ldots, \ell$, $\sum_{h=1}^\ell p_k \dfrac{\partial d_k}{\partial p_h}(p,w) = -d_h(p,w)$.

Proof. Obvious by applying first the homogeneity of degree 0 of the demand and then by computing the partial derivative with respect to w and p_h of the Walras identity $p \cdot d(p,w) = w$. □

2.4 Exercises

Exercise 2.4.1. Let $(p,\bar{p}) \in (\mathbb{R}^\ell_{++})^2$ and $(w,\bar{w}) \in (\mathbb{R}_{++})^2$. Show that the budget set $B(p,w)$ is included in $B(\bar{p},\bar{w})$ if and only if $\frac{w}{p_h} \leq \frac{\bar{w}}{\bar{p}_h}$ for all $h = 1, \ldots, \ell$.

Deduce from the previous question that $B(p,w) = B(\bar{p},\bar{w})$ if and only if (p,w) is positively proportional to $(\bar{p},\bar{w})$.

Exercise 2.4.2. Let $(p,w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$. Show that $\tilde{x} = \frac{w}{\sum_{k=1}^\ell p_k}(1,\ldots,1)$ satisfies $p \cdot \tilde{x} = w$. Let $x' \in \mathbb{R}^\ell_+$ such that $p \cdot x < w$. Show that there exists $\bar{x} \in \mathbb{R}^\ell_{++}$ such that $x \ll \bar{x}$ and $p \cdot \bar{p} = w$.

Let $x \in B(p,w)$. Show that for all $t \in [0,1[$, $p \cdot (tx) < w$.

Exercise 2.4.3. A student has a fixed expense of 5 euro for lunch every day. She buys only french fries and coffee. She is supposed to maximize a monotonic preference relation under her budget constraint. The usual prices are 1.3 euro for the french fries and 1.1 euro for the coffee. This student then buys 3 portions of french fries and one coffee. There is a special offer for the french fries and the new price is 1.0 euro but the price of coffee increases to 1.2 euro. Shows that the student will at least buy $\frac{0.5}{0.46}$ portions of french fries.

Exercise 2.4.4. Compute the demand of a consumer having lexicographic preferences.

Exercise 2.4.5. For the following utility functions, check wether they are continuous, monotonic, strictly monotonic, quasi-concave. Compute the associated demand for $p \gg 0$, $w > 0$ and the indirect utility function. Give also the expenditure function and the compensated demand.

$u(x) = \min\{\alpha_1 x_1, \alpha_2 x_2\}$, $\alpha_h > 0$, $h = 1, 2$;
$u(x) = c_1 x_1 + c_2 x_2$, $c_h > 0$, $h = 1, 2$;
$u(x) = x_1^{\alpha_i} x_2^{\alpha_2}$, $\alpha_h > 0$, $h = 1, 2$;
$u(x) = (x_1^\alpha + x_2^\alpha)^{1/\alpha}$; $\alpha \in]0, 1[$;
$u(x) = x_1(1 + \sqrt{x_2})$;
$u(x) = x_1 + \sqrt{x_2}$;
$u(x) = -e^{-\alpha_1 x_1} - e^{-\alpha_2 x_2}$, $\alpha_h > 0$, $h = 1, 2$;

Exercise 2.4.6. In a two good economy, a consumer has a utility function u defined by : $u(a, b) = a + 2b - b^2$.

1) Draw the indifference curve associated to the utility level $0,5$, 1 and 2.
2) Show that the utility function is concave. Show that it is not monotonic.
3) Let $p_A > 0$ and $p_B > 0$ the prices of the two commodities A and B. Let $w > 0$ the income of the consumer. Show that (a, b) is the demand of the consumer if and only if

$$a = \frac{r - p_B b}{p_A} \text{ and } b \text{ is a solution of } \left\{ \begin{array}{l} \text{maximize } \frac{r}{p_A} + (2 - \frac{p_B}{p_A})b - b^2 \\ b \in [0, \frac{r}{p_B}] \end{array} \right.$$

4) Give the demand and the indirect utility function of this consumer. Draw the Engel's curve for $(p_A = 1, p_B = 1)$.

Exercise 2.4.7. We consider a consumer with an homogeneous utility function u, i.e., $\exists \alpha > 0,\ \forall x \in \mathbb{R}^\ell_+,\ \forall t > 0, u(tx) = t^\alpha u(x)$. Show that the demand function is such that for all $p \in \mathbb{R}^\ell_{++}$, for all income $w > 0$ and for all $t > 0$, the demand of the consumer satisfies $d(p, tw) = td(p, w)$. Show that the indirect utility function is homogeneous with respect to the wealth for a fixed price vector.

Exercise 2.4.8. We consider a consumer whose preferences are represented by a continuous, strictly quasi-concave, strictly increasing utility function from $\mathbb{R}^2_+$ to $\mathbb{R}$. We denote the prices by $p_a > 0$, $p_b > 0$ and the wealth by $w > 0$. The demand of the consumer is denoted by $x(p_a, p_b, w)$.
1) Recall the definition of the demand.
2) Recall the property of the demand called Walras law.
3) We assume that the demand in good a is $x_a(p_a, p_b, w) = \frac{p_b w}{p_a(p_a + p_b)}$. Compute the demand in good b by using the Walras law.
4) We denote by $\Delta(p_a, p_b, \bar{u})$ the indirect (or Hicksian) demand of the consumer for the prices p_a, p_b and the utility level $\bar{u}$. We assume that the indirect demand in good a is equal to $\Delta_a(p_a, p_b, \bar{u}) = \left(\frac{p_b \bar{u}}{p_a + p_b}\right)^2$.

a) What is equal to $\Delta(p_a, p_b, v(p_a, p_b, w))$, where $v(p_a, p_b, w)$ is the indirect utility function of the consumer?
b) Using the result of question 4.a, compute the indirect utility function of the consumer.

Exercise 2.4.9. In a two-commodity economy, we consider a consumer whose preferences are represented by a continuous strictly quasi-concave, monotonic utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$. The price of the second commodity p_2 is normalized to 1. The wealth of the consumer is fixed and denoted w. The demand of this consumer for the first commodity is given by the following formula:

$$d_1(p_1, 1, w) = \begin{cases} 2 - p_1 & \text{if } p_1 \leq 2 \\ 0 & \text{if } p_1 > 2 \end{cases}$$

1) Compute the value of this demand for the first commodity at price p_1, find the maximum of this value for $p_1 \geq 0$, and show that the formula implies that the wealth w is larger or equal to 1.
2) Compute the demand of this consumer for the second commodity $d_2(p_1, 1, w)$.
3) We now assume that $w = 3$. Compute the demand for $p_1 = 1$. Show that the consumption $(1, 2)$ is strictly preferred to the consumption $(2, 1)$.

Exercise 2.4.10. In a two-good economy, we consider a consumer who has a preference relation on $\mathbb{R}^2_{++}$ characterized by the following properties:

- the preferences are strictly monotonic;
- for all $t > 0$, the consumptions $(a, b) \in R^2_{++}$ which are equivalent to the consumption (t, t) are those which satisfy $b = \dfrac{ta}{2a - t}$.

1) Show that the unique utility function u which represents the preferences of the consumer and which satisfies $u(t, t) = t$ for all $t > 0$, is the function :

$$u(a, b) = \frac{2ab}{a + b}$$

2) Draw the indifference curve which contains the consumption $(1, 1)$. Show that the utility function is homogeneous of degree 1.
3) Find the demand with respect to the prices $p_A > 0$, $p_B > 0$ and the wealth $w > 0$. Compute the indirect utility function, the compensated demand and the expenditure function.
4) We assume that the wealth and one price are fixed. Show that the indirect utility function is bounded above whatever is the second price.

Exercise 2.4.11. In a two-good economy, the indirect utility function of a consumer is :

$$v(p_A, p_B, w) = -\frac{1}{2} \log(p_A) - \frac{1}{2} \log(p_B) + \log w.$$

Compute the demand and the utility function of this consumer.

Exercise 2.4.12. In a two-good economy, the indirect utility function of a consumer is :

$$v(p_A, p_B, w) = \frac{w}{(\sqrt{p_A} + \sqrt{p_B})^\alpha}$$

1) Show that $\alpha = 2$.
2) Compute the demand function $d(p_A, p_B, w)$.
3) Show that for all $(x, y) \in R^2_{++}$, there exists $(p_A, p_B) \in R^2_{++}$ such that $(x, y) = d(p_A, p_B, 1)$.
4) Deduce the utility function of the consumer from the previous question.
5) Compute the compensated demand and the expenditure function of this consumer.

Exercise 2.4.13. In a two-commodity economy, the indirect utility function of a consumer is:

$$v(p_1, p_2, w) = \frac{w}{p_1} + \left(\sqrt{\frac{p_2}{p_1}} - 1\right)^2$$

if $p_2 < p_1$ and $w > \sqrt{p_2}(\sqrt{p_1} - \sqrt{p_2})$.

1) Compute the demand function of the consumer for (p_1, p_2, w) satisfying $p_2 < p_1$ and $w > \sqrt{p_2}(\sqrt{p_1} - \sqrt{p_2})$.
2) Show that for all $(x_1, x_2) \in \mathbb{R}^2_{++}$ there exists (p_2, w) satisfying $p_2 < 1$ and $w > \sqrt{p_2}(1 - \sqrt{p_2})$ and such that

$$(x_1, x_2) = d(1, p_2, w)$$

3) Deduce the utility function of the consumer from the previous question.
4) Compute the demand of the consumer for all level of the wealth w when $p_1 = 1$ and $p_2 = \frac{1}{4}$.
5) Draw the Engel's curve for $p_1 = 1$ and $p_2 = \frac{1}{4}$.

Exercise 2.4.14. In a two-good economy, the expenditure function of a consumer is :

$$c(p_A, p_B, u) = (2p_A + p_B)u$$

Compute the compensated demand, the indirect utility function and the demand of this consumer.

Exercise 2.4.15. In a two-good economy, the expenditure function of a consumer is :

$$c(p_a, p_b, u) = p_a^{1/4} p_b^\beta u.$$

What is the value of β ?

Exercise 2.4.16. In a two-good economy, the demand of a consumer is :

$$d(p,w) = (p_B/p_A, \frac{w}{p_B} - 1)$$

when $w > p_B$.
1) Show that for all $x > 0$ and $y > 0$, there exists $(p_A, p_B) \in R^2_{++}$ such that $p_B < 1$ and $(x, y) = d(p_A, p_B, 1)$.
2) Let u be a utility function representing the preferences of this consumer. Show that if φ is a function from $\mathbb{R}_{++}$ to itself satisfying $u(\varphi(y), y)$ is constant, then $\varphi'(y) = -\varphi(y)$.
3) Deduce from the previous question the unique utility function representing the preferences of this consumer and satisfying $u(x, 1) = x$ for all $x \geq 0$.
4) Give the demand of this consumer when $w \leq p_B$.

Exercise 2.4.17. We consider a consumer with a preference relation on $\mathbb{R}^\ell_+$ and we assume that the demand is unique for every $(p, w) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$. Let $(p, p') \in (\mathbb{R}^\ell_{++})^2$. Show that

$$(p' - p) \cdot (d(p', p' \cdot d(p, w)) - d(p, w)) \leq 0$$

Show also that the inequality is strict when $d(p', p' \cdot d(p, w)) \neq d(p, w)$.

Exercise 2.4.18. In a two-commodity economy, we consider a consumer, whose preferences can be represented by a continuous quasi-concave utility function from $\mathbb{R}^2_+$ to $\mathbb{R}$, which is differentiable on $\mathbb{R}^2_{++}$ with positive partial derivatives.

We assume that for all price $p = (p_1, p_2) \in \mathbb{R}^2_{++}$ and for all wealth $w > 0$ the demand of the agent is

$$d(p_1, p_2, w) = (\frac{w}{2p_1}, \frac{w}{2p_2})$$

1) Show that for all $x \in \mathbb{R}^2_{++}$, there exists $p = (p_1, p_2) \in \mathbb{R}^2_+$ such that $x = d(p_1, p_2, 1)$.

2) Deduce from the previous question that for all $x \in \mathbb{R}^2_{++}$, the marginal rate of substitution of the utility function for the commodities 1 and 2 is equal to $\frac{x_2}{x_1}$.

3) Let k be a positive number. We consider the function φ_k from $]0, +\infty[$ to $\mathbb{R}$ defined by $\varphi_k(x) = u(x, \frac{k}{x})$. Deduce from the previous question that φ_k is constant on $]0, +\infty[$.

4) Let x and x' be two consumptions in $\mathbb{R}^2_{++}$ such that $x_1 x_2 = x'_1 x'_2$. Deduce from the previous question that $u(x) = u(x')$.

5) Let x and x' be two consumptions in $\mathbb{R}^2_{++}$ such that $x_1 x_2 > x'_1 x'_2$. Show that there exists $t \in]0, 1[$ such that $u(tx) = u(x')$. Show that $u(x) > u(x')$.

6) Let x and x' be two consumptions in $\mathbb{R}^2_{++}$. Show that x is preferred or indifferent to x' if $x'_1 x'_2 \leq x_1 x_2$.

7) Let x and x' be two consumptions in $\mathbb{R}^2_{++}$. Show that if x is preferred or indifferent to x' then $x'_1 x'_2 \leq x_1 x_2$.

8) Conclude from the previous questions that the preferences on $\mathbb{R}^2_{++}$ of the consumer can be represented by the utility function $\bar{u}(x) = x_1 x_2$.

9) Conclude from the previous questions that the preferences on $\mathbb{R}^2_{+}$ of the consumer can be represented by the utility function $\bar{u}(x) = x_1 x_2$.

3. Producers

3.1 Production set

A producer is an economic agent, who can implement some technological process in order to transform some commodities, called inputs, into other commodities, called output. The technological processes face physical constraints and are also dependent on the available knowledge. The production possibilities will be represented by a production set which is merely a nonempty subset of $\mathbb{R}^\ell$. An element y of Y is called a production. A production is feasible if and only if it belongs to Y.

The commodities h such that $y_h > 0$ are the outputs and the commodities k such that $y_k < 0$ are the inputs. So the producer can produce the quantities y_h of outputs by consuming the quantities $-y_k$ of inputs in the production process.

To illustrate in a more intuitive way the exposition below, we will consider the case where the producer has a unique output, the commodity ℓ, and $\ell - 1$ inputs, the commodities $1, \ldots, \ell$. In this case, one can describe the production possibilities by using a production function: this function associates the maximal quantity ℓ that one can produce with a bundle of inputs $(y_1, \ldots, y_{\ell-1})$.

To simplify the notations, if $y \in \mathbb{R}^\ell$, then $y_{-\ell}$ is the vector of $\mathbb{R}^{\ell-1}$ defined by $y_{-\ell} = (y_1, \ldots, y_{\ell-1})$. The production function is a function f from $\mathbb{R}_-^{\ell-1}$ to $\mathbb{R}_+$. The production set is then defined by:

$$Y = \{(y_1, \ldots, y_\ell) \in \mathbb{R}^\ell \mid y_1 \leq 0, \ldots, y_{\ell-1} \leq 0, y_\ell \leq f(y_1, \ldots, y_{\ell-1})\}$$

For a given level of production $y_\ell \geq 0$, one define the iso-ouput set as follows :

$$Y(y_\ell) = \{(y_1, \ldots, y_{\ell-1}) \in \mathbb{R}_-^{\ell-1} \mid \ y_\ell \leq f(y_1, \ldots, y_{\ell-1})\}.$$

This is the set of input bundles, which allow to produce at least y_ℓ. We will denote by $O \subset \mathbb{R}_+$ the set of attainable production level, that is the set of y_ℓ such that $Y(y_\ell)$ is nonempty. Since f takes its values in $\mathbb{R}_+$, O is an interval of $\mathbb{R}_+$ containing 0.

Transformation function

In the general case, a production set can be represented by a transformation function t from $\mathbb{R}^\ell$ to $\mathbb{R}$. Then the production set Y is defined by:

$$\{y \in \mathbb{R}^\ell \mid t(y) \leq 0\}$$

Examples:
- $\ell = 2$ and $t(y) = e^{6y_1} + e^{y_2} - 2$
- $\ell = 2$ and $t(y) = \frac{-2+y_1+y_2+\sqrt{4+(y_1-y_2)^2}}{2}$
- $\ell = 3$ and $t(y) = y_1 + \beta y_2 - 2\sqrt{\mid y_3 \mid}$

Remark. In many cases, one is only interested in a local representation of the production set, that is in a neighborhood of a given production $\bar{y} \in Y$. Formally, a local transformation function t is a function from the open ball $B(\bar{y}, r)$ with $r > 0$ to $\mathbb{R}$ is such that

$$Y \cap B(\bar{y}, r) = \{y \in B(\bar{y}, r) \mid t(y) \leq 0\}$$

For example, with a continuous production function f around a production $\bar{y}$ such that $\bar{y}_{-\ell} \ll 0$, the production set is defined by the transformation function $t(y) = y_\ell - f(y_{-\ell})$ on an open ball $B(\bar{y}, r)$ for some $r > 0$.

3.1.1 Basic assumptions on the production set

Closedness : Y is a closed subset of $\mathbb{R}^\ell$.

This is equivalent to assume that if y is not feasible ($y \notin Y$), then a small variation of the quantities of inputs or outputs is not enough to obtain a feasible production. In mathematical words, there exists $r > 0$, such that if y' satisfies $|y_h - y'_h| < r$ for all h, then $y' \notin Y$. If the production set is defined by a *continuous* transformation function or a *continuous* production function, then it is closed.

Possibility of inaction : $0 \in Y$.

This assumption merely means that the producer can do nothing. It is satisfied if $t(0) \leq 0$. It is always satisfied when the production set is defined by a production function.

Impossibility of free production : $Y \cap \mathbb{R}^\ell_+ \subset \{0\}$.

This assumption means that it is not possible to produce a positive quantity of an output without consuming a positive quantity of an input in the production process. If the production set is defined by a transformation function, it is satisfied if $y \geq 0$ and $t(y) \leq 0$ imply $y = 0$. If the production set is defined by a production function, it is satisfied if $f(0) = 0$.

Irreversibility : $Y \cap -Y \subset \{0\}$.

This assumption means that it is not possible that a production y and its opposite $-y$ are both feasible, that is that one can obtain the inputs from the outputs if one can obtain the outputs from the inputs. Basically, it translates the thermo-dynamical principle of the growth of the entropy. If the production set is defined by a transformation function, it is satisfied if $t(0) = 0$ and t is a strictly quasi-convex function. If the production set is defined by a production function, it is satisfied if $f(0) = 0$.

Free disposability : $Y - \mathbb{R}^{\ell}_{+} \subset Y$.

This assumption means that one can get ride of extra quantities of inputs or output at no cost. From a mathematical point of view an equivalent formulation is: for all $y \in Y$, for all $y' \in \mathbb{R}^{\ell}$, if $y'_h \leq y_h$ for all h, then $y' \in Y$. So if it is possible to produce a quantity of output with some quantities of inputs, it is possible to produce a smaller quantity of output with the same quantities of inputs or it is possible to produce the same quantity of output with larger quantities of inputs. If the production set is defined by a transformation function, it is satisfied if the transformation function is weakly increasing: for all $(y, y') \in (\mathbb{R}^{\ell})^2$, $y \geq y'$ implies $t(y) \geq t(y')$. If the production set is defined by a production function, it is satisfied if the production function is weakly decreasing: for all $(y_{-\ell}, y'_{-\ell}) \in (\mathbb{R}^{\ell-1}_{-})^2$, $y_{-\ell} \geq y'_{-\ell}$ implies $f(y_{-\ell}) \leq f(y'_{-\ell})$.

Convexity : Y *is convex.*

This assumption means that for all pairs of possible productions $(y, y') \in (Y)^2$, for all $t \in [0, 1]$, the mixed production $ty + (1 - t)y'$ is feasible. If the production set is defined by a transformation function, it is satisfied if the transformation function is quasi-convex. If the production set is defined by a production function, it is satisfied if the production function is concave.

Examples of production sets: In a two commodity economy, A and B, we can consider the three following typical production functions:

$$\begin{array}{ll} f(a) = -\alpha a & \text{with } \alpha > 0; \\ f(a) = \alpha\sqrt{-a} & \text{with } \alpha > 0; \\ f(a) = \alpha a^2 - \beta a & \text{with } \alpha > 0 \text{ and } \beta > 0 \end{array}$$

In the three cases, the production set is closed, satisfies the impossibility of free production, possibility of inactivity, irreversibility and free-disposal. In the two first cases, the production set is also convex but not in the third case.

In a three commodity economy A, B, C, A and B being inputs and C and output. The Cobb-Douglas production function is defined by:

$$f(a, b) = |a|^{\alpha}|b|^{\beta}$$

where α and β are positive real numbers. The production set is then defined by:

$$Y = \{(a,b,c) \in \mathbb{R}^3 \mid a \leq 0, \quad b \leq 0, \quad c \leq |a|^\alpha |b|^\beta\}$$

Y is closed and satisfies the impossibility of free production, possibility of inactivity, irreversibility and free-disposal. If $\alpha + \beta \leq 1$, Y is convex.

The Leontief production function is given by:

$$f(a,b) = \min\{\alpha|a|, \ \beta|b|\}$$

where α and β are positive real numbers. The production set is then defined by:

$$Y = \{(a,b,c) \in \mathbb{R}^3 \mid \ a \leq 0, \ b \leq 0, \ c \leq \min\{\alpha|a|, \ \beta|b|\}\}$$

Y is closed and satisfies the impossibility of free production, possibility of inactivity, irreversibility and free-disposal and it is convex.

We now give some particular properties for the important class of closed convex production sets.

Proposition 3.1.1. *Let Y be a closed convex production set satisfyingt $0 \in Y$.*

(i) Y satisfies the free-disposal assumption if and only if $-\mathbb{R}^\ell_+ \subset Y$.
(ii) If Y satisfies the impossibility of free production, then there exists $p \in \mathbb{R}^\ell_+$, $p \neq 0$, such that for all $y \in Y$, $p \cdot y \leq 0$.
(iii) If there exists $p \in \mathbb{R}^\ell_{++}$ such that for all $y \in Y$, $p \cdot y \leq 0$ then Y satisfies the impossibility of free production.
(iv) If Y satisfies the impossibility of free production, then for all $e \in \mathbb{R}^\ell_+$, $A(e) = \{y \in Y \mid y + e \geq 0\}$ is compact.

Remark. The first assertion means that the free-disposal assumption for a closed convex production set is equivalent to the much weaker condition of free elimination, that is $-\mathbb{R}^\ell_+ \subset Y$.

Note that we can have the impossibility of free production even if Assertion (iii) is false. This is the case for the following example:

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 \leq \sqrt{-y_1}\}$$

Assertion (iv) means that the production is bounded above with finite quantities of input. This is consistent with the physical and economic intuitions.

Proof. (i) If Y satisfies the free-disposal assumption and the possibility of inaction, then, $-\mathbb{R}^\ell_+ = 0 - \mathbb{R}^\ell_+ \subset Y$.

Conversely, let us assume that $-\mathbb{R}^\ell_+ \subset Y$. Let $y \in Y$, $z \in \mathbb{R}^\ell_+$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^\ell$. Let $\varepsilon > 0$ and $t \geq \max\{1, \max\{y_h/\varepsilon \mid h = 1, \ldots, \ell\}\}$. Then $y - t(z + \varepsilon\mathbf{1}) \leq 0$ hence $y - t(z + \varepsilon\mathbf{1}) \in Y$. Since Y is convex and $y -$

$(z + \varepsilon \mathbf{1}) \in [y, y - t(z + \varepsilon \mathbf{1})]$, $y - (z + \varepsilon \mathbf{1}) \in Y$. Since Y is closed, one gets $y - z = \lim_{\varepsilon \to 0} y - (z + \varepsilon \mathbf{1}) \in Y$.

(ii) We apply a separation theorem for the disjoint convex sets Y and $\mathbb{R}^\ell_{++}$. Then there exists $p \in \mathbb{R}^\ell \setminus \{0\}$ such that for all $(y, z) \in Y \times \mathbb{R}^\ell_{++}$, $p \cdot y \leq p \cdot z$. Since $0 \in Y$ and $\mathbb{R}^\ell_+$ is the closure of $\mathbb{R}^\ell_{++}$, one deduces that $0 \leq p \cdot z$ for all $z \in \mathbb{R}^\ell_+$. Consequently, by considering the vectors $(\mathbf{1}^h)_{h=1}^\ell$, which are the vectors of the canonical basis of $\mathbb{R}^\ell$, one deduces that $p_h \geq 0$ for all h or equivalently $p \in \mathbb{R}^\ell_+$. Furthermore, for all $y \in Y$,for all $t > 0$, one has $p \cdot y \leq p \cdot (t\mathbf{1})$. Hence, taking the limit when t converges to 0^+, one gets $p \cdot y \leq 0$.

(iii) We remark that for all $y \in Y$, $\sum_{h=1}^\ell p_h y_h \leq 0$. Since $p_h > 0$ for all h, it is impossible that $y_h \geq 0$ pour tout h with one strict inequality, which means $y \notin \mathbb{R}^\ell_+ \setminus \{0\}$.

(iv) The set $A(e)$ is closed and convex since $A(e) = Y \cap \{-e\} + \mathbb{R}^\ell_+$ and Y is closed and convex. Let us assume that it is not bounded. Then, from a standard result in convex analysis, there exists a vector $z \in \mathbb{R}^\ell \setminus \{0\}$ such that for all $t \geq 0$, $tz \in A(e)$. Hence $tz \geq -e$ for all $t > 0$, which implies that for all h, $z_h \geq -\frac{e_h}{t}$ and at the limit when t converges to $+\infty$, $z_h \geq 0$. So $z \in (\mathbb{R}^\ell_+ \setminus \{0\}) \cap Y$ which contradicts the impossibility of free production. □

3.1.2 Returns to scale

This notion is a measurement of the change of production level to a proportional change of the quantities of inputs. Roughly speaking, the question is to know whether the production level is the double, more than the double or less than the double when one consumes two times more inputs in the production process.

Definition 3.1.1. *Let Y be a production set. The production exhibits increasing (resp. decreasing, resp. constant) returns to scale if $y \in Y$, for all $t \geq 1$ (resp. $t \in [0, 1], t \in \mathbb{R}_+$), $ty \in Y$.*

Proposition 3.1.2. *If the production set is defined by a production function f, then the returns to scale are increasing (resp. decreasing, resp. constant) if for all $t \geq 1$ (resp. $t \in [0, 1]$, $t \in \mathbb{R}_+$), $f(ty_{-\ell}) \geq tf(y_{-\ell})$.*

So, one easily checks that the production has constant return to scale when the production function is $f(a) = -\alpha a$, increasing returns if $f(a) = \alpha a^2 - \beta a$, constant returns to scale if $f(a) = \alpha\sqrt{-a}$.

Proposition 3.1.3. *If the production set is convex and satisfies the possibility of inaction, then the production has decreasing returns to scale. If the production set is convex with constant returns to scale, then the production set is a convex cone.*

3.1.3 Aggregate production set

A producer may have several units of production with similar or different technologies. It is then interesting to know which are the aggregate production possibilities of this producer. The question is the same if one merges to firms. We limit ourself to the case of two units of production but the results may be easily extended to several producers or several production sets.

With our notational convention with the quantities of inputs represented by negative numbers, the mathematical formula is very simple. If Y_1 and Y_2 are the production sets, then the aggregate production set is

$$Y_1 + Y_2 = Y = \{y_1 + y_2 \mid (y_1, y_2) \in Y_1 \times Y_2\}$$

We now investigate the properties of the aggregate production set. We first remark that Y may be not closed even if Y_1 and Y_2 are closed. This is the case with $Y_1 = \{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b \leq \frac{a^2-4a}{2-a}\}$ and $Y_2 = \{(a,b) \in \mathbb{R}^2 \mid b \leq 0, a \leq -b\}$. This example shows also that Y does not satisfies the propeerty of the impossibility of free production even if Y_1 and Y_2 satisfy this property.

We now give some positive properties.

Proposition 3.1.4. *Let Y_1 and Y_2 be two production sets and $Y = Y_1 + Y_2$ be the aggregate production set.*

(i) If Y_1 and Y_2 satisfy the property of the possibility of inactivity, (resp. convexity), then Y satisfies also this property.
(ii) If Y_1 and Y_2 satisfy the property of free disposal, then Y satisfies also this property.
(iii) If Y_1 and Y_2 are closed, convex and contain 0 and if $Y_1 \cap -Y_2 = \{0\}$, then Y is closed.

The third assertion shows that Y is closed if one has irreversibility of the production between y_1 and Y_2. Note that it holds true if $Y \cap -Y = \{0\}$.

Proof of Proposition 3.1.4 The two first assertions are direct consequences of the definition of the aggregate production set as the sum of the individual production sets.

(iii) Let $y \in \mathbb{R}^\ell$ and let $r > 0$. We first show that the set

$$A^r(y) = \{(y_1, y_2) \in Y_1 \times Y_2 \mid y_1 + y_2 \in \bar{B}(y, r)\}$$

is bounded and closed. If it is empty, the result is obvious. If it is nonempty, we remark that $A^r(y)$ is closed and convex since Y_1 and Y_2 are so. If it is not bounded, from a result of convex analysis, there exists $(z_1, z_2) \in (\mathbb{R}^\ell)^2 \setminus \{0\}$ such that for all $(y_1, y_2) \in A^r(y)$, for all $t \geq 0$, $(y_1 + tz_1, y_2 + tz_2) \in \mathcal{A}^r(y)$. Thus, there exists $\zeta^t \in \bar{B}(0, r)$ such that $y_1 + y_2 + t(z_1 + z_2) = y + \zeta^t$. Dividing by t and taking the limit when t tends to $+\infty$, one deduces that $z_1 + z_2 = 0$.

Since $(z_1, z_2) \neq (0,0)$, one deduces that $z_1 \neq 0$. For all $t > 0$, $y_1 + tz_1 \in Y_1$ and $y_2 + tz_2 = y_2 - tz1 \in Y_2$. $z_1 + \frac{1}{t}y_1 = \frac{1}{t}(y_1 + tz_1) + (1 - \frac{1}{t})0$ belongs to Y_1 for all $t \geq 1$ since Y_1 is closed and convex and contains 0. Taking the limit. when t tends to $+\infty$, one deduces that $z_1 \in Y_1$. With the same argument, one gets $-z_1 \in Y_2$ which contradicts the assumption $Y_1 \cap -Y_2 = \{0\}$.

Let $(y^\nu)_{\nu \in \mathbb{N}}$ a sequence of Y converging to $\bar{y}$. Since the sequence (y^ν) is bounded, there exists $r > 0$ such that for all ν, $y^\nu \in \bar{B}(\bar{y}, r)$. It also exists a sequence $(y_1^\nu, u_2^\nu)_{\nu \in \mathbb{N}}$ of $Y_1 \times Y_2$ such that $y^\nu = y_1^\nu + y_2^\nu$. Thus, the sequence $(y_1^\nu, u_2^\nu)_{\nu \in \mathbb{N}}$ remains in the bounded closed set $A^r(\bar{y})$. Consequently, it has a converging subsequence whose limit is $(\bar{y}_1, \bar{y}_2)$. $\bar{y}_1 \in Y_1$ and $\bar{y}_2 \in Y_2$ since Y_1 et Y_2 are closed. We then have $\bar{y} = \bar{y}_1 + \bar{y}_2 \in Y_1 + Y_2 = Y$. Hence Y is closed. □

We remark that Y can be convex even if Y_1 and Y_2 are not convex. For example, this is the case with $Y_1 = \{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b \leq 0 \text{ if } a \geq -1, b \leq -a \text{ if } a \leq -1\}$ and $Y_2 = \{(a,b) \in \mathbb{R}^2 \mid b \leq 0, a \leq -b\} \cup \{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b \leq 0 \text{ if } a \geq -1, b \leq 1 \text{ if } a \leq -1\}$. Nevertheless, we have the following propery.

Proposition 3.1.5. *Let Y_1 and Y_2 be two production sets and $Y = Y_1 + Y_2$ be the aggregate production set. If Y is closed and convex, then $Y = \bar{\text{co}}Y_1 + \bar{\text{co}}Y_2$, where $\bar{\text{co}}Y_j$ is the closed convex hull of Y_j that is the smallest closed convex set containing Y_j.*

Proof. Since $Y_1 \subset \bar{\text{co}}Y_1$ and $Y_2 \subset \bar{\text{co}}Y_2$, one has $Y \subset \bar{\text{co}}Y_1 + \bar{\text{co}}Y_2$. Let $(z_1, z_2) \in \text{co}Y_1 \times \text{co}Y_2$. There exist k elements $(y_1^1, y_1^2, \ldots, y_1^k)$ of Y_1 and $\lambda \in \mathbb{R}_+^k$ such that $\sum_{\kappa=1}^k \lambda_\kappa = 1$ and $z_1 = \sum_{\kappa=1}^k \lambda_\kappa y_1^\kappa$ and m elements $(y_2^1, y_2^2, \ldots, y_1^m)$ of Y_2 and $\mu \in \mathbb{R}_+^m$ such that $\sum_{\nu=1}^m \mu_\nu = 1$ and $z_2 = \sum_{\nu=1}^m \mu_\nu y_2^\nu$. One remarks that

$$\begin{array}{rl} z_1 + z_2 = & \sum_{\kappa=1}^k \lambda_\kappa y_1^\kappa + \sum_{\nu=1}^m \mu_\nu y_2^\nu \\ & \sum_{\kappa=1}^k \sum_{\nu=1}^m (\lambda_\kappa + \mu_\nu)(y_1^\kappa + y_2^\nu) \end{array}$$

and $\sum_{\kappa=1}^k \sum_{\nu=1}^m (\lambda_k + \mu_\nu) = 1$. Consequently $z_1 + z_2$ is convex combination of the elements $(y_1^\kappa + y_2^\nu)_{\kappa=1,\nu=1}^{\kappa=k,\nu=m}$ of Y. Since Y is convex, $z_1 + z_2$ belongs to Y. Finally, if $(z_1, z_2) \in \bar{\text{co}}Y_1 \times \bar{\text{co}}Y_2$, there exists a sequence $(z_1^n, z_2^n)_{n \in \mathbb{N}} \in \text{co}Y_1 \times \text{co}Y_2$, which converges to (z_1, z_2). Hence $z_1 + z_2$ is the limit of $(z_1^n + z_2^n)$, which is a sequence of Y, thus it belongs to Ysince Y is closed. Consequently, $\bar{\text{co}}Y_1 + \bar{\text{co}}Y_2 \subset Y$. □

Aggregate production function. Let us now assume that we have two producers with a unique output, the commodity ℓ. Let f and g be the productions functions of these two producers. We are looking for the production function F of the aggregate producer.

Let $y_{-\ell}$ a given basket of inputs. $F(y_{-\ell})$ is the maximal quantity of commodity ℓ that one can produce using the two technologies represented by f and g. It is then necessary to share the quantities of inputs between the two producers, $y'_{-\ell}$ for the first and $y''_{-\ell}$ for the second. The production of commodity ℓ is then $f(y'_{-\ell}) + g(y''_{-\ell})$. The repartition is feasible if $y_h \leq y'_h + y''_h$, $y'_h \leq 0$ and $y''_h \leq 0$for all $h = 1, \ldots, \ell - 1$. We then deduce that $F(y_{-\ell})$ is the value of the following maximization problem:

$$\mathcal{P}(y_{-\ell}) \left\{ \begin{array}{l} \text{maximize } f(y'_{-\ell}) + g(y''_{-\ell}) \\ y_h \leq y'_h + y''_h,\ y'_h \leq 0,\ y''_h \leq 0,\ h = 1, \ldots, \ell - 1 \end{array} \right.$$

Examples : We give two examples of the computation of the aggregate production function F. We consider a two-commodity economy. The commodity A is an input and the commodity B is an output. We have two producers with the production functions: $f(a) = \sqrt{-a}$ and $g(a) = 2\sqrt{-2a}$. Then $F(a)$ is the value of :

$$\left\{ \begin{array}{l} \text{maximize } \sqrt{-\alpha} + 2\sqrt{-2\alpha'} \\ \alpha + \alpha' \geq a \\ \alpha \leq 0,\ \alpha' \leq 0 \end{array} \right.$$

One remarks that the production function are strictly decreasing, so the optimal solution satisfies $\alpha + \alpha' = a$. Consequently, one can solve the simplest following problem:

$$\left\{ \begin{array}{l} \text{maximize } \sqrt{-\alpha} + 2\sqrt{-2(a-\alpha)} \\ \alpha \in [a, 0] \end{array} \right.$$

The function $\alpha \to \sqrt{-\alpha} + 2\sqrt{-2(a-\alpha)}$ is concave and its derivative is equal to $-\frac{1}{2}(-\alpha)^{-1/2} + 2[-2(a-\alpha)]^{-1/2}$. It vanishes for $\alpha = \frac{a}{9}$, which is the optimal solution. So $F(a) = 3\sqrt{-a}$.

Although the second producer produces a higher quantity of output with the same quantity of input, one remarks that the two producers have to produce in order to get the optimal level of production. One remarks that the marginal productivity, that is the derivative of the production function, are equal at the solution $\frac{a}{9}$ and $\frac{8a}{9}$.

Let us now consider the case where the production functions are $f(a) = -a$ and $g(a) = a^2$. Then, considering again the fact that the production functions are strictly decreasing, the problem is the following:

$$\left\{ \begin{array}{l} \text{maximiser } -\alpha + (a-\alpha)^2 \\ a \leq \alpha \leq 0 \end{array} \right.$$

The function $\alpha \to -\alpha + (a-\alpha)^2$ is convex. Thus, on the interval $[a, 0]$, the maximum is at $\alpha = 0$ if $a \leq -1$ and at $\alpha = a$ if $a \geq -1$. So, we obtain the following production function :

$$F(a) = \left\{ \begin{array}{ll} -a & \text{si } -1 \leq a \leq 0; \\ a^2 & \text{si } a \leq -1. \end{array} \right.$$

In this example, we remark that the whole quantity of input is either given to the first producer of to the second, but it is never shared among the two. This is due to the presence of increasing returns to scale in the production of the second producer.

We now characterize the optimal solution when it is an interior solution, that is when the quantities of inputs are shared in such a way that each producer has a non zero quantity of each input. For this, we assume that the production functions are differentiable on the interior of their domain $-\mathbb{R}_+^{\ell-1}$. The partial derivative of the production function is called the marginal productivity.

Proposition 3.1.6. *Let f and g be two strictly decreasing production functions from $\mathbb{R}^{\ell-1}$ to $\mathbb{R}_+$. We assume that f and g are continuously differentiable on the interior of $\mathbb{R}^{\ell-1}$. Let $y_{-\ell} \ll 0$ and let $(\bar{y}'_{-\ell}, \bar{y}''_{-\ell})$ an optimal solution of the maximization problem $\mathcal{P}(y_{-\ell})$ such that $\bar{y}'_{-\ell} \ll 0$ and $\bar{y}''_{-\ell} \ll 0$. Then*

$$\bar{y}'_{-\ell} + \bar{y}''_{-\ell} = y_{-\ell}$$

and for all $h = 1, \ldots, \ell - 1$,

$$\frac{\partial f}{\partial y_h}(\bar{y}'_{-\ell}) = \frac{\partial g}{\partial y_h}(\bar{y}''_{-\ell})$$

Conversely, if f and g are furthermore concave, if $(\bar{y}'_{-\ell}, \bar{y}''_{-\ell})$ are satisfying $\bar{y}'_{-\ell} \ll 0$ and $\bar{y}''_{-\ell} \ll 0$ and the two above conditions, then it is a solution of the problem $\mathcal{P}(y_{-\ell})$.

The condition on the marginal productivity are completely natural. It means that the two marginal productivity must be equal. Indeed, if, for example, the first is higher than the second, one can increase the global production by transferring a small amount of the input from the second producer to the first. Nevertheless, with a boundary solution, where the whole quantity of an input is given to one producer, the argument is no more valid.

Proof. Since the production functions are strictly decreasing, if for an input h, $y'_h + y''_h > y_h$, then one can strictly increase the production level by considering the distribution $(y'_h - (y'_h + y''_h - y_h), y''_h)$. So, $(y'_{-\ell}, y''_{-\ell})$ is not a solution of $\mathcal{P}(y_{-\ell})$. Consequently, if $(\bar{y}'_{-\ell}, \bar{y}''_{-\ell})$ is a solution of $\mathcal{P}(y_{-\ell})$ it satisfies $\bar{y}'_h + \bar{y}''_h = y_h$.

From above, one deduces that y' is a solution of the auxiliary problem:

$$\mathcal{P}'(y_{-\ell}) \left\{ \begin{array}{l} \text{maximize } f(y'_{-\ell}) + g(y_{-\ell} - y''_{-\ell}) \\ y_h \leq y'_h \leq 0,\ h = 1, \ldots, \ell - 1 \end{array} \right.$$

Since the $\bar{y}' \ll 0$ and $\bar{y}''_{-\ell} \ll 0$, one deduces that for all h, $\bar{y}'_h < 0$ and $y_h < \bar{y}'_h$. So the partial derivative of the objective function with respect to y_h must be equal to 0 at $\bar{y}'$. Hence one gets

$$\frac{\partial f}{\partial y_h}(\bar{y}'_{-\ell}) - \frac{\partial g}{\partial y_h}(y_{-\ell} - \bar{y}'_{-\ell}) = 0$$

which leads to the result since $y_{-\ell} - \bar{y}'_{-\ell} = \bar{y}''_{-\ell}$.

Conversely, if $f$and g are concave, we know that $f(z_{-\ell}) \leq f(\bar{y}'_{-\ell}) + \nabla f(\bar{y}'_{-\ell}) \cdot (z_{-\ell} - \bar{y}'_{-\ell})$ and $g(z_{-\ell}) \leq g(\bar{y}''_{-\ell}) + \nabla g(\bar{y}''_{-\ell}) \cdot (z_{-\ell} - \bar{y}''_{-\ell})$ for all $z_{-\ell} \in -\mathbb{R}_+^{\ell-1}$.

If $\bar{y}'_{-\ell} + \bar{y}''_{-\ell} = y_{-\ell}$ and for all $h = 1, \ldots, \ell - 1$, $\frac{\partial f}{\partial y_h}(\bar{y}'_{-\ell}) = \frac{\partial g}{\partial y_h}(\bar{y}''_{-\ell})$, for all $(y'_{-\ell}, y''_{-\ell}) \in (-\mathbb{R}_+^{\ell-1})^2$ satisfying $y_{-\ell} \leq y'_{-\ell} + y''_{-\ell}$, one obtains:

$$\begin{aligned} f(y'_{-\ell}) + g(y''_{-\ell}) \leq \quad & f(\bar{y}'_{-\ell}) + g(\bar{y}''_{-\ell}) + \nabla f(\bar{y}'_{-\ell}) \cdot (y'_{-\ell} + y''_{-\ell} - (\bar{y}'_{-\ell} + \bar{y}''_{-\ell})) \\ & f(\bar{y}'_{-\ell}) + g(\bar{y}''_{-\ell}) + \nabla f(\bar{y}'_{-\ell}) \cdot (y'_{-\ell} + y''_{-\ell} - y_{-\ell}) \end{aligned}$$

Since f is decreasing, one has $\nabla f(\bar{y}'_{-\ell}) \in -\mathbb{R}_+^{\ell-1}$. Consequently, since $y_{-\ell} \leq y'_{-\ell} + y''_{-\ell}$, one gets $\nabla f(\bar{y}'_{-\ell}) \cdot (y'_{-\ell} + y''_{-\ell} - y_{-\ell}) \leq 0$, which implies $f(y'_{-\ell}) + g(y''_{-\ell}) \leq f(\bar{y}'_{-\ell}) + g(\bar{y}''_{-\ell})$ So, $(\bar{y}'_{-\ell}, \bar{y}''_{-\ell})$ is a solution of $\mathcal{P}(y_{-\ell})$. □

3.1.4 Efficient productions

From a technical point of view, some productions are better than another since they are optimal in the sense that the inputs are optimally used to produce the output. Hence there is no dispose of unused commodities. It is a minimal criterion for an rational producer.

Definition 3.1.2. *Let Y be a production set. A production $y \in Y$ is efficient if it does not exist a production $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. In other words, $(\{y\} + \mathbb{R}_+^\ell) \cap Y = \{y\}$. A production y in Y is weakly efficient if it does not exist a production $y' \in Y$ tel que $y\prime \gg y$. In other words, $(\{y\} + \mathbb{R}_{++}^\ell) \cap Y = \emptyset$.*

We denote by $E_f(Y)$ the set of weakly efficient productions of Y.

An efficient production is obviously weakly efficient. A efficient production is optimal in the sense that it is not possible to produce more outputs with the same quantities of inputs or it is not possible to produce the same quantities of outputs with smaller quantities of inputs.

We now give some characterization of the efficient productions.

Proposition 3.1.7. *Let Y be a production set.*

(i) If y is weakly efficient, then $y \in \partial Y$, where ∂Y denotes the boundary of Y.
(ii) If Y is closed, the set $E_f(Y)$ is closed.
(iii) If Y is closed and satisfies the free-disposal assumption, then $\partial Y = E_f(Y)$.
(iv) If Y is closed and convex, then :

$$E_f(Y) = \{y \in Y \mid \exists p \in \mathbb{R}_+^\ell \setminus \{0\}, \quad p \cdot y \geq p \cdot y', \quad \forall y' \in Y\}.$$

(v) Let $y \in Y$. If it exists $p \gg 0$ such that $p \cdot y \geq p \cdot y'$ for all $y' \in Y$, then y is efficient.

Proof. (i) If $y \in E_f(Y)$, then for all $t > 0$, $y + t\mathbf{1} \notin Y$. So y belongs to the boundary of Y since y is the limit of the sequence $(y + \frac{1}{\nu+1}\mathbf{1})_{\nu \in \mathbb{N}})$.

(ii) Let (y^ν) a sequence of E_f converging to $\bar{y}$. $\bar{y} \in Y$ since Y is closed. Let us assume that $\bar{y}$ is not weakly efficient. Then, it exists $y \in Y$ such that $y \gg \bar{y}$. So, for all $h = 1, \ldots, \ell$, $\bar{y}_h < y_h$. For ν large enough, $|y^\nu_h - \bar{y}_h| < y_h - \bar{y}_h$ for all h. Hence $y^\nu \ll y$ which contradicts the fact that y^ν is weakly efficient. Thus $\bar{y} \in E_f(Y)$ and $E_f(Y)$ is closed.

(iii) From (i), $E_f(Y) \subset \partial Y$. Let $y \in \partial Y$. If $y \notin E_f(Y)$, it exists $y' \in Y$ such that $y' \gg y$. $\{y'\} - \mathbb{R}^\ell_{++}$ is open and included in Y for Y satisfies the free-disposal assumption. Hence $y \in \{y'\} - \mathbb{R}^\ell_{++} \subset \text{int } Y$. This contradicts the fact that $y \in \partial Y$. Hence $y \in E_f(Y)$.

The proofs of Assertion (iv) and (v) are identical of the ones of Assertion (ii) and (iii) of Proposition 3.1.1. Indeed, 0 is efficient if and only if the production set Y satisfies the impossibility of free-production. □

Figure 3.1 gives an example of a production set such that a production on the boundary is not always weakly efficient and the set of efficient production is neither closed nor open.

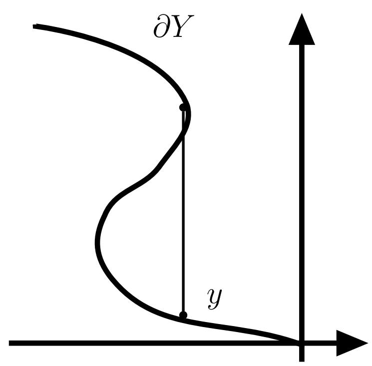

Figure 3.1. y is weakly efficient but not efficient.

When the production set is defined by a production function, we can characterize the efficient production but some weakly efficient production are not intuitively rational. Indeed, if one input is not used at the production, then the production is weakly efficient since no feasible production has a positive quantity of input. But, the other inputs may not be used efficiently in the sense that the level of output may be strictly smaller than the quantity given by the production function and this does not matter for the weak efficiency.

We now give two additional remarks when the inputs are all used in the production or when the production constraint is binding at the production.

Proposition 3.1.8. *Let Y be a production set defined by a continuous production function f. If y is a weakly efficient production plan such that $y_h < 0$ for all $h = 1, \dots, \ell - 1$, then $y_\ell = f(y_{-\ell})$. If f is decreasing and if $y \in Y$ satisfies $y_\ell = f(y_{-\ell})$, then y is weakly efficient.*

Proof. If $y_\ell < f(y_{-\ell})$, then the continuity of f implies that there exists $t < 1$ and $\varepsilon > 0$ such that $y_\ell + \varepsilon < f(ty_{-\ell})$. Thus the production $y' = (ty_{-\ell}, y_\ell + \varepsilon)$ belongs to Y and $y \ll y'$, which contradicts that y is weakly efficient.

Conversely, if $y_\ell = f(y_{-\ell})$, then the definition of Y by the production function implies that y belongs to the boundary of Y since y is the limit of $(y_{-\ell}, y_\ell + \frac{1}{\nu+1})_{\nu \in \mathbb{N}}$. Furthermore, as already remarked, Y satisfies the free-disposal Assumption since f is decreasing. Hence Proposition 3.1.7 (iii) implies that y is weakly efficient. □

We will now show how we can characterize the production pair (y_1, y_2) of $Y_1 \times Y_2$ such that $y_1 + y_2$ is weakly efficient in $Y = Y_1 + Y_2$. This is a fundamental question since it concerns the coordination of two units or two producers in order to get the global efficiency.

Proposition 3.1.9. *Let Y_1 and Y_2 be two production sets of $\mathbb{R}^\ell$ Let $(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2$. We assume that Y_1 and Y_2 are locally representable by transformation functions g_1 and g_2 in a neighborhood of $\bar{y}_1$ and $\bar{y}_2$ and that g_1 and g_2 be continuously differentiable and satisfies $\nabla g_1(\bar{y}_1) \neq 0$ and $\nabla g_2(\bar{y}_2) \neq 0$. Then, if $\bar{y}_1 + \bar{y}_2$ is weakly efficient in $Y = Y_1 + Y_2$ then $g_1(\bar{y}_1) = g_2(\bar{y}_2) = 0$ and it exists $\lambda > 0$ such that $\nabla g_1(\bar{y}_1) = \lambda \nabla g_2(\bar{y}_2)$.*

Conversely, if we furthermore assume that g_1 and g_2 are convex, if $(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2$ satisfies $g_1(\bar{y}_1) = g_2(\bar{y}_2) = 0$, $\nabla g_1(\bar{y}_1) \in \mathbb{R}^\ell \setminus \{0\}$ and $\nabla g_2(\bar{y}_2) \in \mathbb{R}^\ell \setminus \{0\}$, and it exists $\lambda > 0$ such that $\nabla g_1(\bar{y}_1) = \lambda \nabla g_2(\bar{y}_2)$, then $\bar{y}_1 + \bar{y}_2$ is weakly efficient in $Y = Y_1 + Y_2$.

This proposition means that the global production $\bar{y}_1 + \bar{y}_2$ is efficient, if the marginal productivities and the marginal rate of substitution between inputs are equal for the two units of production represented by Y_1 and Y_2.

In the case where the production sets are represented by production functions, we can derive the following result.

Proposition 3.1.10. *Let Y_1 and Y_2 be two convex closed production set in $\mathbb{R}^\ell$. We assume that they satisfy the free disposal assumption and that they produce the commodity ℓ using the other goods $1, \dots, \ell - 1$ as inputs. We denotes by f^1 and f^2 from $-\mathbb{R}^{\ell-1}_+$ to $\mathbb{R}_+$ the two production functions, which are assumed to be concave, différentiable on $-\mathbb{R}^{\ell-1}_{++}$. Let $(\bar{y}^1, \bar{y}^2) \in Y_1 \times Y_2$ such that $\bar{y}^1_{-\ell} \in -\mathbb{R}^{\ell-1}_{++}$ and $\bar{y}^2_{-\ell} \in -\mathbb{R}^{\ell-1}_{++}$, $f^1(\bar{y}^1_{-\ell}) = \bar{y}^1_\ell)$ and $f^2(\bar{y}^2_{-\ell}) = \bar{y}^2_\ell)$. Then, $\bar{y}^1 + \bar{y}^2$ is weakly efficient in $Y = Y_1 + Y_2$ if and only if for all $h = 1, \dots, \ell - 1$,*

$$\frac{\partial f^1}{\partial y_h}(y^1_{-\ell}) = \frac{\partial f^2}{\partial y_h}(y^2_{-\ell})$$

3.2 Economic behavior of the producer: profit maximization

In this section, we study the competitive behavior of the producer. Nevertheless, we will show that this behavior is incompatible with the presence of producers with increasing returns to scale on the market.

The competitive behavior is characterized by the fact that the producer takes the price as given. This means that the producer does not take into account the effect of her supply on the output markets and of her demand on the input markets. Then, she maximizes the profit over the production set. The competitive behavior can be justified by the fact that the producer is small with respect to the size of the market, which is not satisfactory with large companies. But, we can also think that a large producer is not able to calculate accurately her effect on the prices, so a cautious behavior is to take the price as given.

Definition 3.2.1. *Let $p \gg 0$, a price vector. The profit of the firm with the production set Y is $\pi(p) = \sup\{p \cdot y | y \in Y\}$. The supply of the producer is the set $s(p)$ of productions $y \in Y$ such that for all $y' \in Y$, $p \cdot y \geq p \cdot y'$ or equivalently $p \cdot y = \pi(p)$.*

Remark. It is possible that for a given price, the profit be infinite, the supply be empty and the supply may also be multi-valued.

Examples. Let us find the profit and the supply for a producer having the following production set:

$$Y = \{(a,b) \in \mathbb{R}_- \times \mathbb{R} \mid \ b \leq \sqrt{2 \mid a \mid}\}$$

We have to solve the following maximization problem.

$$\begin{cases} \text{maximize } p_a a + p_b b \\ a \leq 0 \\ b \leq \sqrt{2|a|} \end{cases}$$

Clearly, the solution satisfies $b = \sqrt{2|a|}$, so we have to solve the one variable problem:

$$\begin{cases} \text{maximize } p_a a + p_b \sqrt{2|a|} \\ a \leq 0 \end{cases}$$

One then easily find the solution since the objective function is concave. $s(p) = \left\{ -\dfrac{p_b^2}{2p_a^2}, \dfrac{p_b}{p_a} \right\}$ and $\pi(p) = \frac{p_b^2}{2p_a}$.

Let us now consider the case where

$$Y = \{(a,b) \in \mathbb{R}_- \times \mathbb{R} \mid \; b \leq a^2\}$$

The problem to be solved after reduction is:

$$\begin{cases} \text{maximize } p_a a + p_b a^2 \\ a \leq 0 \end{cases}$$

One remarks that there is no solution and the profit is always equal to $+\infty$.

Now if we consider a producer having constant returns to scale, the supply may be multi-valued. If

$$Y = \{(a,b) \in \mathbb{R}_- \times \mathbb{R} \mid \; b \leq -\gamma a\}$$

where $\gamma > 0$ is the constant marginal productivity. The supply is then defined as:

$$s(p) = \begin{cases} (0,0) & \text{si } p_b < p_a/\gamma \\ \{(-t, \gamma t) \mid t \geq 0\} & \text{si } p_b = p_a/\gamma \\ \emptyset & \text{si } p_b > p_a/\gamma \end{cases}$$

So, for $p_b = p_a/\gamma$, the supply is an half line. One checks that the profit is either 0 when $p_b \leq p_a/\gamma$ or $+\infty$.

To end this list of examples, let us consider the case where

$$Y = \{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b \leq \frac{a^2 - 4a}{2-a}\}$$

Then for $p_a = p_b$, the profit is finite and equal to $2p_b$ but the supply is empty.

In the next proposition, we state some elementary properties of the supply and profit.

Proposition 3.2.1. *Let $p \gg 0$ and Y a nonempty production set.*

a) Every element of $y \in s(p)$ is efficient.
b) If Y is closed and convex, $s(p)$ is closed and convex. For all $t > 0$, $s(tp) = s(p)$.
c) If Y' is a second nonempty production set. Let $Y'' = Y + Y'$ be the aggregate production set. Then, let π, s, π', s' et π'', s'' the profit functions and supply associated to the production sets Y, Y' and Y''. Then $\pi''(p) = \pi(p) + \pi'(p)$. For all $(y, y') \in s(p) \times s'(p)$, $y + y' \in s''(p)$; for all $y'' \in s''(p)$, for all $(y, y') \in Y \times Y'$ such that $y'' = y + y'$, then $y \in s(p)$ and $y' \in s'(p)$.
d) The profit function π is convex, homogeneous of degree 1. Its domain of definition is a convex cone et it is continuous on the interior of its domain.

Proof. Assertion (a) is identical to Assertion (v) of Proposition 3.1.7. Assertion (b) comes from the fact that $s(p)$ is the intersection of Y with the hyperplan $\{y \in \mathbb{R}^\ell \mid p \cdot y = \pi(p)\}$ when $\pi(p)$ is finite. Otherwise, $s(p)$ is empty. The homogeneity of s is obvious.

c) Let $y'' \in Y''$. From the definition of Y'', it exists $(y, y') \in Y \times Y'$ such that $y'' = y + y'$. Thus $p \cdot y'' = p \cdot y + p \cdot y' \leq \pi(p) + \pi'(p)$. Taken the supremum on Y'' in the right side of the equality, one gets the inequality $\pi''(p) \leq \pi(p) + \pi'(p)$. Conversely if $\pi(p) = +\infty$, then there exists a sequence $(y^\nu)_{\nu \in \mathbb{N}}$ of Y such that the sequence $(p \cdot y^\nu)$ converges to $+\infty$. Let y' be any element of Y'. Then we have $\lim_\nu p \cdot (y^\nu + y') = +\infty$ and since $y^\nu + y' \in Y''$, one deduces that $\pi''(p) = +\infty$. A symmetric argument shows that the result is identical if $\pi'(p) = +\infty$ If $\pi(p)$ and $\pi'(p)$ are finite, for all $\varepsilon > 0$, it exists $(y, y') \in Y \times Y'$ such that $p \cdot y \geq \pi(p) - \varepsilon$ and $p \cdot y' \geq \pi'(p) - \varepsilon$. Hence $p \cdot (y + y') \geq \pi(p) + \pi'(p) - 2\varepsilon$. Since $y + y' \in Y''$, one deduces that $\pi''(p) \geq \pi(p) + \pi'(p) - 2\varepsilon$. Since the inequality holds true for every $\varepsilon > 0$, one can conclude that $\pi''(p) \geq \pi(p) + \pi'(p)$.

Let $(y, y') \in s(p) \times s'(p)$. Hence $p \cdot y = \pi(p)$ and $p \cdot y' = \pi'(p)$. So $p \cdot (y + y') = \pi(p) + \pi'(p) = \pi''(p)$. Since $y + y' \in Y''$, this implies that $y + y' \in s''(p)$. Conversely let $y'' \in s''(p)$ and let $(y, y') \in Y \times Y'$ such that $(y, y') = y + y'$. Then, $p \cdot (y + y') = \pi''(p) = \pi(p) + \pi'(p)$. Since $p \cdot y \leq \pi(p)$ and $p \cdot y' \leq \pi'(p)$, this implies that $p \cdot y = \pi(p)$ and $p \cdot y' = \pi'(p)$. Consequently $y \in s(p)$ and $y' \in s(p')$.

d) It is easy to check that π is homogeneous of degree 1. π is convex since it is the supremum of linear functions $p \to p \cdot y$ for $y \in Y$. Hence, the results of convex analysis shows that the domain of π is a convex cone and π is continuous on the interior of its domain. □

Assertion (c) shows that the maximization of profit can be done in a decentralized way. So a producer with several units of production is not obliged to compute the aggregate production set to compute the supply. It suffices to compute the supply for each unit and then to aggregate the supplies of the units. This property is fundamental since it shows that the optimal decision can be found with the knowledge of the individual production sets without knowing the production possibilities of the production sector as a whole.

As we have shown in the previous examples, the profit function may be not defined for some prices. The next proposition gives us a criterion to determine the interior of the domain of the profit function. The domain is denoted $\mathcal{D}(\pi)$.

Proposition 3.2.2. *Let Y be a convex, closed production set containing* 0*. A price $p \in \mathbb{R}^\ell_{++}$ is in the interior of the domain of π, $\mathrm{int}\mathcal{D}(\pi)$, if and only if the set $Y_p = \{y \in Y \mid p \cdot y \geq 0\}$ is bounded. In that case, $s(p)$ is nonempty.*

In the example above with the constant marginal productivity, one remarks that the profit is finite for the price $(p_a, p_a/\gamma)$ but the set of production having a non-negative value for this price is unbounded. The profit is infinite and the supply is empty if one slightly increase the price of the second commodity.

Proof. We first show that Y_p is bounded if the profit is finite in a neighborhood of p. Let us assume by contraposition that Y_p is not bounded. Since Y_p is a closed convex nonempty subset of $\mathbb{R}^\ell$, there exists $z \neq 0$ such that for all $t \geq 0$, $tz \in Y_p$. Let $\tau > 0$ and $p^\tau = p + \tau z$. For τ samll enough, p^τ is in $\mathbb{R}^\ell_{++}$. Furthermore $p^\tau \cdot tz = p \cdot tz + \tau t\|z\|^2$. Since $z \neq 0$, one deduces that the limit of $p^\tau \cdot tz$ when t tends to $+\infty$ is $+\infty$. Hence, $\pi(p^\tau) = +\infty$ for all $\tau > 0$, which contradicts the fact that π is finite in a neighborhood of p.

Let us now prove the converse implication. We assume that $Y(p)$ is bounded. If the profit is not finite in a neighborhood of p, there exists a sequence (r^ν, y^ν, p^ν) in $\mathbb{R}_{++} \times Y \times \mathbb{R}^\ell_{++}$ such that the sequence (r^ν) is decreasing and converges to 0, for all ν, $\|p^\nu - p\| \leq r^\nu$ and $p^\nu \cdot y^\nu \geq \nu$. Since the sequence (p^ν) is bounded, this implies that the sequence $(\|y^\nu\|)$ converges to $+\infty$. The sequence $(r^\nu, (1/\|y^\nu\|)y^\nu, p^\nu)$ is bounded and it has a converging subsequence $(r^{\psi(\nu)}, (1/\|y^{\psi(\nu)}\|)y^{\psi(\nu)}, p^{\psi(\nu)})$ whose limit is $(0, z, p)$ with $\|z\| = 1$. We remark that $p^{\psi(\nu)} \cdot y^{\psi(\nu)} \geq 0$. Hence dividing by $\|y^{\psi(\nu)}\|$ and taking the limit, we obtain $p \cdot z \geq 0$. We now show that for all $t > 0$, $tz \in Y_p$, which contradicts the assumption that Y_p is bounded. Since $p \cdot z \geq 0$, one has $p \cdot tz \geq 0$. For ν large enough, $\|y^{\psi(\nu)}\| \geq t$ and then $\tau^{\psi(\nu)} = 1 - (t/\|y^{\psi(\nu)}\|) \in [0,1]$. Hence, since Y is convex and contains 0, $(1 - \tau^{\psi(\nu)})y^{\psi(\nu)} \in Y$. The sequence $((1 - \tau^{\psi(\nu)})y^{\psi(\nu)})$ converges to tz which belongs to Y since Y is closed.

The non-emptiness of $s(p)$ when $Y(p)$ is bounded comes from the fact that it suffices to maximize the profit on $Y(p)$ to get the productions in $s(p)$. Then, since $y \to p \cdot y$ is continuous, there exists a solution, which means that $s(p)$ is nonempty. □

We now provide sufficient conditions for which if the supply is non-empty, then it is single valued.

Proposition 3.2.3. *Let Y be a production set. We assume that Y is representable by a transformation function t and t is continuous and strictly quasi-convex. Let $p \in \mathbb{R}^\ell_{++}$. If $s(p) \neq \emptyset$, then $s(p)$ is a singleton.*

We give sufficient conditions, which insure that the supply is single valued and continuous on the interior of the domain of definition of the profit function. This allows us to show that the profit function is then continuously differentiable and that one can recover the supply function from the partial derivative of the profit function.

Proposition 3.2.4. *Let Y be a convex, closed production set containing 0. We assume that for all $t \in]0,1[$, for all $(y, y') \in Y^2$, $y \neq y'$, if y and y' are efficient, then $ty + (1-t)y'$ is not efficient. Then, for all $p \in \mathbb{R}^\ell_{++}$, $s(p)$ contains at most one element. The supply function s is continuous on the interior of the domain of definition of the profit function. The profit function is continuously differentiable and $\nabla \pi(p) = s(p)$.*

The assumption on the efficient productions is obviously satisfied when the production set is strictly convex. Nevertheless, the strict convexity never holds true when one has a production function or, more generally, a separation among inputs and outputs. Our assumption holds true when the production function is strictly concave.

Proof. If $s(p)$ is not single valued, let y and y' in $s(p)$ with $y \neq y'$. From Proposition 3.2.1 (a), y and y' are efficient. From our assumption, $ty+(1-t)y'$ is not efficient so, again from Proposition 3.2.1 (a), it does not belong to $s(p)$. But this contradicts the fact that $s(p)$ is convex from Proposition 3.2.1 (b).

From Proposition 3.2.2, we know that $s(p)$ is nonempty on $\text{int}\mathcal{D}(\pi)$. We now show that s is continuous. Let (p^ν) a sequence of $\text{int}\mathcal{D}(\pi)$, which converges to $p \in \text{int}\mathcal{D}(\pi)$. Using the same argument as in the proof of Proposition 3.2.2, one shows that $(s(p^\nu))$ is a bounded sequence. It is now enough to show that each cluster point y of this sequence is equal to $s(p)$. Let ψ a strictly increasing function from $\mathbb{N}$ to itself such that $(s(p^{\psi(\nu)}))$ converges to y. Since Y is closed, $y \in Y$. Since π is continuous on $\text{int}\mathcal{D}(\pi)$, $p \cdot y = \lim_\nu p^{\psi(\nu)} \cdot s(p^{\psi(\nu)}) = \lim_\nu \pi(p^{\psi(\nu)}) = \pi(y)$. So $y = s(p)$.

We now show that π est continûment différentiable. Let $\bar{p}$ in $\text{int}\mathcal{D}(\pi)$. For all $p \in \text{int}\mathcal{D}(\pi)$, one has $\pi(p) - \pi(\bar{p}) \geq s(\bar{p}) \cdot (p - \bar{p})$ and $\pi(\bar{p}) - \pi(p) \geq s(p) \cdot (\bar{p} - p)$. Consequently,

$$0 \leq \pi(p) - \pi(\bar{p}) - s(\bar{p})(p - \bar{p}) \leq (s(p) - s(\bar{p})) \cdot (p - \bar{p})$$

But $\|(s(p)-s(\bar{p}))\cdot(p-\bar{p})\| \leq \|s(p)-s(\bar{p})\|\|p-\bar{p}\|$ and $\lim_{p\to\bar{p}} \|s(p)-s(\bar{p})\| = 0$ since s is continuous. This shows that π is differentiable at $\bar{p}$ and its gradient vector is $s(\bar{p})$. Thus π is continuously differentiable on $\text{int}\mathcal{D}(\pi)$. □

The following result shows that one can recover the supply from the profit function when it is differentiable.

Proposition 3.2.5. *If the profit function π is differentiable on a neighborhood U of $\bar{p}$, then $s(\bar{p})$ is single valued and*

$$\nabla\pi(\bar{p}) = s(\bar{p})$$

Proof. From Proposition 3.2.2, $s(p)$ is nonempty for all $p \in U$. Let us consider the function g defined on U by $g(p) = \pi(p) - p \cdot \bar{y}$ where $\bar{y} \in s(\bar{p})$. For all $p \in U$, $p \cdot \bar{y} \leq \pi(p)$. Thus, for all $p \in U$, $g(p) \geq 0$ and $g(\bar{p}) = 0$. Consequently, $\bar{p}$ is the minimum of g on U. Hence $\nabla g(\bar{p}) = 0$. A simple calculus gives us $\nabla\pi(\bar{p}) = \bar{y}$, which implies that $s(\bar{p})$ is single valued and equal to $\{\nabla\pi(p^*)\}$. □

Example. Let π the function from $\mathbb{R}^2_{++}$ to $\mathbb{R}$ defined by :

$$\pi(p_1, p_2) = \frac{p_2^2}{4p_1}$$

The computation of the partial derivative leads to $s(p_1, p_2) = (-\frac{p_2^2}{4p_1^2}, \frac{p_2}{2p_1})$, which is the profit function associated to the production set $\{(a,b) \in \mathbb{R}^2 \mid a \leq 0, b \leq \sqrt{-a}\}$.

We now state a proposition, which shows that the production set can be recovered from the profit function. From an economic point of view, this result is important since the production sets are not observable whereas the profit function are.

Proposition 3.2.6. *Let Y be a convex, nonempty closed production set satisfying the free-disposal asumption. We assume that there exists $\bar{p} \in \mathbb{R}^\ell_{++}$ such that $\pi(\bar{p})$ is finite. Then an element y of $\mathbb{R}^\ell$ belongs to Y if and only if* $\sup\{p \cdot y - \pi(p) \mid p \in \mathbb{R}^\ell_{++}\} \leq 0$.

Proof. For all $y \in Y$ and for all $p \in \mathbb{R}^\ell_{++}$, $p \cdot y \leq \pi(p)$ thus $\sup\{p \cdot y - \pi(p) \mid p \in \mathbb{R}^\ell_{++}\} \leq 0$. We now show the converse implication. Let $y \in \mathbb{R}^\ell$ such that $\sup\{p \cdot y - \pi(p) \mid p \in \mathbb{R}^\ell_{++}\} \leq 0$. By contraposition, if $y \notin Y$, we apply a separation theorem between $\{y\}$ and Y. Thus, there exists $q \in \mathbb{R}^\ell \setminus \{0\}$ such that $q \cdot y > \sup\{q \cdot z \mid z \in Y\} = \pi(q)$. Since Y satisfies the free-disposal assumption, q belongs to $\mathbb{R}^\ell_{+}$.

We now consider $q^t = tq + (1-t)\bar{p}$ for $t \in [0,1]$. We remark that for all $t < 1$, $q^t \in \mathbb{R}^\ell_{++}$. Since the profit function π is convex, we have $\pi(q^t) \geq t\pi(q) + (1-t)\pi(\bar{p})$. Taking into account the facts that $\pi(\bar{p})$is finite and that $q^t \cdot y$ converges to $q \cdot y$ when t converges to 1, one deduces that there exists $\bar{t} \in [0,1[$ such that $q^{\bar{t}} \cdot y > \pi(q^{\bar{t}})$. Since $q^{\bar{t}} \in \mathbb{R}^\ell_{++}$, we get a contradiction with $\sup\{p \cdot y - \pi(p) \mid p \in \mathbb{R}^\ell_{++}\} \leq 0$ □

Remark. The competitive behavior for the producers is incompatible with the presence of producers having increasing returns to scale in production. Indeed, let us assume that Y is a production set satisfying $0 \in Y$. If one has strictly increasing returns, that is, for all $y \in Y$, $y \neq 0$ and y efficient, for all $t > 1$, $ty \in \text{int}\, Y$. Let $p \in \mathbb{R}^\ell_{++}$. $s(p)$ cannot contains another production than 0. In other words, $s(p) = \{0\}$ or $\emptyset$. Indeed, if $y \neq 0 \in s(p)$, then $2y \in \text{int}Y$, hence $\pi(p) = p \cdot y > p \cdot (2y) = 2p \cdot y$. This implies $\pi(p) < 0$ which is impossible since $0 \in Y$.

We can now give a first order characterization of the supply when the production set is represented by a transformation function.

Proposition 3.2.7. *Let Y be a production set of $\mathbb{R}^\ell$. Let $\bar{y} \in Y$ and $p \in \mathbb{R}^\ell_{++}$. We assume that Y is locally representable by a transformation function t in a neighborhood of $\bar{y}$, t is differentiable and there exists at least one commodity k such that $D_{y_k} t(\bar{y}) > 0$.*

1) *If $\bar{y} \in s(p)$, then there exists $\mu > 0$ such that $p = \mu \nabla t(\bar{y})$ and $t(\bar{y}) = 0$.*

2) *Conversely, we furthermore assume that t is quasi-convex. If there exists $\mu > 0$ such that $p = \mu \nabla t(\bar{y})$ and $t(\bar{y}) = 0$, then $\bar{y} \in s(p)$.*

This result means that $\bar{y}$ is the supply of the firm if the marginal productivities and the marginal rate of substitution between inputs are equal to the relative prices.

3.2.1 Production cost and profit maximization

In the particular case of a production set defined by a production function, the behavior of the producer can be decomposed into two successive steps: given a production level and the prices of input, the producer chooses the basket of inputs, which minimizes the cost, then, given the price of the output, she chooses the optimal level of production.

Let us define the cost function and the demand in inputs.

Definition 3.2.2. *Let Y be a production set defined by a production function f from $\mathbb{R}^{\ell-1}$ to $\mathbb{R}_+$, i.e.,*

$$Y = \{y \in \mathbb{R}^\ell \mid y_{-\ell} \leq 0,\ y_\ell \leq f(y_{-\ell})\}.$$

Let $O = \{y_\ell \in \mathbb{R}_+ \mid Y(y_\ell) \neq \emptyset\}$ be the set of attainable production level. The cost function C of $\mathbb{R}^{\ell-1}_{++} \times O$ in $\mathbb{R}_+$ of this producer is defined by:

$$C(p_{-\ell}, y_\ell) = \inf\{-p_{-\ell} \cdot y_{-\ell} \mid y_{-\ell} \in Y(y_\ell)\}$$

The set $D(p_{-\ell}, y_\ell) = \{y_{-\ell} \in Y(y_\ell) \mid -p_{-\ell} \cdot y_{-\ell} = C(p_{-\ell}, y_\ell)\}$ is called the demand in inputs.

For a given production level and input prices, the cost function is the minimum of the values of the baskets of commodities, with which it is possible to produce the chosen production level. The demand in inputs is the set of baskets of commodities, which minimizes the cost.

The following proposition gives some basic properties of the cost function and demand in inputs.

Proposition 3.2.8. *We assume that f is continuous and weakly decreasing on $\mathbb{R}^{\ell-1}_-$.*

a) For all $(p_{-\ell}, y_\ell) \in \mathbb{R}^{\ell-1}_{++} \times O$, $C(p_{-\ell}, y_\ell)$ is finite and non negative. For all $(y_{-\ell}) \in D(p_{-\ell}, y_\ell)$, $(y_{-\ell}, y_\ell)$ is weakly efficient.
b) For all $(p_{-\ell}, y_\ell) \in \mathbb{R}^{\ell-1}_{++} \times O$, the set $D(p_{-\ell}, y_\ell)$ is nonempty. For all $y_{-\ell} \in D(p_{-\ell}, y_\ell)$, $y_{-\ell} \neq 0$ implies $y_\ell = f(y_{-\ell})$. If f is strictly quasi-concave, then $D(p_{-\ell}, y_\ell)$ is single valued.
c) For a fixed y_ℓ in O, the function $p_{-\ell} \to C(p_{-\ell}, y_\ell)$ is continuous, homogeneous of degree 1, concave and weakly increasing.
d) For a fixed $p_{-\ell}$, the cost function is weakly increasing with respect to y_ℓ. If f is concave, C is convex with respect to the production level.

e) *If f is decreasing in the sense that for all $(y_{-\ell}, y'_{-\ell}) \in (\mathbb{R}_-^{\ell-1})^2$, $y_{-\ell} \ll y'_{-\ell}$ implies $f(y_{-\ell}) > f(y'_{-\ell})$, then the cost function is continuous. If furthermoref is strictly quasi-concave, then D is continuous.*

Proof. The properties of the cost function and the demand in inputs are very similar to the ones of the expenditure functions and compensated demand, so we refer to the proof of Proposition 2.3.6. □

The next proposition show the relation between the shape of the cost function and the returns to scale.

Proposition 3.2.9. *Let C be the cost function associated to the production function f. Let $p_{-\ell} \gg 0$ an input price. If the production exhibits increasing (resp. constant, decreasing) return to scale, for all $y_\ell \in O$, we have:*

$$\forall \lambda \in [0,1], C(p_{-\ell}, \lambda y_\ell) \geq (\text{resp. } =, \leq) \lambda C(p_{-\lambda}, y_\ell)$$

We provide a first order characterization of the demand of inputs when the production function is differentiable.

Proposition 3.2.10. *Let f be a production function and Y be the associated production set. We assume that the production function f is differentiable on $-\mathbb{R}_{++}^{\ell-1}$ and for every $y_{-\ell} \in -\mathbb{R}_{++}^{\ell-1}$ there is at least one input k such that $D_{y_k} f(y_{-\ell}) < 0$. Let $p_{-\ell} \in \mathbb{R}_{++}^{\ell-1}$ and $y_\ell \geq 0$.*

a) *If $y_{-\ell} \in D(p_{-\ell}, y_\ell)$, then there exists $\gamma > 0$ such that $p_{-\ell} = -\gamma \nabla f(y_{-\ell})$ and $f(y_{-\ell}) = y_\ell$.*
b) *If f is quasi-concave and if $y_{-\ell} \in -\mathbb{R}_{++}^{\ell-1}$ satisfies $f(y_{-\ell}) = y_\ell$ and $p_{-\ell} = -\gamma \nabla f(y_{-\ell})$ for some $\gamma > 0$, then $y_{-\ell} \in D(p_{-\ell}, y_\ell)$.*

Let us now come back to the profit maximization.

Proposition 3.2.11. *Let f be a production function and let Y be the associated production set. Let $y \in Y$, $y \in s(p)$ if and only if $y_{-\ell} \in D(p_{-\ell}, y_\ell)$ and y_ℓ solves the maximization problem* $\max\ p_\ell y'_\ell - C(p_{-\ell}, y'_\ell)$ *on O.*

Proof. If $y \in s(p)$, then, for all $y'_{-\ell} \in Y(y_\ell)$, $(y'_{-\ell}, y_\ell) \in Y$ and thus $p \cdot y = p_{-\ell} \cdot y_{-\ell} + p_\ell y_\ell \geq p_{-\ell} \cdot y'_{-\ell} + p_\ell y_\ell$. Consequently $-p_{-\ell} \cdot y_{-\ell} \leq -p_{-\ell} \cdot y'_{-\ell}$ which means that $y_{-\ell} \in D(p_{-\ell}, y_\ell)$. Let $y'_\ell \in O$. Then, for all $\varepsilon > 0$, it exists $y^\varepsilon_{-\ell} \in Y(y'_\ell)$ such that $C(p_{-\ell}, y'_\ell) \leq -p_{-\ell} y^\varepsilon_{-\ell} + \varepsilon$. Consequently, $(y^\varepsilon_{-\ell}, y'_\ell) \in Y$ and $p \cdot y = p_{-\ell} \cdot y_{-\ell} + p_\ell y_\ell = p_\ell y_\ell - C(p_{-\ell}, y_\ell) \geq p \cdot (y^\varepsilon_{-\ell}, y'_\ell) \geq p_\ell y'_\ell - C(p_{-\ell}, y'_\ell) - \varepsilon$.Since it holds true for all $\varepsilon > 0$, one deduces that $p_\ell y_\ell - C(p_{-\ell}, y_\ell) \geq p_\ell y'_\ell - C(p_{-\ell}, y'_\ell)$ and y_ℓ is a maximum of $p_\ell y'_\ell - C(p_{-\ell}, y'_\ell)$ on O.

Conversely, if $y_{-\ell} \in D(p_{-\ell}, y_\ell)$ and y_ℓ is a maximum of $p_\ell y'_\ell - C(p_{-\ell}, y'_\ell)$ on O, then for all $y' \in Y$, one has $y'_{-\ell} \in Y(y'_\ell)$ and thus $-p_{-\ell} \cdot y'_{-\ell} \geq C(p_{-\ell}, y'_\ell)$.

Hence $p \cdot y' = p_\ell y'_\ell + p_{-\ell} \cdot y'_{-\ell} \leq p_\ell y'_\ell - C(p_{-\ell}, y'_\ell) \leq p_\ell y_\ell - C(p_{-\ell}, y_\ell) = p \cdot y$, d'où $y \in s(p)$. □

We can now characterize the elements in the supply using the partial derivatives of the cost function.

Proposition 3.2.12. *The production set Y is defined by a production function f. We assume that the cost function C is differentiable on $-\mathbb{R}^{\ell-1}_{++} \times (O \setminus \{0\})$. Let $p \in \mathbb{R}^\ell_{++}$.*

a) If $y \in s(p)$ with $y_\ell > 0$, then $y_{-\ell} = -\nabla_{p_{-\ell}} C(p_{-\ell}, y_\ell)$ and $p_\ell = C'(p_{-\ell}, y_\ell)$ where C' denotes the partial derivative of the cost function with respect to the production level, which is called the marginal cost.
b) If the cost function C is convex with respect to the production level and if y satisfies $y_{-\ell} = -\nabla_{p_{-\ell}} C(p_{-\ell}, y_\ell)$ and $p_\ell = C'(p_{-\ell}, y_\ell)$, then $y \in s(p)$.

This proposition shows that the output price is equal to the marginal cost at the optimal production plan.

Proof. a) Using the same argument as in the proof of Proposition 2.3.7, one proves that $y_{-\ell} = \nabla_{p_{-\ell}} C(p_{-\ell}, y_\ell)$ since $y_{-\ell}$ minimizes the cost. So the result is a direct consequence of the previous proposition.

b) We just have to check that the function $p_\ell y'_\ell - C(p_{-\ell}, y'_\ell)$ is concave when the cost function is convex. So the first order necessary condition is sufficient. □

3.3 Exercises

Exercise 3.3.1. Compute the cost function for the following production sets.

- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \alpha a^2 + \beta |a|\}, \quad \alpha > 0, \beta \geq 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \alpha(1 - e^{Ka})\}, \quad \alpha > 0, K > 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \alpha |a|\}, \quad \alpha > 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \alpha |a|^\beta\}, \quad \alpha > 0,\ \beta > 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \max\{|a|, 2|a| - 2\}\},$
- $Y = \{(a,b,c) \in R^3 \mid a \leq 0, b \leq 0, c \leq |a|^\alpha |b|^{1-\alpha}\}, \quad 0 < \alpha < 1.$
- $Y = \{(a,b,c) \in R^3 \mid a \leq 0, b \leq 0, c \leq \sqrt{|\alpha a + \beta b|}\}, \quad \alpha > 0,\ \beta > 0.$
- $Y = \{x \in R^\ell \mid x_1 \leq 0, \ldots, x_{\ell-1} \leq 0, x_\ell \leq \min\{\alpha_i |x_i|, i = 1, \ldots, \ell - 1\}\}$
- $Y = \{(a,b,c,d) \in R^4 \mid a \leq 0, b \leq 0, c \leq 0, d \leq |a| + \min\{|b|, |c|\}\}.$
- $Y = \{x \in R^3 \mid x_1 \leq 0, x_2 \leq 0, x_3 \leq \alpha_1 |x_i| + \alpha_2 |x_2|\}$

Exercise 3.3.2. In a three-good economy, a firm has two production units. The first one produces the commodity C using the commodity A as input. The production set of this unit is $\{(a,b,c) \in R^3 \mid a \leq 0, b \leq 0, c \leq |a|\}$. The second unit produces the commodity C using the commodity B as input. The production set of this unit is $\{(a,b,c) \in R^3 \mid a \leq 0, b \leq 0, c \leq b^2\}$. Determine the iso-output set, that is the set of baskets of inputs allowing to produce at least a given quantity $c \geq 0$ of output. Compute the cost function and the demand of inputs. Compute the supply of this firm.

Exercise 3.3.3. In a three-good economy, a firm produces the commodity C using the commodities A and B as inputs. The production function is :

$$f(a,b) = a(b-1)$$

1) Show that the cost function is given by :

$$C(p_A, p_B, c) = \begin{cases} 2\sqrt{cp_A p_B} - p_B \text{ if } \frac{cp_A}{p_B} \geq 1 \\ cp_A \text{ if } \frac{cp_A}{p_B} < 1 \end{cases}$$

and compute the cost function and the demand of inputs.
2) Give the demand of inputs and the cost function when the production function is :

$$g(a,b) = \sqrt{a(b-1)}$$

3) The price of commodity C is supposed to be equal to 1. What is the supply of the producer having the production function g.

Exercise 3.3.4. Compute the production function associated to the following cost functions :

- $C(y, p_1, p_2) = 2y^{1/2} p_1^{1/2} p_2^{1/2}$
- $C(y, p_1, p_2) = y(p_1 + p_1^{1/2} p_2^{1/2} + p_2)$

Exercise 3.3.5. Compute the supply and the profit of the producers having the following production sets :

- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \alpha|a|\}, \quad \alpha > 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq k(1 - e^{\alpha a})\}, \quad k > 0, \ \alpha > 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \alpha|a|^\beta\}, \quad \alpha > 0, \ \beta > 0.$
- $Y = \{(a,b) \in R^2 \mid a \leq 0, b \leq \max\{0, \text{Log}(-a)\}\}.$
- $Y = \{(a,b,c) \in R^3 \mid a \leq 0, b \leq 0, c \leq k(|a|^\alpha |b|^{1-\alpha})^\beta\}, \quad \alpha \in]0,1[, \ \beta \in]0,1[.$
- $Y = \{(a,b,c) \in R^3 \mid a \leq 0, b \leq 0, c \leq (\beta|a|^\rho + \gamma|b|^\rho)^{\frac{1}{\rho}}\}, \quad \beta > 0, \gamma > 0, \rho > 0.$

- $Y = \{(a,b,c) \in R^3 \mid a \leq 0, \ b \leq 0, \ c \leq \min\{-\alpha a, -\beta b\}^\gamma\}, \ \alpha > 0, \ \beta > 0, \ \gamma > 0.$

Exercise 3.3.6. We consider the following production set :

$$Y = \{a, b, c) \in R^3 \mid a \leq 0, b \leq 0, c \leq 100|a|^{1/2}|b|^{1/4}\}.$$

a) Compute the cost function $c(p_a, p_b, c)$.
b) Compute the supply function and the profit function.

Exercise 3.3.7. In a two-commodity economy, we consider a producer having the following production set :

$$Y_1 = \{(a, b) \in R^2 \mid a \leq 0,\ b \leq k_1|a|^\alpha\}$$

where k_1 is a positive real number and $\alpha \in]0, 1[$. We denote by $p_A > 0$ and $p_B > 0$ the prices of the two commodities.
1) Give the profit and the supply of this producer with respect to the price vector (p_A, p_B).
2) We assume now that they are n producers having the same production set Y_1. Give the aggregate supply of the production sector with respect to the price vector (p_A, p_B).

Exercise 3.3.8. In a two-commodity economy, we consider a producer having the following production set :

$$Y_\alpha = \{(a, b) \in R^2 \mid a \leq 0,\ b \leq \sqrt{-a + \alpha} - \sqrt{\alpha}\}$$

où α est un paramètre réel positif ou nul.
1) Give the profit and the supply of this producer with respect to the price vector (p_A, p_B).
2) We now assume that there is a second producer whose production set is Y_β avec $\beta \geq \alpha$. What is the aggregated supply of the economy?

Exercise 3.3.9. In a two commodity economy, we consider a producer, which uses the first commodity as input to produce the second one. The production possibilities are represented by a production function f from $]-\infty, 0]$ to $[0, +\infty[$. We assume that f is concave, continuous and decreasing and that $f(y_1) \geq -y_1$ for all $y_1 \leq 0$. Let $p = (p_1, p_2) \in \mathbb{R}^2_{++}$. We assume that $\frac{p_2}{p_1} > 1$.

1) For all $y_1 \leq 0$, show that the value of the production $(y_1, f(y_1))$ at the price p is larger than $y_1(p_1 - p_2)$.

2) Recall what is the definition of the supply of the producer with respect to the price p.

3) Show that the supply for the price p is empty as a consequence of the first question.

Exercise 3.3.10. In a three-dommodity economy, we consider a producer whose cost function is

$$C(p_A, p_B, c) = 2c^2 p_A^{\frac{2}{3}} p_B^{\frac{1}{3}}$$

Compute the supply and the profit of this producer with respect to the prices p_A, p_B and p_C.

Exercise 3.3.11. Let Y be a closed convex production set of $\mathbb{R}^\ell$. Let $p \in \mathbb{R}^\ell_{++}$. Let $\bar{y} \in Y$ such that there exists $r > 0$; for all $y \in Y$, if $\|y - y'\| \leq r$, $p \cdot y \geq p \cdot y'$. Show that $y \in s(p)$.

Exercise 3.3.12. Let Y be a closed convex production set of $\mathbb{R}^\ell$. Let $(p, p') \in (\mathbb{R}^\ell_{++})^2$ and $(y, y') \in s(p) \times s(p')$. Show that $(p - p') \cdot (y - y') \geq 0$. Show that the inequality is strict when $y \notin s(p)$ or $y' \notin s(p')$.

Exercise 3.3.13. Let Y be a nonempty closed convex production set of $\mathbb{R}^\ell$ satisfying the free-disposal assumption. We assume that there exists $\underline{p} \in \mathbb{R}^\ell_{++}$ such that $\pi(\underline{p}) < +\infty$. Let $\bar{p} \in \mathbb{R}^\ell_{++}$. Let $\bar{y} \in \mathbb{R}^\ell$ such that for all $p \in \mathbb{R}^\ell_{++}$, $\pi(p) \geq \pi(\bar{p}) + \bar{y} \cdot (p - \bar{p})$. Using Proposition 3.2.6, show that $\bar{y} \in Y$ and then that $\bar{y} \in s(\bar{p})$.

4. Optimality in exchange economies

4.1 Pareto optimal allocations

We consider an exchange economy with a finite number ℓ of commodities labeled by the superscript $h = 1, \ldots, \ell$ and a finite number m of consumers labeled by the subscript $i = 1, \ldots, m$. The preferences of each consumer i ($i = 1, \ldots, m$) are represented by a utility function u_i from $\mathbb{R}^\ell_+$ to $\mathbb{R}$. Each consumer has an initial endowments $e_i \in \mathbb{R}^\ell_+$. The total initial endowments of the economy is then $e = \sum_{i=1}^m e_i \in \mathbb{R}^\ell_+$.

The consumer will trade to get a final allocation $(x_i) \in (\mathbb{R}^\ell_+)^m$. Since there is no production, the final allocation must be feasible with respect to the total initial endowments e, that is $\sum_{i=1}^m x_i = e$. If (x_i) is a feasible allocation, one remarks that $0 \leq x_i \leq e$. Thus, the set of feasible allocations is closed, convex and bounded. It is also nonempty since $(e, 0, \ldots, 0)$ is a feasible allocation.

In an economy with two goods and two consumers, one can use the Edgeworth box to represent the set of feasible allocation. The Edgeworth box is built by two orthogonal set of axis having opposite directions. The origin of the first set of axis is e in the second set of axis and conversely. Every feasible allocation is represented by a point in the box and conversely a point in the box represents a feasible allocation. The allocation of the first consumer is given by the coordinates of the point in the first set of axis and the allocation of the second consumer by the coordinates of the point in the second set of axis. Since the point is in the box, the coordinates in both sets of axis are non-negative. The fact that the same point represents the allocations of the two agents means that these allocations are feasible.

We now come to the Pareto criterion of optimality, which allows us to compare feasible allocations. This criterion is a minimal condition, which eliminates the feasible allocations, which are unanimously rejected by the consumers.

Among the feasible allocations, we consider the following order relation. An allocation (x_i) is preferred in the sense of Pareto to an allocation (x'_i) if for all i, $u_i(x_i) \geq u_i(x'_i)$ and if for at least one consumer i_0, $u_{i_0}(x_{i_0}) > u_{i_0}(x'_{i_0})$. In other words, an allocation is preferred in the sense of Pareto if all consumers prefer this allocation in the weak sense and at least one consumer prefers strictly this allocation.

Definition 4.1.1. *An allocation* $(x_i) \in (\mathbb{R}^\ell_+)^m$ *is a Pareto optimum if it is feasible and if there does not exist a feasible allocation* (x'_i) *which is preferred to* (x_i) *in the sense of Pareto.*

An allocation is then Pareto optimal if it does not exist an allocation unanimously preferred by all consumers. The set of Pareto optimal allocations can be represented in the Edgeworth box when the preferences of the consumers are linear or of the Leontieff type.

Pareto optima can also be represented graphically in the space of utilities $\mathbb{R}^m$. Let $A = \{(x_i) \in (\mathbb{R}^\ell_+)^m \mid \sum_{i=1}^m x_i = e\}$ be the set of feasible allocations. Let U from $(\mathbb{R}^\ell_+)^m$ to $\mathbb{R}^m$ defined by

$$U(x_1, \ldots, x_m) = (u_1(x_1), \ldots, u_m(x_m))$$

The set of feasible utilities is $U(A) \subset \mathbb{R}^m$. If the utility functions are continuous, then $U(A)$ is compact. A utility vector v in $U(A)$ is the image by U of a Pareto optimal allocation if it does not exist an element $v' \in U(A)$ such that $v' \geq v$ and $v' \neq v$. Hence the set of Pareto optimal utility vectors is the north-east frontier of the set $U(A)$ when we have two consumers.

An allocation can be Pareto optimal but not be consistent with a fairness criterion or a criterion of justice. Indeed, if the preferences of all consumers are strictly monotonic, then the allocation where the total initial endowments is given to a unique consumer and nothing to the others is Pareto optimal.

We now characterize Pareto optima as solutions of an appropriate optimization problem. This allows us to easily obtain the existence and the characterization of Pareto optimal allocations.

Proposition 4.1.1. *Let us assume that for all* $i = 1, \ldots, m$, u_i *is continuous and strictly increasing on* $\mathbb{R}^\ell_+$. $(\bar{x}_i)$ *is a Pareto optimal allocation if and only if* $(\bar{x}_i)$ *is a solution of the following problem.*

$$\max_{(x_i) \in (\mathbb{R}^\ell_+)^m} \quad u_1(x_1)$$
$$\textit{subject to} \quad \begin{cases} u_i(x_i) \geq u_i(\bar{x}_i) \textit{ for all } i = 2, \ldots, m, \\ \sum_{i=1}^m x_i = e \end{cases}$$

From this proposition, we can derive the existence of a family of Pareto optima.

Proposition 4.1.2. *We assume that for all* $i = 1, \ldots, m$, u_i *is continuous and strictly increasing on* $\mathbb{R}^\ell_+$. *For all* $v \in U(A)$, *the set of feasible utilities, the following optimization problem has at least one solution and every solution is a Pareto optimum :*

$$\max_{(x_i) \in (\mathbb{R}^\ell_+)^m} \quad u_1(x_1)$$
$$\textit{subject to} \quad \begin{cases} u_i(x_i) \geq v_i \textit{ for all } i = 2, \ldots, m, \\ \sum_{i=1}^m x_i = e \end{cases}$$

Furthermore, if v and v' in $U(A)$ are such that $v_i \neq v'_i$ for at least one $i = 2, \ldots, m$, then the solutions of the associated problems to v and v' are different.

Note that A is nonempty, hence $U(A)$ is so, which implies that there exists Pareto optimal allocations under the assumptions of this proposition.

Using Proposition 4.1.1, we now derive a first order characterization of Pareto optimal allocations for quasi-concave and differentiable utility functions. This result is the fundamental result of this section. In the Edgeworth box, when the indifference curves are smooth, one remarks that if an allocation is Pareto optimal then the two indifference curves have the same tangent at the point representing the allocation. If the preferences are furthermore convex, then this condition is also sufficient for Pareto optimality.

Proposition 4.1.3. *Let us assume that for all $i = 1, \ldots, m$, u_i is continuous and quasi-concave on $\mathbb{R}^\ell_+$, differentiable on $\mathbb{R}^\ell_{++}$ and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$. Then, $(\bar{x}_i) \in (\mathbb{R}^\ell_{++})^m$ is a Pareto optimal allocation if and only if $(\bar{x}_i) \in (\mathbb{R}^\ell_{++})^m$ is a feasible allocation, that is $\sum_{i=1}^m \bar{x}_i = e$, and all gradient vectors $(\nabla u_i(\bar{x}_i))$ are positively collinear, that is there exists $(\lambda_2, ..., \lambda_i, ..., \lambda_m) \in \mathbb{R}^{m-1}_{++}$ such that*

$$\nabla u_1(\bar{x}_1) = \lambda_i \nabla u_i(\bar{x}_i), \;\; \forall\, i = 2, \ldots, m$$

Proof. Let $z \in \mathbb{R}^\ell$ such that $\nabla u_1(\bar{x}_1) \cdot z < 0$. Then for all $t > 0$ small enough, $u_1(\bar{x}_1 - tz) > u_1(\bar{x}_1)$. Consequently, for all $i > 1$, $u_i(\bar{x}_i + tz) < u_i(\bar{x}_i)$ otherwise $(\bar{x}_1 - tz, \bar{x}_2, \ldots, \bar{x}_{i-1}, \bar{x}_i + tz, \bar{x}_{i=1}, \ldots, \bar{x}_m)$ would be Pareto preferred to $(\bar{x}_1, \ldots, \bar{x}_m)$, which is incompatible with the Pareto optimality of this allocation.

Hence, $\varphi(t) = u_i(\bar{x}_i + tz)$ satisfies $\varphi(0) = u_i(\bar{x}_i)$ and $\varphi(t) < \varphi(0)$ for all t small enough. Consequently, $\varphi'(0) \leq 0$, that is, $\nabla u_i(\bar{x}_i) \cdot z \leq 0$. Since it holds true for all z such that $\nabla u_1(\bar{x}_1) \cdot z < 0$, this implies that $\nabla u_1(\bar{x}_1)$ and $\nabla u_i(\bar{x}_i)$ are positively proportional. □

Remark. Under the assumptions made in Proposition 4.1.3, one easily deduces the Pareto optimality conditions in terms of *marginal rate of substitutions*. That is, $(\bar{x}_i) \in (\mathbb{R}^\ell_{++})^m$ is a Pareto optimal allocation if and only if $(\bar{x}_i) \in (\mathbb{R}^\ell_{++})^m$ is a feasible allocation and all consumers' marginal rates of substitution between every pair of commodities must be equalized at $(\bar{x}_i)$.

Using Proposition 4.1.3, we give now an example of the computation of all Pareto optima in a simple economy.

Example. Let us consider an economy with two consumers :

$u_1(x_1^1, x_1^2) = (x_1^1)^{1/3}(x_1^2)^{2/3}$

$u_2(x_2^1, x_2^2) = (x_2^1)^{1/2}(x_2^2)^{1/2}$

$e = (3, 3)$

$\widetilde{u}_1(x_1^1, x_1^2) = \ln(u_1(x_1^1, x_1^2))$ and $\widetilde{u}_2(x_2^1, x_2^2) = \ln(u_2(x_2^1, x_2^2))$

$\nabla\widetilde{u}_1(x_1^1, x_1^2) = (\frac{1}{3x_1^1}, \frac{2}{3x_1^2})$, $\nabla\widetilde{u}_2(x_2^1, x_2^2) = \lambda_2(\frac{1}{2x_2^1}, \frac{1}{2x_2^2})$

Hence the Pareto optima are the points given by the following formula :

$$\big(x_1^1, \frac{6x_1^1}{3 + x_1^1}, 3 - x_1^1, 3 - \frac{6x_1^1}{3 + x_1^1}\big), \quad x_1^1 \in [0, 3].$$

Using the characterization of the demand, one can reformulate Proposition 4.1.3 as follows.

Proposition 4.1.4. *Let us assume that for all $i = 1, \dots, m$, u_i is continuous and quasi-concave on $\mathbb{R}^\ell_+$, differentiable on $\mathbb{R}^\ell_{++}$ and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$. Then a feasible allocation $(\bar{x}_i) \in (\mathbb{R}^\ell_{++})^m$ is Pareto optimal if and only if there exists a price $p \in \mathbb{R}^\ell_{++}$, such that $\bar{x}_i = d_i(p, p \cdot \bar{x}_i)$.*

For a Pareto optimum, the price p given by the previous definition is called a supporting price. In the Edgeworth box, the price p is a vector orthogonal to the common tangent to the two indifference curves at the optimal allocation.

4.2 Exercises

Exercise 4.2.1. Find all Pareto optima of two-good, two-consumer economy when the preferences are linear, when the preferences are of Leontieff type, when the preferences of one consumer are linear and the one of the other consumers are of Leontieff type.

Exercise 4.2.2. We consider a two-good two-consumer economy. The utility functions are given by :

$u_1(a, b) = a^{\frac{1}{3}} b^{\frac{2}{3}}$

$u_2(a, b) = a^{\frac{1}{2}} b^{\frac{1}{2}}$

Find all Pareto optimal allocations whatever are the initial endowments of the economy and draw the set of Pareto optimal allocation in the Edgeworth box when the initial endowments are $e = (4, 2)$.

Exercise 4.2.3. We consider a two-good two-consumer economy. The utility functions are given by :

$u_1(a, b) = u_2(a, b) = a + \sqrt{b}$

Find all Pareto optimal allocations whatever are the initial endowments of the economy and draw the set of Pareto optimal allocation in the Edgeworth box when the initial endowments are $e = (4, 2)$.

Exercise 4.2.4. We consider a two-good two-consumer economy. The utility functions are given by :

$u_1(a,b) = a + b$
$u_2(a,b) = a^{\frac{2}{3}} b^{\frac{1}{3}}$

The initial endowments of the economy are $e = (4,3)$. Find all Pareto optimal allocations. First look for the Pareto optimal allocations such that $(a^2, b^2) \in \mathbb{R}^2_{++}$.

Exercise 4.2.5. We consider an economy with m consumers and ℓ commodities. We assume that the preferences of all consumers are represented by the same utility function u, which is supposed to be continuous, concave and homogeneous, that is $u(tx) = tu(x)$ for all $t > 0$ and $x \in \mathbb{R}^\ell_+$ and continuously differentiable on $\mathbb{R}^\ell_{++}$. We denote by $e \in \mathbb{R}^\ell_{++}$ the vectors of the total initial endowments. Show that the Pareto optimal allocations in $(\mathbb{R}^\ell_{++})^m$ are the allocations $(t_i e)$ where $t_i > 0$ and $\sum_{i=1}^m t_i = 1$.

5. Competitive equilibrium in an exchange economy

5.1 Definition and first properties

We consider an exchange economy with a finite number ℓ of commodities labeled by the superscript $h = 1, \ldots, \ell$ and a finite number m of consumers labeled by the subscript $i = 1, \ldots, m$. Their preferences are represented by utility function u_i from $\mathbb{R}^\ell_+$ to $\mathbb{R}$. Each consumer has an initial endowments $e_i \in \mathbb{R}^\ell_+$.

We will consider an exchange process between the consumers corresponding to an organized market with posted prices that determine the ratio of possible trades between the goods.

For a price $p \gg 0$, a consumer can exchange a quantity $x^h > 0$ of commodity h against a quantity $x^k > 0$ of commodity k if the ratio of the quantities are the inverse of the ratio of prices, that is, $\frac{x^h}{x^k} = \frac{p^k}{p^h}$, or again $p^h x^h = p^k x^k$. This means that the net trade $(-x^h, x^k)$ has a zero value when it is evaluated with the price p.

Taken the price as given, the consumer can compute her wealth $r_i(p)$ as the value of her initial endowments $p \cdot e_i$. This allows her to determine all affordable bundles of commodities for her on the market, that is the budget set for the price p and the wealth $r_i(p)$. She can then compute her demand $d_i(p, r_i(p))$, which is the preferred consumption under the budget constraint. We actually posit an important assumption. Indeed, the consumers do not take into account the influence of the demand on the price formation. They are supposed to have a competitive behavior. This assumption is often justified by the fact that the consumers are negligible with respect to the market. It can also be coming from the fact that the consumers have no way in terms of information and computation to determine their influence on the price formation.

A Walras (or competitive) equilibrium is realized when the individual demands are compatible with the feasibility constraint. This condition is necessary in order to realize the trades that each agents want to do. For this economic model, a price p can be observed on a market if demand and supply are balanced, which allows the exchange to take place since they are physically feasible. Thus, the price formation is explained by the matching between supply and demand, the later being defined by the consumer's preferences and the initial endowments.

We can now give the definition of a Walras equilibrium.

Definition 5.1.1. *A Walras equilibrium of the economy* $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ *is a price* $p^* \in \mathbb{R}_+^\ell$ *and allocations* $(x_i^*) \in (\mathbb{R}_+^\ell)^m$ *satisfying :*

a) For all $i = 1, \ldots, m$, x_i^* *is a solution of the optimization problem :*

$$\begin{cases} \text{Maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i \\ x_i \geq 0 \end{cases}$$

and

b) (Market Clearing Conditions) $\sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i$.

We can represent a Walras equilibrium in the Edgeworth box. We first remark that the two budget lines of the two consumers are identical. Indeed, they go through the point representing the initial endowments and they are orthogonal to the price. In the two sets of axis, the price has the same direction since the axis have opposite directions. To get an equilibrium, the two demands must have the same representation in both sets of axis. Hence an equilibrium is represented by a budget line going through the point of initial endowments and a point on this line such that in the sets of axis, the indifference curves associated to this point have the budget line as tangent.

For example, one can graphically find the equilibrium price and the equilibrium allocations when the two consumers have linear or Leontieff type preferences.

We now give some properties of Walras equilibrium, which follows immediately from the definition.

Proposition 5.1.1. *Let* $(p^*, (x_i^*))$ *be a Walras equilibrium of the economy* $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$.

i) For all $t > 0$, $(tp^*, (x_i^*))$ *is a Walras equilibrium of the economy* $\mathcal{E}$.
ii) If the preferences of one consumer are strictly monotonic, then the equilibrium price p^* *belongs to* $\mathbb{R}_{++}^\ell$.
iii) For all $i = 1, \ldots, m$, $p^* \cdot x_i^* = p^* \cdot e_i$.
iv) For all $i = 1, \ldots, m$, $u_i(x_i^*) \geq u_i(e_i)$.

The first assertion shows that there always exists an indetermination on the equilibrium price, that is a nominal indetermination, which does not change the relative prices and has no influence on the equilibrium allocation. So, when one look for an equilibrium price with at least one consumer having monotonic preferences, one can only consider the simplex of $\mathbb{R}^\ell$, that is the set $\{p \in \mathbb{R}_+^\ell \mid \sum_{h=1}^\ell p^h = 1\}$. This corresponds to a normalization, which allows us to avoid the multiple solution due to the nominal indetermination.

When one consumer has strictly monotonic preferences, one can normalize the prices by considering that one commodity is a numéraire good, that is has a price equal to 1.

Assert in (iii) shows that the value of the equilibrium allocation is always equal to the value of the initial endowments, hence the net trade $x_i^* - e_i$ has a zero value. We also note that the consumer gets a final allocation, which is always weakly preferred to the initial endowments. Thus, the participation to the market leads never a consumer to a worse situation than the autarkic solution where she keeps her initial endowments and does not participate to the exchange.

The next proposition dates back to Léon Walras. It means that the equilibrium on $\ell - 1$ markets are enough to get an equilibrium on the remaining market when the equilibrium price is positive.

Proposition 5.1.2. *Let us assume that the preferences are monotonic. Let* $(p^*, (x_i^*)) \in \mathbb{R}^\ell_{++} \times (\mathbb{R}^\ell_+)^m$ *such that*

a) for all $i = 1, \ldots, m$, x_i^* *is a solution of the problem :*

$$\begin{cases} \text{Maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i \\ x_i \geq 0 \end{cases}$$

and

b) for all commodities $h = 1, \ldots, \ell - 1$, $\sum_{i=1}^m x_i^{*h} = \sum_{i=1}^m e_i^h$.

Then, $(p^*, (x_i^*))$ *is a Walras equilibrium of the economy* $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$.

The proof relies on the fact that the market clearing condition for the commodity ℓ is satisfied as a consequence of the fact that all budget constraints are binding. The previous proposition allows us to remark that an equilibrium is a solution of a system of $\ell - 1$ equations with $\ell - 1$ unknowns when the demand is single valued. Indeed, to find an equilibrium price p in the simplex of $\mathbb{R}^\ell$, it suffices to find a solution of $\sum_{i=1}^m (d_i^h(p) - e_i^h) = 0$ for all $h = 1, \ldots, \ell - 1$. Since in the simplex, the price of the commodity ℓ is determined by the prices of the other commodities, we are actually in presence of $\ell - 1$ unknowns.

Example. The previous propositions allow us to use a simplified system of equation to find an equilibrium in particular when the number of commodities is 2. Indeed, one can often fix the price of one commodity to 1 thanks to the strict monotonicity of the preferences and one can look for the equality between supply and demand on only one market. So one gets one equation with one unknown. We now give an example of such computation for an economy with two commodities and two consumers :

$u_1(x_1^1, x_1^2) = (x_1^1)^{1/3}(x_1^2)^{2/3}, \quad e_1 = (1, 2)$;

$u_2(x_2^1, x_2^2) = (x_2^1)^{1/2}(x_2^2)^{1/2}, \quad e_2 = (2,1)$;
The price of the first commodity p^1 is chosen equal to 1;

$d_1(1, p^2, 1+2p^2) = \big(\frac{1+2p^2}{3}, 2\frac{1+2p^2}{3p^2}\big)$;

$d_2(1, p^2, 2+p^2) = \big(\frac{2+p^2}{2}, \frac{2+p^2}{2p^2}\big)$.

$(1, p^{*2})$ is an equilibrium price if one has an equality on the market of the first commodity, that is, if

$$\frac{1+2p^{*2}}{3} + \frac{2+p^{*2}}{2} = 3$$

Thus $p^{*2} = \frac{10}{7}$. Hence $\left(p^* = \left(1, \frac{10}{7}\right), x_1^* = \left(\frac{9}{7}, \frac{9}{5}\right), x_2^* = \left(\frac{12}{7}, \frac{6}{5}\right)\right)$ is the unique Walras equilibrium of this economy satisfying $p^{*1} = 1$.

Using the characterization of the demand when the utility functions are differentiable (see Proposition 2.3.3) and Proposition 5.1.2 we can characterize the equilibrium as follows.

Proposition 5.1.3. *We consider an exchange economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ and we assume that the utility functions are continuous on $\mathbb{R}_+^\ell$, differentiable on $\mathbb{R}_{++}^\ell$ and that $\nabla u_i(x) \in \mathbb{R}_{++}^\ell$, for all i and for all $x \in \mathbb{R}_{++}^\ell$.*

a) If $(p^*, (x_i^*)) \in \mathbb{R}_{++}^\ell \times (\mathbb{R}_{++}^\ell)^m$ is a Walras equilibrium of the economy, then there exists $(\lambda_i) \in \mathbb{R}_{++}^m$ such that

- *for all i, $\nabla u_i(x_i^*) = \lambda_i p^*$ and $p^* \cdot x_i^* = p^* \cdot e_i$;*
- *for all commodities $h = 1, \ldots, \ell - 1$, $\sum_{i=1}^m x_i^{*h} = \sum_{i=1}^m e_i^h$.*

b) Conversely, we furthermore assume that the utility functions are quasi-concave on $\mathbb{R}_{++}^\ell$. If $(p^*, (x_i^*)) \in \mathbb{R}_{++}^\ell \times (\mathbb{R}_{++}^\ell)^m$ is such that there exists $(\lambda_i) \in \mathbb{R}_{++}^m$ for which the two conditions above are satisfied, then $(p^*, (x_i^*))$ is a Walras equilibrium of the economy.

We end this first list of properties of Walras equilibrium by studying the effect of some particular modifications of the initial endowments on the equilibrium.

Proposition 5.1.4. *In an exchange economy $(u_i, e_i)_{i=1}^m$, let $(p^*, (x_i^*))$ be a Walras equilibrium. Then $(p^*, (x_i^*))$ is also a Walras equilibrium of the economy $(u_i, \widetilde{e}_i)_{i=1}^m$ for all $(\widetilde{e}_i) \in (\mathbb{R}_+^\ell)^m$ satisfying $p^* \cdot \widetilde{e}_i = p^* \cdot e_i$ and $\sum_{i=1}^m \widetilde{e}_i = \sum_{i=1}^m e_i$. In particular, $(p^*, (x_i^*))$ is a Walras equilibrium of the economy $(u_i, x_i^*)_{i=1}^m$.*

This proposition asserts that a redistribution of the initial endowments, which does not modify the value of the initial endowments with respect to the equilibrium price vector does not affect the equilibrium. In particular, this is true for the equilibrium allocation. Hence, if the consumers want to retrade after the distribution of equilibrium allocations, the same price is an equilibrium price and no trade take place since no consumer wants to modify her allocation.

5.2 Optimality of equilibrium allocation

Using Propositions 4.1.3 and 5.1.3, one easily deduces that the following fundamental property of the equilibrium allocations, that is the First Theorem of welfare economics.

Proposition 5.2.1. *Let us assume that for all* $i = 1, \dots, m$, u_i *is continuous and quasi-concave on* $\mathbb{R}^\ell_+$, *differentiable on* $\mathbb{R}^\ell_{++}$ *and* $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ *for all* $x \in \mathbb{R}^\ell_{++}$. *If* $(p^*, (x_i^*)) \in \mathbb{R}^\ell_{++} \times (\mathbb{R}^\ell_{++})^m$ *is a Walras equilibrium of the economy* $\mathcal{E} = (u_i, e_i)_{i=1}^m$, *then the equilibrium allocation* (x_i^*) *is Pareto optimal.*

The First Theorem of welfare economics can be restated under very mild assumptions.

Proposition 5.2.2. *If* $((x_i^*), p^*)$ *is a Walras equilibrium of the economy* $\mathcal{E} = (u_i, e_i)_{i=1}^m$ *and for all* i, u_i *is monotonic, then the equilibrium allocation* (x_i^*) *is Pareto optimal.*

The proof follows the same steps as the proof of Proposition 6.2.2.

We note that the exchange process on the market selects an allocation which is Pareto optimal and individually rational (see Proposition 5.1.1).

If the initial endowments is a Pareto optimal allocation, we remark that the unique equilibrium is a no-trade equilibrium where each consumer keeps her initial endowments and the price vector is the unique (up to a scaling parameter) supporting price of the Pareto optimal allocation. We now give the statement of the Second Theorem of welfare economics in a pure exchange economy.

Proposition 5.2.3. *Let us assume that for all* $i = 1, \dots, m$, u_i *is continuous, strictly quasi-concave on* $\mathbb{R}^\ell_+$, *differentiable on* $\mathbb{R}^\ell_{++}$ *and* $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ *for all* $x \in \mathbb{R}^\ell_{++}$. *If* $(x_i^*) \in (\mathbb{R}^\ell_{++})^m$ *is a Pareto optimal allocation of the economy* $\mathcal{E} = (u_i, e_i)_{i=1}^m$, *then there exists* $p^* \in \mathbb{R}^\ell_{++}$ *such that, up to a normalization of the price,* $(p^*, (x_i^*)) \in \mathbb{R}^\ell_{++} \times (\mathbb{R}^\ell_{++})^m$ *is the unique Walras equilibrium of the economy* $\mathcal{E}^* = (u_i, x_i^*)_{i=1}^m$ *where the equilibrium price is given by*

$$p^* = \frac{\nabla u_1(x_1^*)}{D_{x_1^{*\ell}} u_1(x_1^*)}$$

Remark 1. In this case, we say that the Pareto optimal allocation $(x_i^*) \in (\mathbb{R}^\ell_{++})^m$ can be *decentralized* as a Walras equilibrium using transfers (taxes or subsidies) $(t_i) \in \left(\mathbb{R}^\ell\right)^m$ such that $\sum_{i=1}^m t_i = 0$ and $p^* \cdot (e_i + t_i) = p^* \cdot x_i^*$ for all $i = 1, \ldots, m$. The equilibrium price p^* given by Proposition 5.2.3 is called the supporting price.

Remark 2. If the Pareto optima of an economy are known as well as the associated supporting prices, one can compute the equilibrium for all initial endowments. Indeed, the equilibrium associated to the initial endowments (e_i) are the elements $(p^*, (x_i^*))$ such that (x_i^*) is a Pareto optimum, p^* is the supporting price and $p^* \cdot x_i^* = p^* \cdot e_i$ for all $i = 1, \ldots, m-1$.

5.3 Existence of an equilibrium

We now give without proof sufficient conditions for the existence of a Walras equilibrium. See the proof in Debreu (1984).

Theorem 5.3.1. *The exchange economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ has a Walras equilibrium with a non-zero equilibrium price if the following sufficient condition are satisfied :*

(i) For all $i = 1, \ldots, m$, u_i is continuous, quasi-concave and monotonic;
(ii) For all $i = 1, \ldots, m$, $e_i \gg 0$.

This theorem shows that under our basic assumptions on the utility functions, an equilibrium exists if the initial endowments of each consumer is strictly positive, which means that each consumer has a positive quantity of each commodity before the market. This assumption is quite restrictive and it can be weaken at the price of a stronger assumption on the preferences.

Corollary 5.3.1. *The exchange economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ has a Walras equilibrium with a strictly positive equilibrium price if the following sufficient condition are satisfied :*

(i) For all $i = 1, \ldots, m$, u_i is continuous, quasi-concave and strictly monotonic;
(ii) For all $i = 1, \ldots, m$, $e_i \geq 0$ and $\sum_{i=1}^m e_i \gg 0$.

We now give an example due to D. Gale. We consider a two-commodity two-consumer linear exchange economy, which has no equilibrium since the initial endowments are not strictly positive and the preferences not strictly monotonic.

$u_1(x_1^1, x_1^2) = x_1^2$, $u_2(x_2^1, x_2^2) = x_2^1 + x_2^2$; $e_1 = (2,2)$, $e_2 = (2,0)$.

If one perturbs this economy taken the same utility functions and the initial endowments $e_1^\varepsilon = (2, 2-\varepsilon)$ and $e_2^\varepsilon = (2, \varepsilon)$ for $\varepsilon > 0$, one gets an equilibrium. When ε is small enough, the equilibrium allocations are $((0,2),(4,0))$ with an equilibrium price p^ε in the simplex which converges to $(0,1)$ when ε tends to 0. Note that the limit of the equilibrium price is not an equilibrium price.

If one perturbs the utility functions keeping the initial endowments fixed, with the same preferences for the second consumer and $u_1^\varepsilon(x_1^1, x_1^2) = \varepsilon x_1^1 + x_1^2$, one gets an equilibrium allocation $((2,2),(2,0))$ when ε is small enough and the equilibrium price$(\frac{\varepsilon}{1+\varepsilon}, \frac{1}{1+\varepsilon})$ also converges to $(0,1)$.

In a two-commodity economy, we can give a simple proof of the existence of an equilibrium by studying the excess demand function of the first commodity with respect to price of the first commodity, the second one being normalized to 1. One can shows that the excess demand is positive is the price is sufficiently small and negative if the price is sufficiently large. Since the excess demand is continuous with respect to the price, one deduces that the excess demand vanishes for some positive price. Using Proposition 5.1.2, one then concludes that the excess demand for the second commodity also vanishes for the same price and that one has find an equilibrium price.

In this simple framework, one can also show that the number of equilibrium is finite and odd when the derivative of the excess demand function does not vanish for the equilibrium prices.

5.4 Exercises

Exercise 5.4.1. Figure 1 below represents an Edgeworth box with the initial endowments e^1 of the first agent, the two indifference curves of the two agents associated to the initial endowments. Using graphical arguments and complementary explanations, answer to the following questions:

1) Is the allocation (e^1, e^2) Pareto optimal?

2) In which area of the Edgeworth box lies the equilibrium allocation?

3) Are the dotted lines possible equilibrium budget lines?

Figure 2 below represents an Edgeworth box with the initial endowments e^1 of the first agent, the curve of Pareto optimal allocations and the dotted line represents the tangent to the indifference curve associated to x^1, which is on the Pareto optimal curve. Using graphical arguments and complementary explanations, answer to the following questions:

1) Is it possible that x^1 be the equilibrium allocation of the first agent?

2) Where are the initial endowments $e^{1\prime}$ of the first agent in the Edgeworth box such that x^1 is an equilibrium allocation associated to $e^{1\prime}$?

Exercise 5.4.2. We consider an economy with two commodities A et B and 2 consumers. The utility functions are of the Cobb-Douglas type, that is : $u_i(a,b) = a^{\alpha_i} b^{1-\alpha_i}$ with $\alpha_i \in]0,1[$ for $i = 1;2$. The initial endowments are $e_1, e_2 \gg 0$.

1. Compute the excess demand function of this economy and show that it satisfies the gross substitute property : if the price of one commodity increases and the other one is kept fixed, then the excess demand of the other commodity increases strictly.
2. Show that if all consumers have the same initial endowments e, then the global excess demand is the same as the one of unique consumer having initial endowments $2e$ and a utility function :
   $u(a,b) = a^{\alpha_1+\alpha_2} b^{2-\alpha_1-\alpha_2}$
   Give the equilibrium price as a function of e, α_1 and α_2.
3. Show that if all consumers have the same utility function, then the global excess demand is the same as the one of a unique consumer with the same utility function and initial endowments $e_1 + e_2$.
   Give the equilibrium price as a function of $e_1 + e_2$ and α_1.

Exercise 5.4.3. We consider an economy with two commodities A et B and 2 consumers. The utility functions are given by :

$u_1(a,b) = a^{\frac{1}{3}} + b^{\frac{2}{3}}$
$u_2(a,b) = a^{\frac{2}{3}} + b^{\frac{1}{3}}$

The initial endowments are $e_1 = (1,2)$ and $e_2 = (1,1)$. Show that the trade to get the equilibrium allocation is such that the first consumer sells a positive quantity of commodity A and buys a positive quantity of commodity B. Do not try to compute the equilibrium allocation.

Exercise 5.4.4. We consider an economy with two commodities A et B and 2 consumers. The global initial endowments are $e = (3,2)$. The Pareto optimal allocations of this economy are $((a_1,b_1),(a_2,b_2)) \in (\mathbb{R}^2_+)^2$ such that :

$$\begin{cases} a_1 + a_2 = 3 \\ b_1 + b_2 = 2 \\ 9b_1 = 12a_1 - 2a_1^2 \end{cases} .$$

For all Pareto optimal allocation, $((a_1,b_1),(a_2,b_2))$, the unique supporting price is $p = (2,1)$. Find the Walras equilibrium of the economy where the initial endowments of the two consumers are $e_1 = (\frac{5}{2}, \frac{7}{9})$ and $e_2 = (\frac{1}{2}, \frac{11}{9})$.

Exercise 5.4.5. We consider an economy with two commodities A et B and 2 consumers. The initial endowments are $e_1 = (1,1)$ and $e_2 = (1,1)$. The utility functions are given by :

$u_1(a,b) = a^{\frac{1}{3}} b^{\frac{2}{3}}$
$u_2(a,b) = a^{\frac{1}{4}} b^{\frac{3}{4}}$

1. Find all Pareto optimal allocations.

2. For equity of treatment, the planner wishes to obtain a Pareto optimal allocation which guarantees the same allocation in commodity A for both consumers. Which is this optimum?
3. To decentralize this optimum, the planner has the possibility to implement some transfers between the initial endowments of commodity A. Determine the transfer which leads to a Walras equilibrium satisfying the equity of treatment.

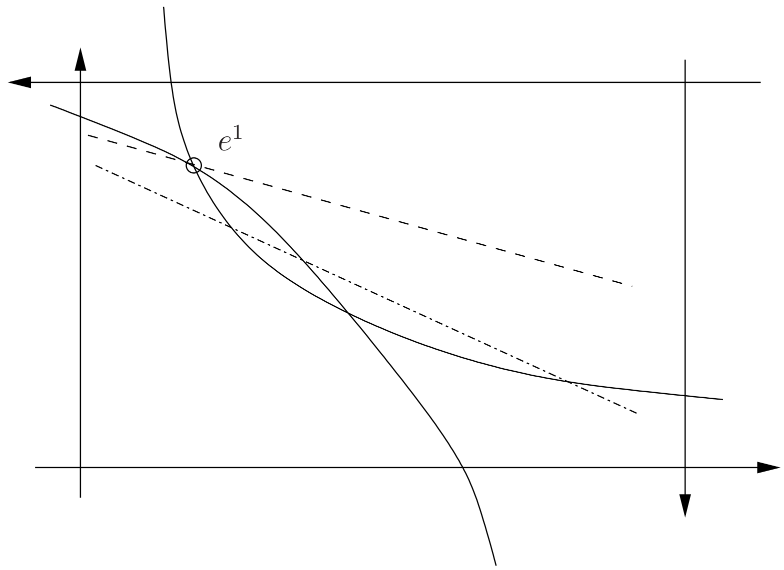

Figure 1

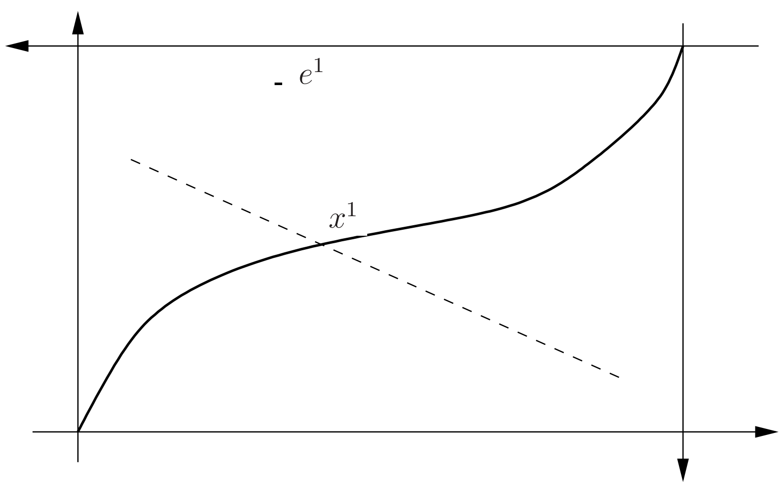

Figure 2

6. Production economy

6.1 Competitive equilibrium

We consider a production economy with a finite number ℓ of commodities labeled by the superscript $h = 1, \ldots, \ell$, a finite number m of consumers labeled by the subscript $i = 1, \ldots, m$, and a finite number n of producers labeled by the subscript $j = 1, \ldots, n$. The preferences of each consumer i are represented by a utility function u_i from $\mathbb{R}^\ell_+$ to $\mathbb{R}$. Each producer j is represented by a production set Y_j included in $\mathbb{R}^\ell$.

We now consider a private ownership economy. This means that each consumer i has an initial endowments $e_i \in \mathbb{R}^\ell$ and a portfolio of shares in the firms $(\theta_{ij})_{j=1}^n$ where $\theta_{ij} \in [0,1]$. Globally these portfolios satisfy the condition $\sum_{i=1}^m \theta_{ij} = 1$ for all j. This means that given the price p^* and the production plan $(y_j^*) \in \prod_{j=1}^n Y_j$, the wealth of the consumer i is given by

$$p^* \cdot e_i + \sum_{j=1}^{n} \theta_{ij} p^* \cdot y_j^*$$

Note that

$$\sum_{i=1}^{m} \left(p^* \cdot e_i + \sum_{j=1}^{n} \theta_{ij} p^* \cdot y_j^* \right) = p^* \cdot \sum_{i=1}^{m} e_i + \sum_{j=1}^{n} y_j^*$$

.

To summarize, a production economy is a collection

$$\mathcal{E} = \left(\mathbb{R}^\ell, (u_i, e_i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_{ij})_{i=1,j=1}^{i=m,j=n} \right)$$

We can now define a Walras (or competitive) equilibrium of a production economy. With respect to an exchange economy, we add a further condition that the production are taken in the supply for each producer and the wealth of the consumers is computed by taking into account the share of the profits of the firms.

Definition 6.1.1. *A Walras equilibrium of the private ownership economy $\mathcal{E}$ is an element $((x_i^*), (y_j^*), p^*)$ of $(\mathbb{R}^\ell_+)^m \times (\mathbb{R}^\ell)^n \times \mathbb{R}^\ell_+$ such that* *(a) [Profit maximization] for every j, y_j^* is a solution of*

$$\begin{cases} \text{maximize } p^* \cdot y_j \\ y_j \in Y_j \end{cases}$$

(b) [Preference maximization] for every i, x_i^* is a solution of

$$\begin{cases} \text{maximize } u_i(x_i) \\ p^* \cdot x_i \le p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^* \\ x_i \ge 0 \end{cases}$$

(c) [Market Clearing Conditions]

$$\sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i + \sum_{j=1}^n y_j^*$$

We now give the basic properties of a Walras equilibrium.

Proposition 6.1.1. *If $((x_i^*), (y_j^*), p^*)$ is a Walras equilibrium of the economy $\mathcal{E}$, then*
(i) for every $t > 0$, $((x_i^*), (y_j^*), tp^*)$ is also a Walras equilibrium;
(ii) for every i, $p^* \cdot x_i = p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^*$;
(iii) if for every $j \in J$, $0 \in Y_j$, then $u_i(x_i^*) \ge u_i(e_i)$ for all i;
(iv) for all $\tau \in \mathbb{R}^I$, $((x_i^*), (y_j^*), p^*)$ is a Walras equilibrium of the economy $\mathcal{E}^\tau = \Big((u_i, \tau_i e_i + (1-\tau_i)(x_i^* - \sum_{j=1}^n \theta_{ij} y_j^*))_{i=1}^m, (Y_j)_{j=1}^n, (\theta_{ij})_{i=1,j=1}^{i=m,j=n}\Big)$.

We also remark that if $((x_i^*), (y_j^*), p^*)$ is a Walras equilibrium of the economy $\mathcal{E}$, then $((x_i^*), p^*)$ is a Walras equilibrium of the pure exchange economy $\tilde{\mathcal{E}} = \Big(u_i, e_i + \sum_{j=1}^n \theta_{ij} y_j^*\Big)_{i=1}^m$

Example. Let us consider an economy with two commodities, one producer and one consumer :

$u_1(x_1^1, x_1^2) = (x_1^1)^{1/2}(x_1^2)^{1/2}, e_1 = (2,1), \theta_{11} = 1$;

$Y_1 = \{(y_1^1, y_1^2) \in \mathbb{R}^2 \mid y_1^1 \le 0; y_1^2 \le \sqrt{|y_1^1|}\}$;

$d_1(p^1, p^2, w) = \big(\frac{w}{2p^1}, \frac{w}{2p^2}\big)$;

$s_1(p^1, p^2) = \big(-\frac{(p^2)^2}{4(p^1)^2}, \frac{p^2}{2p^1}\big), \pi_1(p^1, p^2) = \frac{(p^2)^2}{4p^1}$;

(p^{*1}, p^{*2}) is an equilibrium price if $\dfrac{2p^{*1} + p^{*2} + \dfrac{(p^{*2})^2}{4p^{*1}}}{2p^{*1}} = -\dfrac{(p^{*2})^2}{4(p^{*1})^2} + 2.$

Let $\pi^* = \dfrac{p^{*2}}{p^{*1}}$, we get the equation $\dfrac{2 + \pi^* + \dfrac{(\pi^*)^2}{4}}{2} = -\dfrac{(\pi^*)^2}{4} + 2$.

Hence $\pi^* = 2(\sqrt{3} - 1)$. Consequently

$\left((1, 2(\sqrt{3}-1)), (\frac{2+3\sqrt{3}}{2}, \frac{2+3\sqrt{3}}{4(\sqrt{3}-1)}), (-\frac{2-\sqrt{3}}{2}, \sqrt{3}-1)\right)$ is the unique equilibrium of this economy up to price normalization.

As in an exchange economy, the following result asserts that, at equilibrium, there is one redundant equation.

Proposition 6.1.2. *Let us assume that all the preferences are monotonic. Let* $(p^*, (x_i^*), (y_j^*)) \in \mathbb{R}^\ell_{++} \times (\mathbb{R}^\ell_+)^m \times \prod_{j=1}^n Y_j$ *such that*

(a) [Profit maximization] for every j, y_j^* *is a solution of*

$$\begin{cases} \text{maximize } p^* \cdot y_j \\ y_j \in Y_j \end{cases}$$

(b) [Preference maximization] for every i, x_i^* *is a solution of*

$$\begin{cases} \text{maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^* \\ x_i \geq 0 \end{cases}$$

(c) for all commodities $h = 1, \ldots, \ell - 1$, $\sum_{i=1}^m x_i^{*h} = \sum_{i=1}^m e_i^h + \sum_{j=1}^n y_j^{*h}$.

Then, $(p^*, (x_i^*))$ *is a Walras equilibrium of the economy.*

6.2 Optimality in production economies

We consider a production economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i)_{i=1}^m, (Y_j)_{j=1}^n, e\right)$, where e denotes the total initial endowments. We denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations of the economy $\mathcal{E}$, that is

$$\mathcal{A}(\mathcal{E}) = \left\{ ((x_i), (y_j)) \in (\mathbb{R}^\ell_+)^m \times \prod_{j=1}^n Y_j \mid \sum_{i=1}^m x_i = e + \sum_{j=1}^n y_j \right\}$$

Among the feasible allocations, we consider the following order relation. An allocation $((x_i), (y_j))$ is preferred in the sense of Pareto to an allocation $((x'_i), (y'_j))$ if for all i, $u_i(x_i) \geq u_i(x'_i)$ and if for at least one consumer i_0, $u_{i_0}(x_{i_0}) > u_{i_0}(x'_{i_0})$. In other words, an allocation is preferred in the sense of Pareto if all consumers prefer this allocation in the weak sense and at least one consumer prefers strictly this allocation.

Definition 6.2.1. *An allocation $((x_i),(y_j)) \in \mathcal{A}(\mathcal{E})$ is a Pareto optimum if there does not exist an allocation $((x'_i),(y'_j)) \in \mathcal{A}(\mathcal{E})$ which is preferred to $((x_i),(y_j))$ in the sense of Pareto.*

We remark that if $((\bar{x}_i),(\bar{y}_j))$ is Pareto optimal, then $(\bar{x}_i)$ is a Pareto optimum of the pure exchange economy $\mathcal{E}^c = \left((u_i)_{i=1}^m, e + \sum_{j=1}^n \bar{y}_j\right)$.

We now give a characterization of the Pareto optimal allocations in a production economy, which is the extension of the result given in Proposition 4.1.3.

Proposition 6.2.1. *We consider a production economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i)_{i=1}^m, (Y_j)_{j=1}^n, e\right)$ and we assume that*

a) for all $i = 1,\ldots,m$, u_i is continuous and quasi-concave on $\mathbb{R}^\ell_+$, differentiable on $\mathbb{R}^\ell_{++}$ and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$.
b) $Y = \sum_{j=1}^n Y_j$ is convex.

Then a feasible allocation $((\bar{x}_i),(\bar{y}_j)) \in (\mathbb{R}^\ell_{++})^m \times \prod_{j=1}^n Y_j$ is Pareto optimal if and only if

1) all gradient vectors $(\nabla u_i(\bar{x}_i))$ are positively collinear, that is for all $i = 2,\ldots,m$, there exists $\lambda_i > 0$ such that $\nabla u_1(\bar{x}_1) = \lambda_i \nabla u_i(\bar{x}_i)$ and
2) for all $j = 1,\ldots,n$, $\bar{y}_j \in s_j(\nabla u_1(\bar{x}_1))$.

Proof. From a remark above, $(\bar{x}_i)$ is a Pareto optimum of the exchange economy $\mathcal{E}^c = \left((u_i)_{i=1}^m, e + \sum_{j=1}^n \bar{y}_j\right)$. Then, from Proposition 4.1.3, all gradient vectors $(\nabla u_i(\bar{x}_i))$ are positively collinear.

We end the proof by contradiction. Let $\bar{p} = \nabla u_1(\bar{x}_1)$. Let us now assume that there exists j_0 such that $\bar{y}_{j_0} \notin s_{j_0}(\bar{p})$. Then, from Proposition 3.2.1 (c), $\bar{y} \sum_{j=1}^n \bar{y}_j \notin s(\bar{p})$ for the global production set $Y = \sum_{j=1}^n Y_j$. Consequently, there exists $y \in Y$ such that $\bar{p} \cdot y > \bar{p} \cdot \bar{y}$. Let $z = y - \bar{y}$. For all $t \in [0,1]$, $\bar{y} + tz \in Y$ since Y is convex. For all i, $\nabla u_i(\bar{x}_i) \cdot z > 0$ since $\nabla u_i(\bar{x}_i)$ is positively colinear to $\bar{p}$. Consequently, for all i, there exists $\tau_i > 0$ such that $u_i(\bar{x}_i + \tau z) > u_i(\bar{x}_i)$ for all $\tau \in]0,\tau_i[$. Let $t \in]0,1[$ such that $t/m < \min_i\{\tau_i\}$. Let $y^t = \bar{y} + tz$. Since $y^t \in Y$, there exists $(y_i) \in \prod_{j=1}^n Y_j$, such that $y^t = \sum_{j=1}^n y_j$. For all i, let $x_i = \bar{x}_i + (t/m)z$. Then, $((x_i),(y_j))$ is a feasible allocation, which Pareto dominates $((\bar{x}_i),(\bar{y}_j))$. This contradicts the fact that $((\bar{x}_i),(\bar{y}_j))$ is a Pareto optimal allocation. □

Remark (first order conditions for Pareto optimality). Assume that (u_i, e_i) satisfies the assumptions of Proposition 6.2.1. We remark that if in addition each production set Y_j is represented by a quasi-convex differentiable transformation function t_j which satisfies all the assumptions of Proposition 3.2.7, then Proposition 6.2.1 can be reformulated in terms of first order conditions in the following way:

A feasible allocation $((\bar{x}_i),(\bar{y}_j)) \in (\mathbb{R}^\ell_{++})^m \times (\mathbb{R}^\ell)^n$ is Pareto optimal if and only if

1. $t_j(\bar{y}_j) = 0$ for all $j = 1, \dots, n$, and
2. all gradient vectors $(\nabla u_i(\bar{x}_i))$ and $(\nabla t_j(\bar{y}_j))$ are positively collinear, i.e. for all $i = 2, \dots, m$ there exists $\lambda_i > 0$ such that

$$\nabla u_1(\bar{x}_1) = \lambda_i \nabla u_i(\bar{x}_i)$$

and for all $j = 1, \dots, n$ there exists $\delta_j > 0$ such that

$$\nabla u_1(\bar{x}_1) = \delta_j \nabla t_j(\bar{y}_j)$$

In particular, from the conditions given above one easily deduces the Pareto optimality conditions in terms of *marginal rate of substitutions* and *marginal rates of transformations*. That is, a feasible allocation $((\bar{x}_i),(\bar{y}_j)) \in (\mathbb{R}^\ell_{++})^m \times (\mathbb{R}^\ell)^n$ is Pareto optimal if and only if $t_j(\bar{y}_j) = 0$ for all $j = 1, \dots, n$ and:

a) all consumers' marginal rates of substitution between every pair of commodities must be equalized at $(\bar{x}_i)$,

b) all firms' marginal rates of transformation between every pair of commodities must be equalized at $(\bar{y}_j)$,

c) every consumer i's marginal rates of substitution at $\bar{x}_i$ must equal every firm j's marginal rates of transformation at $(\bar{y}_j)$ for all pair of commodities.

We now state the First Theorem of welfare economics, which shows that the equilibrium allocations are Pareto optimal under very mild assumptions. In particular, this result implies that the global production is weakly efficient in the global production set.

Proposition 6.2.2. *If $((x_i^*),(y_j^*),p^*)$ is a Walras equilibrium of the private ownership economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i, e_i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_{ij})_{i=1,j=1}^{i=m,j=n}\right)$ and for all i, u_i is monotonic, then the equilibrium allocation $((x_i^*),(y_j^*))$ is Pareto optimal.*

Proof. By contraposition. If $((x_i^*),(y_j^*))$ is not Pareto optimal, then there exists a feasible allocation $((x_i),(y_j))$ such that $u_i(x_i) \geq u_i(x_i^*)$ for every i and a consumer i_0 such that $u_{i_0}(x_{i_0}) > u_{i_0}(x_{i_0}^*)$. From the very definition of a Walras equilibrium, this implies that $p^* \cdot x_{i_0} > p^* \cdot x_{i_0}^*$. We now show that $p^* \cdot x_i \geq p^* \cdot x_i^*$ for all i. If $p^* \cdot x_i < p^* \cdot x_i^*$, since u_i is monotonic, there exists $x_i' \in \mathbb{R}^\ell_+$, satisfying $p^* \cdot x_i' \leq p^* \cdot x_i^*$ and $u_i(x_i') > u_i(x_i) \geq u_i(x_i^*)$. This contradicts the fact that x_i^* maximizes the utility under the budget constraint. So $p^* \cdot x_i \geq p^* \cdot x_i^*$ for all i. Consequently,

$$p^* \cdot \sum_{i=1}^m x_i > p^* \cdot \sum_{i=1}^m x_i^* = p^* \cdot (\sum_j y_j^* + e)$$

But $\sum_{i=1}^m x_i = \sum_j y_j + e$ implies $p^* \cdot \sum_{i=1}^m x_i = p^* \cdot (\sum_j y_j + e)$, hence $p^* \cdot \sum_j y_j > p^* \cdot \sum_j y_j^*$. This implies that for at least one producer j_0, $p^* \cdot y_{j_0} > p^* \cdot y_{j_0}^*$. But this contradicts the fact that the producer j_0 maximizes its profit for the price p^* at $y_{j_0}^*$. □

We now give below the statement of the Second Theorem of welfare economics in a production economy, which is a consequence of the result given in Proposition 6.2.1. Note that we need stronger assumptions than in the First Theorem of welfare economics, in particular the quasi-concavity of the utility functions and the convexity of the global production set. We can state the following proposition in other words showing that a Pareto optimal allocation of a production economy is an equilibrium allocation if we allow for some transfers of wealth among the consumers, that is some redistribution of the initial endowments and an appropriate assignment of shares.

Proposition 6.2.3. *Under the assumptions of Proposition 6.2.1, if* $((\bar{x}_i), (\bar{y}_j)) \in (\mathbb{R}^\ell_{++})^m \times \prod_{j=1}^n Y_j$ *is Pareto optimal, there exists a price* $\bar{p} \in \mathbb{R}^\ell_{++}$ *such that*

a) for every j, $\bar{y}_j$ *is a solution of*

$$\begin{cases} \text{Maximize } \bar{p} \cdot y_j \\ y_j \in Y_j \end{cases}$$

b) for all i, $\bar{x}_i$ *is a solution of*

$$\begin{cases} \text{Maximize } u_i(x_i) \\ \bar{p} \cdot x_i \leq \bar{p} \cdot \bar{x}_i \\ x_i \geq 0 \end{cases}$$

c) $\sum_{i=1}^m \bar{x}_i = e + \sum_{j=1}^n \bar{y}_j$.

The existence of Pareto optimal allocations in a production economy is slightly more demanding than in an exchange economy. Indeed, it is necessary to have a bounded set of attainable allocations, which is always the case in an exchange economy with non-negative consumptions.

Proposition 6.2.4. *The production economy* $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i)_{i=1}^m, (Y_j)_{j=1}^n, e\right)$ *has a Pareto optimal allocation if the utility functions are continuous and* $(\sum_{j=1}^n Y_j + e) \cap \mathbb{R}^\ell_+$ *is bounded and closed.*

Proof. It suffices to remark that the following maximization problem has a solution $((\bar{x}_i), \bar{y})$.

$$\begin{cases} \text{maximize } \sum_{i=1}^m u_i(x_i) \\ \sum_{i=1}^m x_i = y + e \\ y \in \sum_{j=1}^n Y_j \\ x_i \in \mathbb{R}^\ell_+ \text{ for all } i \end{cases}$$

Furthermore, there exists $(\bar{y}_j) \in \prod_{j=1}^n Y_j$ such that $\bar{y} = \sum_{j=1}^n \bar{y}_j$. Finally, one easily checks that $((\bar{x}_i), (\bar{y}_j))$ is a Pareto optimal allocation. □

6.3 Existence of an equilibrium in production economies

We are now giving the existence result for a Walras equilibrium in a production economy.

Theorem 6.3.1. *The economy $\mathcal{E} = ((u_i, e_i), (Y_j), (\theta_{ij}))$ has a Walras equilibrium if (u_i, e_i) satisfies the assumptions of Theorem 5.3.1 and*

(i) for all $j = 1, \ldots, n$, Y_j is closed, convex and satisfies the possibility of inactivity.
(ii) The total production set $Y = \sum_{j=1}^{n} Y_j$ satisfies the irreversibility condition $Y \cap -Y = \{0\}$ and the impossibility of free production $Y \cap \mathbb{R}^\ell_+ \subset \{0\}$.

We can remark that the convexity assumption is a fundamental requirement for the existence of a Walras equilibrium. We now give an example of a two-commodity economy with one producer and one consumer with no equilibrium only because the production set is not convex.

$\ell = 2$, $m = n = 1$. $u_1(x^1, x^2) = x^1 x^2$, $Y = \{(y^1, y^2) \in \mathbb{R}^2 \mid y^1 \leq 0, y^2 \leq 0 \text{ if } y^1 > -1, y^2 \leq 1 \text{ if } y^1 < -1\}$, $e_1 = (2, 1)$.

The proof of Theorem 6.3.1 is beyond the scope of this course. We just give a sketch of the proof in a two-commodity economy A and B with one producers producing commodity B using commodity A as input.

Let us consider a production economy $\mathcal{E} = ((u_i, e_i), (Y_1), (\theta_i))$ with $\ell = 2$. We assume that the utility functions are continuous, strictly quasi-concave, continuously differentiable on $\mathbb{R}^2_{++}$, with $\nabla u_i(x_i) \gg 0$ for all i and $x_i \in \mathbb{R}^2_{++}$. We also assume that $e_i \gg 0$ for all i. We finally assume that the production set is defined by a continuous, concave production function from $\mathbb{R}_-$ to $\mathbb{R}_+$ satisfying $f(0) = 0$ and f is differentiable on $\mathbb{R}_{--}$ with $f'(a) < 0$ for all $a < 0$.

Since f is concave, f has a negative left derivative at 0 denoted $f'(0)$, which can be equal to $-\infty$. We normalize the price by $p^A = 1$. From Proposition 2.3.1, the demand functions of the consumers are continuous on $\mathbb{R}^2_{++} \times \mathbb{R}_{++}$. For each i, we let

$$p_i^B = \frac{\frac{\partial u_i(e_i)}{\partial b}}{\frac{\partial u_i(e_i)}{\partial a}}$$

From the characterization of the demand in Proposition 2.3.3, one easily checks that $d_i^B((1, p^B), e_i^A + p^B e_i^B) > e_i^B$ if $p^B < p_i^B$ and $d_i^B((1, p^B), e_i^A + p^B e_i^B) < e_i^B$ if $p^B > p_i^B$.

Consequently, if $p^B \leq \min_i \{p_i^B\}$, then $z^B(p^B) = \sum_{i=1}^{m} d_i^B((1, p^B), e_i^A + p^B e_i^B) - e_i^B$ is positive and if $p^B \geq \max_i \{p_i^B\}$, then $z^B(p^B)$ is negative.

We now consider a price $p^B \in]0, p_0^B = \frac{-1}{f'(0)}]$. Note that this case has to be considered only if $f'(0)$ is finite. If there exists p^{*B} in this interval such that

$z^B(p^{*B}) = 0$, then $\big((1, p^{*B}), (d_i((1, p^{*B}), e_i^A + p^{*B} e_i^B)), 0\big)$ is an equilibrium of the production economy as well as an equilibrium of the exchange economy $((u_i, e_i))$.

If such p^{*B} does not exists, note that $z^B(p_0^B) > 0$. We now consider for each $a \in [\underline{a} = -\sum_{i=1}^m e_i^A, 0]$[1] the following functions:

$$p^B(a) = \frac{-1}{f'(a)}, \quad w_i(a) = e_i^A + p^B(a) e_i^B + \theta_i(a + p^B(a) f(a))$$

and

$$\tilde{z}^B(a) = \sum_{i=1}^m \left(d_i^B((1, p^B(a)), w_i(a)) - e_i^B \right) - f(a)$$

We remark that $\tilde{z}^B$ is continuous. If $p_0^B = 0$, one remarks that $\tilde{z}^B(a)$ is positive when a is small enough since $p^B(a)$ tends to 0 when a tends to 0 and the profit $a + p^B(a) f(a)$ also tends to 0 . If $p_0^B > 0$, one remarks that $\tilde{z}^B(0)$ is positive since $\tilde{z}^B(0) = z^B(p_0^B)$. For $a = \underline{a}$, the Walras law implies:

$\sum_{i=1}^m \left(d_i^A((1, p^B(\underline{a})), w_i(\underline{a})) - e_i^B \right) +$
$p^B(\underline{a}) \sum_{i=1}^m \left(d_i^B((1, p^B(\underline{a})), w_i(\underline{a})) - e_i^B \right) =$
$\sum_{i=1}^m (e_i^A + p^B(\underline{a}) e_i^B) + \underline{a} + p^B(\underline{a}) f(\underline{a})$

Hence, since $\underline{a} = -\sum_{i=1}^m e_i^A$, one gets,

$$p^B(\underline{a}) \tilde{z}^B(\underline{a}) + \sum_{i=1}^m \left(d_i^A((1, p^B(\underline{a})), w_i(\underline{a})) - e_i^B \right) = 0$$

which implies $\tilde{z}^B(\underline{a}) < 0$. Consequently, there exists $a^* \in]\underline{a}, 0[$ such that $\tilde{z}^B(a^*) = 0$. So, one easily checks that $(1, p^B(a^*), (x_i(a^*)), (a^*, f(a^*))$ is a Walras equilibrium with $x_i(a^*) = d_i((1, p^B(a^*)), w_i(a^*))$.

6.4 Exercises

Exercise 6.4.1. Determine graphically the unique equilibrium of an economy with two commodities, two consumers, one producer when the consumers have linear utility functions, the producer has a constant return technology using the first commodity as input and the second one as output.

Exercise 6.4.2. We consider an economy with two goods A and B, two consumers and one producer with the following characteristics:

$Y = \{(a, b) \in R^2 \mid a \leq 0,\ b \leq \sqrt{-a}\}$

$u_1(a, b) = a^2 b$, $u_2(a, b) = a b^2$.

The initial endowments are $e_1 = e_2 = (1, 0)$ and each consumer receives half of the profit of the unique firm. We normalize the price by choosing $p^A = 1$.

---

[1] If $p_0^B = 0$, we do not consider the value $a = 0$.

1) Compute the supply function of the producer and the demand functions of the consumers with respect to the price p^B.
2) Determine the unique Walras equilibrium of this economy. Is the allocation Pareto optimal?
3) We now suppose that the economy has n identical producers with the same production set Y. Each consumer receives half of the profit of each firm. Determine the unique Walras equilibrium with respect to n. What is the limit of the equilibrium price when n converges to $+\infty$. How vary the equilibrium utility levels when n increases? Is the entry of new producers profitable to the consumers?

Exercise 6.4.3. We consider an economy with two goods, two producers and one consumer. The characteristics of the agents are the following:

$$Y_1 = \{(a,b) \in R^2 \mid a \leq 0,\ b \leq \sqrt{-a+1} - 1\}$$

and

$$Y_2 = \{(a,b) \in R^2 \mid a \leq 0,\ b \leq \sqrt{-a+2} - \sqrt{2}\}$$

$$u(a,b) = a^{\gamma} b^{1-\gamma}$$

where $\gamma \in]0,1[$. The initial endowments is $e = (\frac{39}{2}, \sqrt{2})$. We consider a Walras equilibrium of this economy (x^*, y_1^*, y_2^*, p^*). We know the global production $y_1^* + y_2^* = (-\frac{3}{2}, 2 - \sqrt{2})$ and we let $p^{*A} = 1$. Find p^{*B} then y_1^*, y_2^*, x^* and finally γ.

Exercise 6.4.4. We consider an economy with two commodities A and B, two producers and one consumer. The characteristics of the agents are the following:

$$Y_1 = \{(a,b) \in R^2 \mid a \leq 0, b \leq -a\};$$

$$Y_2 = \{(a,b) \in R^2 \mid a \leq 0, b \leq \max\{-2a-2, 0\}, b \leq 4\}.$$

$$u(a,b) = ab.$$

The initial endowments are $e = (4,4)$.
1) Determine the total production set $Y = Y_1 + Y_2$.
2) The aim of this question is to show that this economy has no Walras equilibrium. We assume by contraposition that there exists a Walras equilibrium (p^*, x^*, y_1^*, y_2^*) with $p^{*A} = 1$.
2a) Show that the four following cases are the only ones possible:

1) $p^{*B} = 1, y_1^* = (-a, a), a \geq 0, y_2^* = (-3, 4)$;
2) $p^{*B} \in]\frac{3}{4}, 1[, y_1^* = (0,0), y_2^* = (-3,4)$;
3) $p^{*B} = \frac{3}{4}, y_1^* = (0,0), y_2^* = (-3,4)$ ou $(0,0)$;
4) $p^{*B} < \frac{3}{4}, y_1^* = (0,0), y_2^* = (0,0)$..

2b) Show that $\dfrac{x^{*A}}{x^{*B}} = p^{*B}$.
2c) Using the market clearing condition, show that there is no equilibrium
3) Can you explain why there is no equilibrium?
4) Show that there exists an equilibrium in this economy if the second producer follow the average pricing rule, that is $p^* \cdot y_2^* = 0$ instead of maximizing the profit.

Exercise 6.4.5. We consider an economy with two commodities A and B, one producer and one consumer. The production set is:

$$Y = \{(a,b) \in R^2 \mid a \leq 0,\ b \leq \left\{ \begin{array}{l} 0 \text{ if } a \in [-1,0] \\ \sqrt{-a-1} \text{ if } a \leq -1 \end{array} \right\}$$

The utility function of the unique consumer is $u(a,b) = \min\{a,b\}$ and his initial endowments are $e = (3,1)$. We normalize the price by choosing $p^A = 1$.
1) Draw the production set and the cost function.
2) Draw one indifference curve and determine the demand of the consumer.
3) Compute the supply of the producer with respect to the price p^B.
4) Let (x^*, y^*) a feasible allocation and $p^{*B} > 0$. We assume that x^* is the demand of the consumer for the price p^{*B} and the wealth $w^* = (1, p^{*B}) \cdot (e + y^*)$ and that y^* is weakly efficient. Show that :

$$x_A^* = x_B^* = y_A^* + 3 = y_B^* + 1 = \sqrt{-y_A^* - 1} + 1$$

5) Compute the values of $(x_A^*,\ x_B^*,\ y_A^*,\ y_B^*)$.
6) Show graphically that (x^*, y^*) is a Pareto optimal allocation.
7) Show that if $(\bar{x}, \bar{y}, \bar{p}^B)$ is a Walras equilibrium, then $(\bar{x}, \bar{y}) = (x^*, y^*)$.
8) Deduce from the previous questions that this economy has no Walras equilibrium. Show that it has an equilibrium for the average pricing rule where $p^* \cdot y^* = 0$.

Exercise 6.4.6. We consider an economy with two commodities A and B, two consumers and one producer. The price of the commodity A is normalized to 1. The two consumers have the same preferences represented by the utility function $u(a,b) = ab$. The initial endowments are $e_1 = (1,2)$ and $e_2 = (4,1)$. The producer produces the commodity B using the commodity A as input with constant return technology. The production set is:

$$Y = \{(a,b) \in R^2 \mid a \leq 0,\ b \leq -\alpha a\}$$

with $\alpha > 0$.
1) Compute the demand of the consumers with respect to p^B and the wealth $w > 0$.
2) Compute the supply and the profit of the producer with respect to the price p^B and the marginal productivity α.

3) Show why the share of the consumers on the profit of the firm has no influence on the Walras equilibrium in this economy.
4) Compute the unique Walras equilibrium of this economy with respect to α.
5) Give the utility level of the consumers with respect to the marginal productivity α.
6) Show that the utility of the second consumer is increasing.
7) Show that the utility of the first consumer is constant, then decreasing and finally increasing.
8) Can you explain the differences of the behavior of the utility levels of the consumers with respect to the marginal productivity?

Exercise 6.4.7. Find all Pareto optimal allocation of the economy with two commodities A and B, two consumers and one producer with the following characteristics:
$u_1(a,b) = b$, $u_2(a,b) = \min\{6a, b\}$, $e = (20, 50)$,
$Y_1 = \{(a,b) \in R^2 \mid a \leq 0,\ b \leq -a\}$,

Exercise 6.4.8. We consider an economy with 2 goods, 2 consumers and one producer. The utility functions are:

$$u_1(a,b) = a^{\frac{1}{3}} b^{\frac{2}{3}} \quad \text{and} \quad u_2(a,b) = a^{\frac{1}{2}} + b^{\frac{1}{2}}$$

The production set is :

$$Y = \{(a,b) \in R^2 \mid a \leq 0,\ b \leq -a\}$$

and the global initial endowments are $e = (2,1)$. One looks for all Pareto optimal allocations $((a_1,b_1),(a_2,b_2),y)$ of this economy such that $y \neq 0$ and a_1, b_1, a_2, b_2 are positive.
1) Show that $y = (-t,t)$ with $t > 0$ and that $\nabla u_1(a_1,b_1)$ and $\nabla u_2(a_2,b_2)$ are positively proportional to $(1,1)$.
2) Show that $b_1 = 2a_1$ and $a_2 = b_2$.
3) Show that all Pareto optimal allocations satisfying the required conditions are $((-1+2t, -2+4t),(3-3t, 3-3t),(-t,t))$ avec $t \in]\frac{1}{2}, 1[$.

Bibliography

1. Balasko Y., Fondements de la théorie de l'équilibre général, Economica, (1988).
2. Barthelemy P., Hammad. P., et J.V. Le Sueur, Microéconomie mathématique, Edition Cujas, Paris (1989).
3. Champsaur P. et J.C. Milleron, Exercices de microéconomie, Dunod, (1971).
4. Debreu G., Théorie de la valeur, Dunod, (1984).
5. Guerrien B. et B. Nezeys, Microéconomie et Calcul Economique, Economica (1981).
6. Gabszewicz J.J., Théorie Microéconomique, CLET De Boeck, (1986).
7. Guesnerie, R. Théorie économique et organisation sociale, Leçon inaugurale au Collège de France, (2000).
8. Jacquemin A. et H. Tulkens, Fondements d'économie politique, CLET de Boeck, (1986).
9. Kirman A.P. et A. Lapied, Microéconomie, PUF, (1991).
10. Laffont J.J., Cours de théorie microéconomique I et II, Economica, (1982).
11. Malinvaud E., Leçons de théorie microéconomique, Dunod, (1982).
12. Mas-Colell, A., Whinston, M.D., and Green, J.R., Microeconomic Theory, Oxford University Press, (1995).
13. Tallon, J.M., Equilibre général. Une introduction, Vuibert, (1997).
14. Varian H.R., Microeconomic Analysis, Norton, (1984).