Microeconomics 1 – Part B: Equilibria and Optimality
Master M1 IMMAEF

TD – Wednesday, December 6, 2023

Pareto optimality in a production economy

Exercise 1. Consider a production economy with two commodities, two consumers and one firm. The firm produces commodity 2 by using commodity 1 as an input. The production set of the firm is

$$Y = \{y = (y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } y_1 + y_2 \leq 0\}$$

The utility functions are

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}} + (x_{22})^{\frac{1}{2}}$$

and the aggregate initial endowment is $r = (2, 1)$.

$\longrightarrow$ The purpose of this exercise is to determine all the Pareto optimal allocations $(x_1^*, x_2^*, y^*) = ((x_{11}^*, x_{12}^*), (x_{21}^*, x_{22}^*), (y_1^*, y_2^*))$ with $(x_1^*, x_2^*) \gg 0$ and $y^* \neq 0$.

1. Remind the proposition on the characterization of Pareto optimal allocations in a differentiable framework.
2. Show that $\nabla u_1(x_{11}^*, x_{12}^*)$ and $\nabla u_2(x_{21}^*, x_{22}^*)$ are positively proportional to $(1, 1)$ and that $y^* = (-t, t)$ with $t > 0$.
3. Show that $x_{12}^* = 2x_{11}^*$ and $x_{21}^* = x_{22}^*$.
4. Show that all the required Pareto optimal allocations are given by

$$((-1 + 2t, -2 + 4t), (3 - 3t, 3 - 3t), (-t, t)) \text{ with } t \in \left]\frac{1}{2}, 1\right[$$

Exercise 2 (Existence of Pareto optimal allocations). Let $\mathcal{E} = ((Y_j)_{j=1,\dots,n}, (u_i)_{i=1,\dots,m}, r)$ be a production economy. Assume that u_i is continuous on $\mathbb{R}^L_+$ for all $i = 1, ..., m$, and that the set of feasible allocations $F = \{(x, y) = ((x_i)_{i=1,\dots,m}, (y_j)_{j=1,\dots,n}) \in \mathbb{R}^{Lm}_+ \times \prod_{j=1}^n Y_j \colon \sum_{i=1}^m x_i = r + \sum_{j=1}^n y_j\}$ is non-empty and compact. Consider $\alpha = (\alpha_i)_{i=1,\dots,m} \in \mathbb{R}^I_{++}$ and the following maximization problem (P_α) :

$$(P_\alpha) \quad \max_{(x,y) \in \mathbb{R}^{Lm}_+ \times \mathbb{R}^{Ln}_+} \sum_{i=1}^m \alpha_i u_i(x_i)$$
$$\text{subject to} \quad \begin{cases} y_j \in Y_j, \ \forall\, j = 1, \dots, n \\ \sum_{i=1}^m x_i = r + \sum_{j=1}^n y_j \end{cases}$$

1. The problem (P_α) has at least a solution $(\bar{x}, \bar{y})$. Why so?
2. Show that if $(\bar{x}, \bar{y})$ solves problem (P_α), then $(\bar{x}, \bar{y})$ is a Pareto optimal allocation of $\mathcal{E}$.