Logic and Sets
Mid-term exam 2021 (1h)

Name: QEM/MMEF

Exercise 1 (9pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. If 2+2=5, then Paris is in France T
2. London is in France if and only if 2+2=5 T
3. The negation of $(p \rightarrow \neg q) \vee r$ is $p \wedge q \wedge (\neg r)$ T
4. The negation of $p \leftrightarrow q$ is $\neg q \leftrightarrow \neg p$ F
5. The negation of $[\forall x \in \mathbb{R}, \forall y \in \mathbb{N}, \exists z \in \mathbb{R}, x + y + z \geq 2 \text{ or } x - y - z \leq 0]$ is $[\exists x \in \mathbb{R}, \exists y \in \mathbb{N}, \forall z \in \mathbb{R}, x + y + z < 2 \text{ or } x - y - z > 0]$ F
6. The negation of $[\forall x \in \mathbb{Z}_+, \exists y \in \mathbb{Q}, y = \frac{1}{x}]$ is $[\exists x \in \mathbb{Z}_-, \forall y \in \mathbb{Q}, y \neq \frac{1}{x}]$. F
7. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 = 2.$ F
8. $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 = 2$ F
9. $(p \rightarrow q) \rightarrow r \Leftrightarrow \neg r \rightarrow \neg(\neg q \rightarrow \neg p)$ T
10. $(A \setminus B) \setminus A = \emptyset$ T
11. $A \times \emptyset = A$ F
12. $\{1\} \in \mathcal{P}(\{0, 1\})$ T
13. $\emptyset \subseteq A \times B$ T
14. $A \cup B = B \Rightarrow A \cap B = \emptyset$ F
15. $\{1\} \in \{1, \{2\}\}$ F
16. $\mathcal{P}(\emptyset) = \{\emptyset\}$ T
17. If $C = \{\emptyset, \{1, 2\}, \{\{1\}\}\}$ then $\bigcup C = \{1, 2, \{1\}\}$ T
18. $A \times A = \{(a, a) : a \in A\}$ F

Exercise 2 (8pts)

1. Show by contraposition that for any integer $n \geq 1$, $n^4 - 2n^2 + 1$ even implies n odd.

Suppose n even. Then $n = 2k$ with $k \in \mathbb{N}$.
$n^4 - 2n^2 + 1 = (2k)^4 - 2(2k)^2 + 1 = 16k^4 - 8k^2 + 1 = 2\underbrace{(8k^4 - 4k^2)}_{\in \mathbb{N}} + 1$,
which is odd.

2. Show by contradiction that there is no integer $a, b \in \mathbb{Z}$ such that $16a + 4b = 3$.

Suppose $\exists\, a, b \in \mathbb{Z}$ s.t. $16a + 4b = 3$.
Then $16a + 4b = 4(4a + b) = 3$, which implies
$4a + b = 3/4$. However, $4a + b \in \mathbb{Z}$ while $3/4 \notin \mathbb{Z}$, a contradiction.

3. Show by induction that for any $n \geq 1$, $1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$.

- basis: $1 \times 1! = 1$, $2! - 1 = 1$, true.
- induction step: assume $p(n)$ and prove $p(n+1)$
$1 \times 1! + \dots + n \times n! + (n+1) \times (n+1)! = (n+1)! - 1 + (n+1) \times (n+1)!$
$= (n+1)!\,(1 + n + 1) - 1 = (n+2)! - 1$.

4. Show that $A \cap B = \emptyset \Leftrightarrow A \cup B^c = B^c$.

$\Rightarrow$)
- $x \in A \cup B^c \Leftrightarrow (x \in A) \vee (x \in B^c)$. If $x \in B^c$, we are done.
If $x \in A$, since $A \cap B = \emptyset$, we have $x \notin B$, i.e., $x \in B^c$.
- $x \in B^c \Rightarrow x \in A \cup B^c$, trivial.

$\Leftarrow$) By contradiction, pick $x \in A \cap B$. Then $x \in A$ and $x \in B$. Therefore $x \in A \cup B^c$ and $x \notin B^c$. This contradicts $A \cup B^c = B^c$.

Exercise 3 (3pts)

Write the truth tables of $(p \to q) \wedge \neg(q \to \neg p)$ and $((p \to q) \to r) \to (p \vee r)$.

p	q	$\neg p$	$p \to q$	$q \to \neg p$	$(p\to q) \wedge \neg(q \to \neg p)$
T	T	F	T	F	T
T	F	F	F	T	F
F	T	T	T	T	F
F	F	T	T	T	F

p	q	r	$p \to q$	$\cdot \to r$	$p \vee r$	$\cdot \to p \vee r$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	F	T