Python for finance and optimization
Optimization of a stopping strategy
Due Nov 17th 2023

We consider a stock whose daily prices are given by:

$$S_{n+1} = S_n \exp\left(-\frac{1}{2}\sigma^2\Delta t + \sigma\sqrt{\Delta t}\epsilon_{n+1}\right), \qquad S_0 \text{ given},$$

where $\Delta t = \frac{1}{252}$ and $(\epsilon_{n+1})_{n\geq 0}$ are i.i.d. standard Gaussian variables.

We also define

$$A_n = \frac{S_0 + S_1 + \ldots + S_n}{n+1}.$$

Basic functions and Monte-Carlo

1. Write a function `simul_s(S0, sigma, dt, N, nb_traj, seed = 42)` that sets the seed of `numpy.random` and returns an array `trajectories` of shape `(nb_traj, N+1)` containing `nb_traj` samples of $(S_n)_{0\leq n\leq N}$ where `dt` stands for Δt and `S0` for S_0.

2. Write a function `s_to_a(trajectories)` that takes an array of trajectories as returned by `simul_s` and returns an array of shape `(nb_traj, N+1)` containing the `nb_traj` samples of $(A_n)_{0\leq n\leq N}$ corresponding to the trajectories of $(S_n)_{0\leq n\leq N}$ represented by `trajectories`.

3. Plot a few graphs to check that your functions work.

4. Write a function `montecarlo(trajectories)` that takes trajectories as returned by `simul_s` and approximate (using a Monte-Carlo method) the value of $\mathbb{E}\left[\frac{A_N}{S_N}\right]$. Make sure that your function also provides a confidence interval.

5. *(Math question)* What is the value of the above expectation?

6. Check your code with a numerical application: $S_0 = 10, \sigma = 0.2, \Delta t = \frac{1}{252}, N = 22$.

Strategy and optimization

1. Let $a \geq 1$. We consider

   $$\tau_a = \min\left(\min\{n \in \{0,\ldots,N\}, A_n \geq aS_n\}, N\right)$$

   Write a function `strat_ratio(a, trajectories)` that takes trajectories as returned by `simul_s` and approximates

   $$\mathbb{E}\left[\frac{A_{\tau_a}}{S_{\tau_a}}\right]$$

   using a Monte-Carlo method.

2. Propose a code that tries to find a value of `a` maximizing $\mathbb{E}\left[\frac{A_{\tau_a}}{S_{\tau_a}}\right]$.

3. Illustrate your code with a numerical application: $S_0 = 10, \sigma = 0.2, \Delta t = \frac{1}{252}, N = 22$.

Remark: In the Python notebook, avoid loops as much as you can: you should need none! Also, please use Markdown cells and comments to explain what you did. Use Chat GPT if you wish, but smartly.