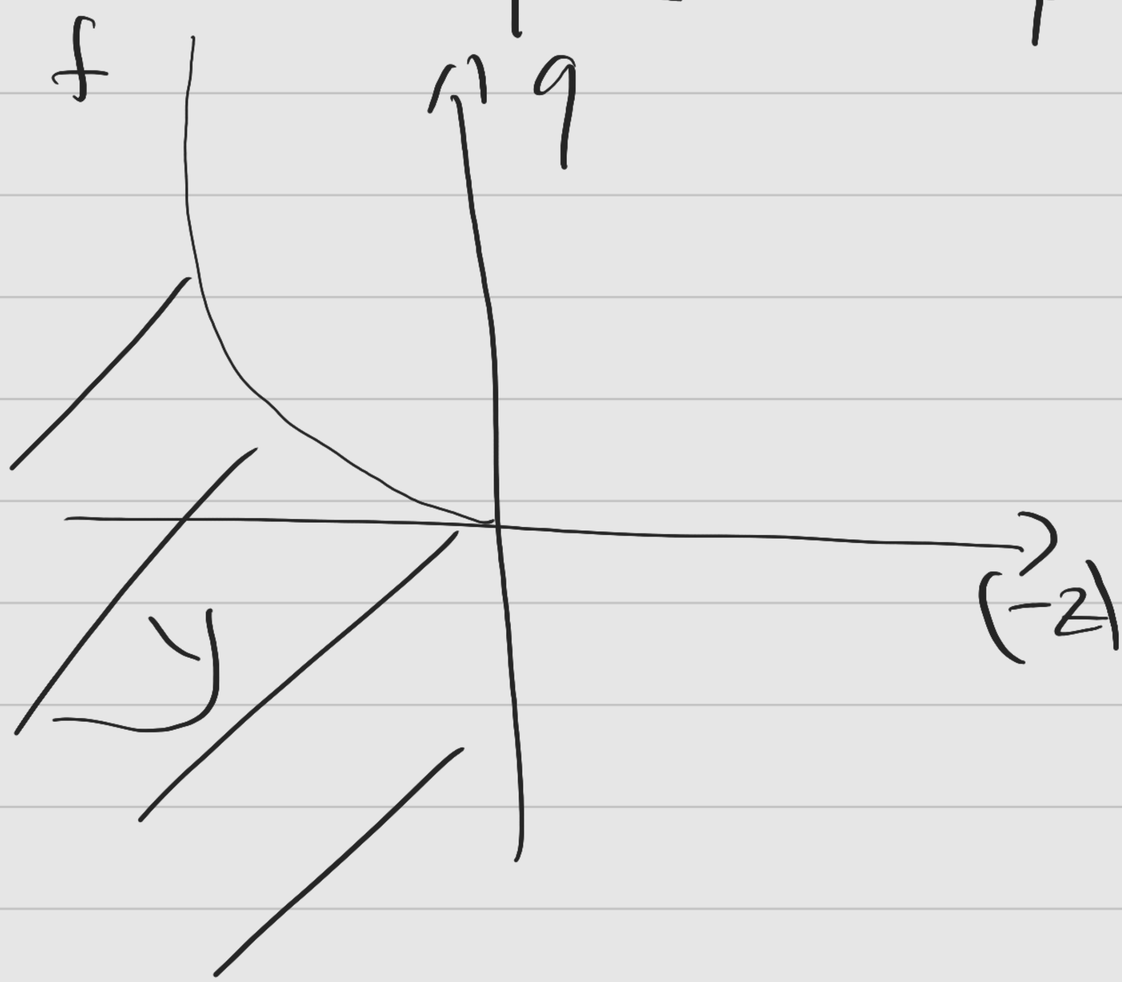
Ex 8 :

$$f(z) = \alpha (z)^2 + \beta z$$

$$\mathcal{Y} = \{(-z, q) \mid z \geq 0 \text{ and } q \leq f(z)\}$$

$$\mathcal{Y} = \{(-z, q) \mid z \geq 0 \text{ and } q \leq \alpha z^2 + \beta z\}$$



+) $0 \in \mathcal{Y}$ $\Rightarrow$ $\mathcal{Y}$ satisfies POI

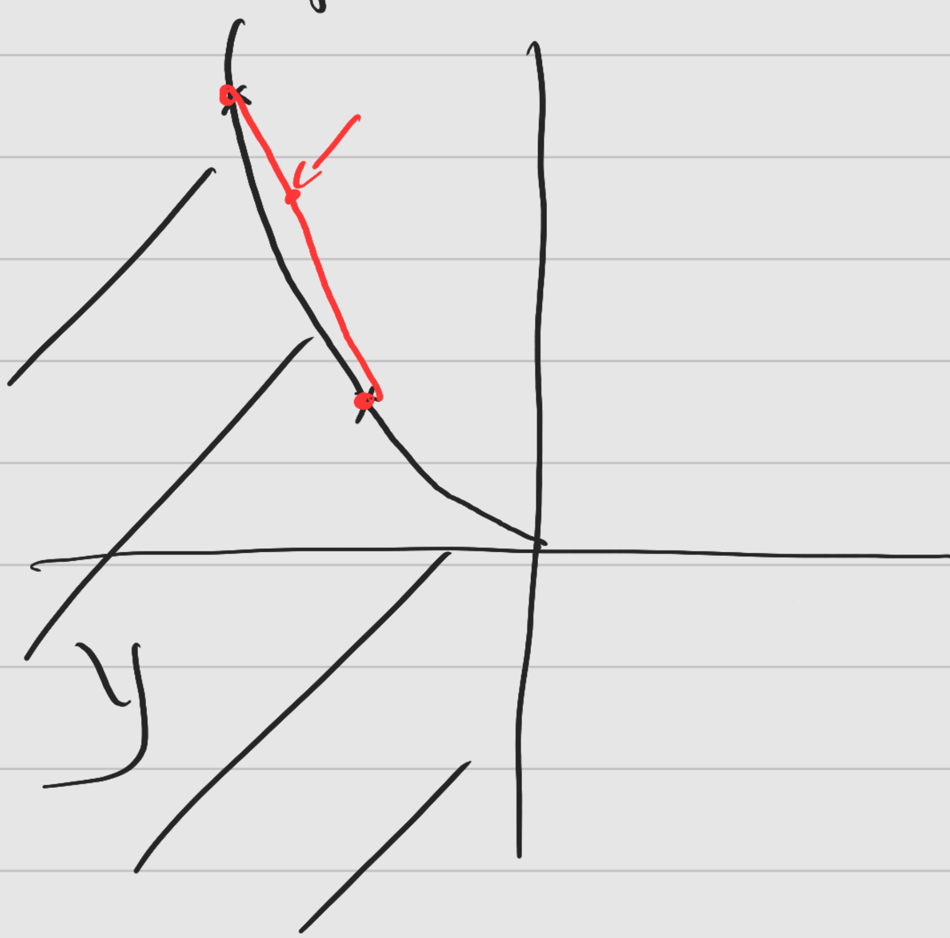
+) If $(-z, q) \in \mathcal{Y} \cap \mathbb{R}^l_+$

$\Rightarrow$ $(-z, q) = (0, 0)$

$\Rightarrow$ It satisfies IOFP

+) It is easily to verify (I) and (FD)

+) Convexity



Consider $(-z, f(z))$, $(-z', f(z'))$.
These 2 points belong to the boundary of $\mathcal{Y}$. $(z \neq z')$

Let $\lambda \in (0,1)$

+) $\lambda(-z, f(z)) + (1-\lambda)(-z', f(z'))$

$$= \left(-\left(\lambda z + (1-\lambda) z\right),\ \lambda f(z) + (1-\lambda) f(z')\right)$$

We have:

$$f(z) = \alpha z^2 + \beta z \quad , \alpha, \beta > 0$$

$$f'(z) = 2\alpha z + \beta$$

$$f''(z) = 2\alpha > 0$$

$\Rightarrow$ f : is a strictly convex function.

$\Rightarrow$ $\lambda f(z) + (1-\lambda) f(z') > f(\lambda z + (1-\lambda) z')$

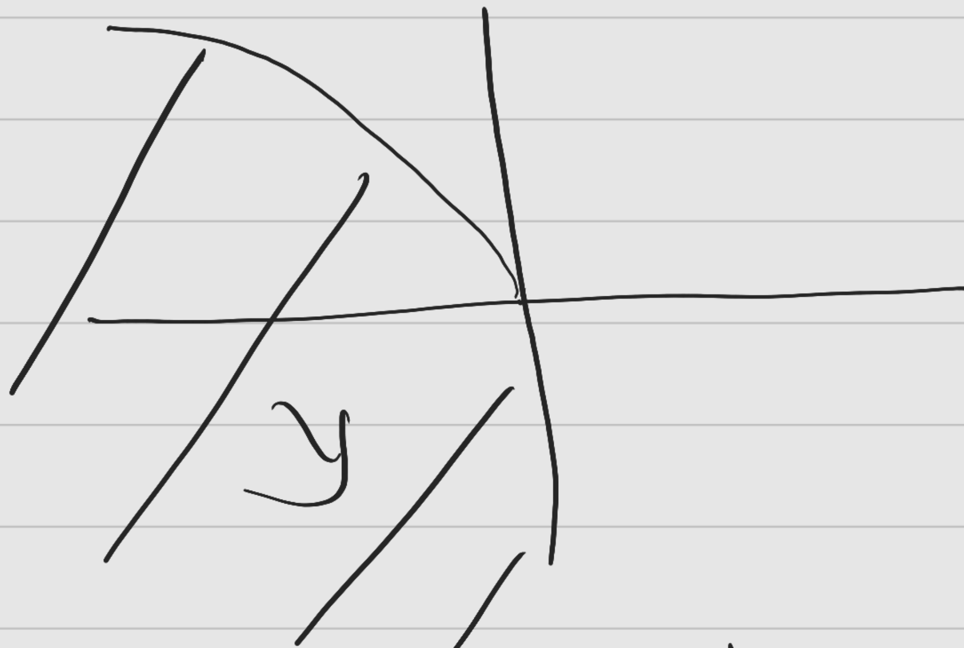
$$\Rightarrow \left(-\left(\lambda z + (1-\lambda) z\right),\ \underbrace{\lambda f(z) + (1-\lambda) f(z')}\right)$$

$$\notin Y$$

$\Rightarrow$ Y is NOT convex

1) $f = \alpha \sqrt{z}$, $\alpha > 0$

Convexity :



Let : $(-z, q)$ and $(-z', q') \in Y$,
i.e.,

$$\begin{cases} z \geqslant 0 \ , \ z' \geqslant 0 \\ q \leq f(z) \\ q' \leq f(z') \end{cases} \qquad (*)$$

Let $\lambda \in \underline{(0,1)}$ / $[0,1]$

$$\lambda(-z, q) + (1-\lambda)(-z', q')$$
$$= \left(-(\lambda z + (1-\lambda) z'), \ \lambda q + (1-\lambda) q'\right)$$

① $\lambda z + (1-\lambda) z' \geqslant 0$ (Because

$z, z' \geq 0$)

② $\lambda q + (1-\lambda) q' \leq f(\lambda z + (1-\lambda) z')$

$f(z) = \alpha\sqrt{z}$

$f'(z) = \dfrac{\alpha}{2\sqrt{z}} = \dfrac{\alpha}{2} \cdot z^{-1/2}$

$f''(z) = \dfrac{\alpha}{2} \cdot \left(-\dfrac{1}{2}\right) z^{-3/2} < 0$

$\Rightarrow$ f is a (strictly) concave functo

We have:

$f(\lambda z + (1-\lambda) z') \geq \lambda f(z) + (1-\lambda) f(z')$

$\geq \lambda q + (1-\lambda) q'$

(Because of (*))

$\Rightarrow (\lambda z + (1-\lambda) z', \lambda q + (1-\lambda) q') \in Y$

$\Rightarrow$ Y is convex.

Ex 10 : L commodities

L : output.

$1, \dots, L-1$: inputs.

$$\boxed{t_f(y) = y_L - f(z)}$$

where $y = (-z_1, \dots, -z_{L-1}, y_L)$

f is concave.

Prove that $t_f : A \longrightarrow \mathbb{R}$ is quasi-convex, where

$$A = \{ y = (-z, y_L) \mid z \geq 0, \; y_L \geq 0 \}$$

$t_f : A \to \mathbb{R}$ is quasi convex if $\{ y \in A \mid t_f(y) \leq \alpha \}$ is convex $\forall \alpha \in \mathbb{R}$

$t_f : A \to \mathbb{R}$ is quasi convex if

$$t_f(\lambda y + (1-\lambda) y') \leq \max \{ t_f(y), t_f(y') \}$$

Let : $(-z, y_L)$, $(-z', y_L') \in A$

and $\lambda \in (0,1)$

$$\lambda(-z, y_L) + (1-\lambda)(-z', y_L')$$

$$= \left(-\left(\lambda z + (1-\lambda) z'\right), \underbrace{\lambda y_L + (1-\lambda) y_L'}_{\equiv\, y^*}\right)$$

$$t_f(y^*) = \lambda y_L + (1-\lambda) y_L' - f\left(\lambda z + (1-\lambda) z'\right)$$

Since f is concave,

$$f(\lambda z + (1-\lambda) z') \geq \lambda f(z) + (1-\lambda) f(z')$$

$$\rightarrow t_f(y^*) \leq \lambda y_L + (1-\lambda) y_L' - \left[\lambda f(z) + (1-\lambda) f(z')\right]$$

$$\Leftrightarrow t_f(y^*) \leq \lambda\left(y_L - f(z)\right) + (1-\lambda)\left(y_L' - f(z')\right)$$

$\Leftrightarrow \ t_f(y^*) \leq \lambda\, t_f(y) + (1-\lambda)\, t_f(y')$

where : $\begin{cases} y = (-z, y_2) \\ y' = (-z', y_2') \end{cases}$

$y^* = \lambda y + (1-\lambda) y'$

$\rightarrow t_f(\cdot)$ is convex

$\Rightarrow t_f(\cdot)$ is quasi-convex

Rk :

A is convex to make sure that

$$y^* = \lambda \underbrace{y}_{\in A} + (1-\lambda) \underbrace{y'}_{\in A} \in A$$

Ex 11 $\quad L = 2$

Commodity 1: input

Commodity 2: output

$$f(z) = \alpha z, \quad \alpha > 0$$

1 The profit maximization problem:

$$\text{Max} \{ p_2 \, q - p_1 z \}$$

st:

$$\begin{cases} z \geq 0 \\ q \leq f(z) \end{cases}$$

where p_1 is the price of input

and p_2 ———————— output

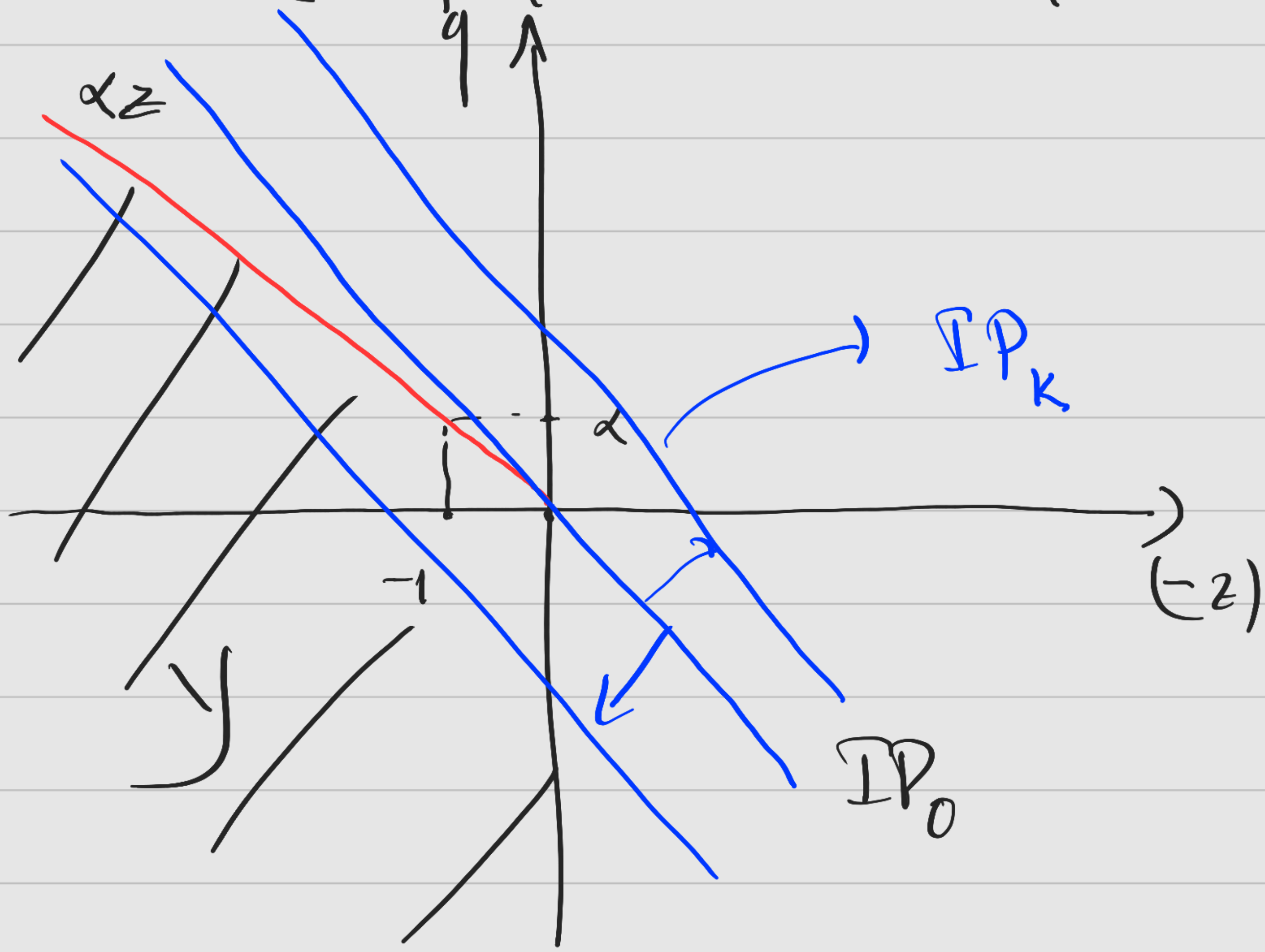
Rewrite:

$$\text{Max} \{ p_2 f(z) - p_1 z \}$$

$$\text{st: } z \geq 0$$

2) $Y = \{ (-z, q) \mid z \geq 0 \text{ and } q \leq f(z) \}$

$Y = \{(-z, q) \mid z \geq 0 \text{ and } q \leq \alpha z\}$



Iso-profit line is a set that all production plans in the set give the same profit.

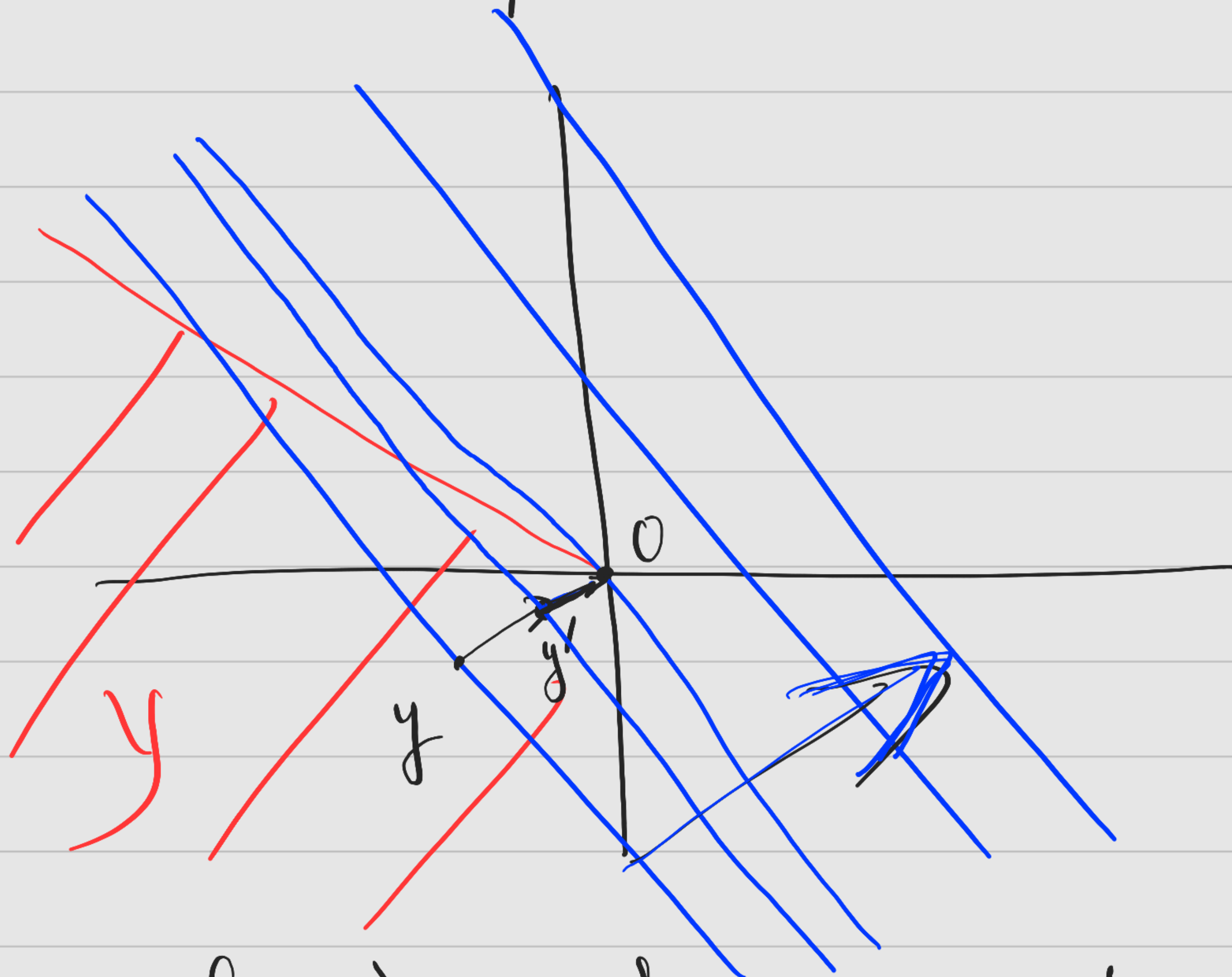
$$IP_K = \{ y \in Y \mid p \cdot y = K \}$$

Iso profit line

$p_1(-z) + p_2 q = K$

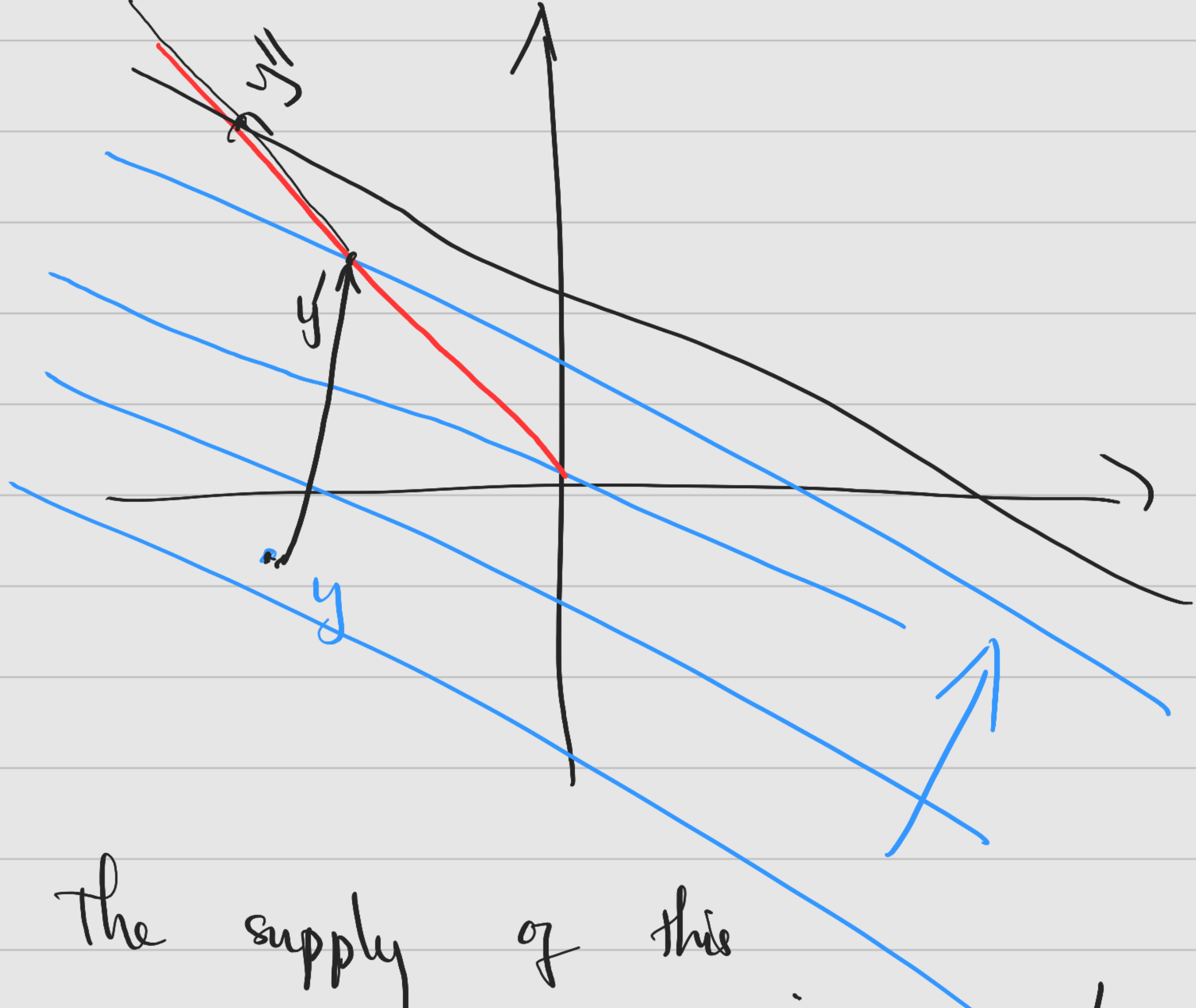
$p_1 y_1 + p_2 y_2 = K$

Case 1 : $-\frac{p_1}{p_2} < -\alpha$



$\Rightarrow$ $(0,0)$ is the optimal plan

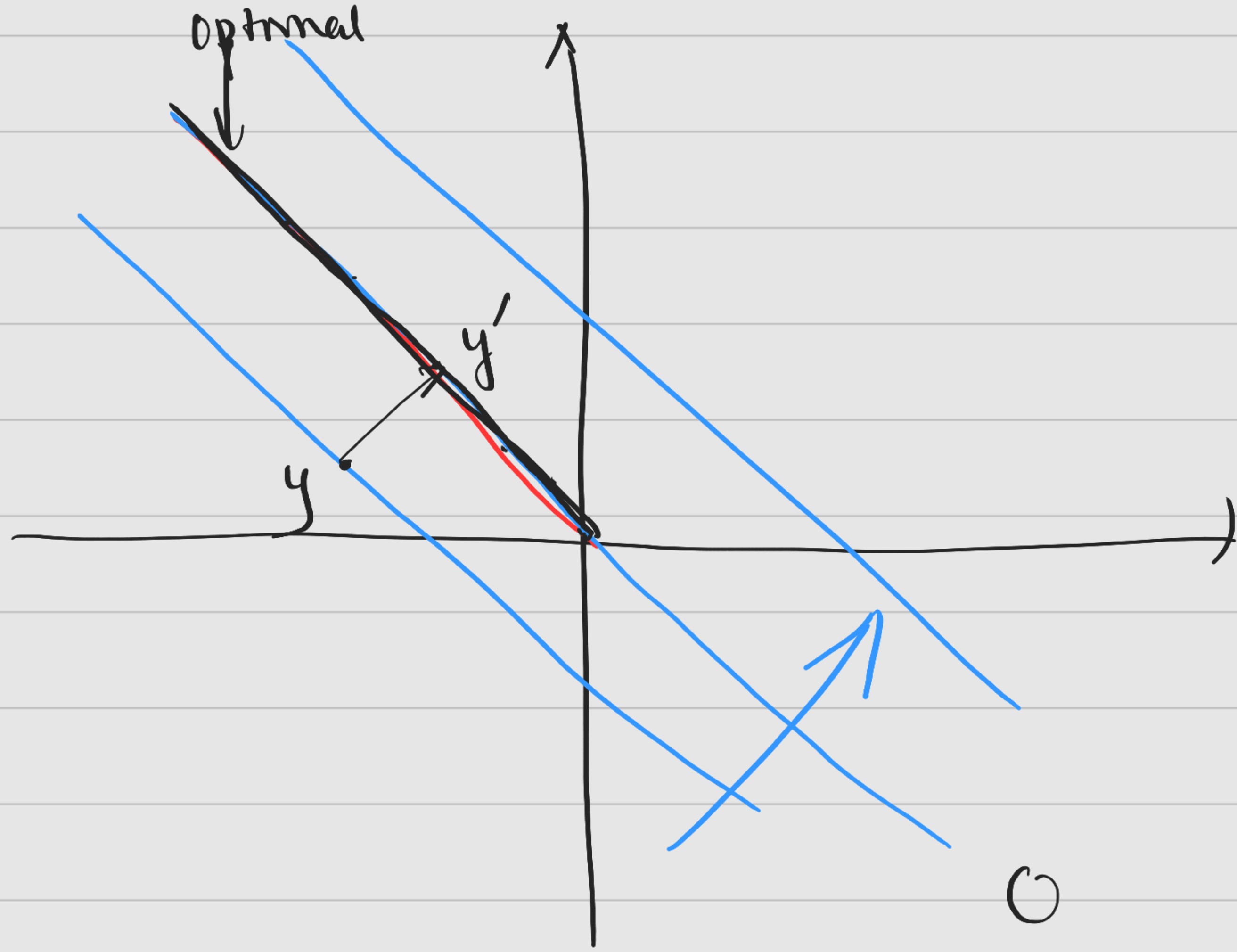
Case 2 : $-\frac{p_1}{p_2} > -\alpha$



The supply of this firm is empty

Case 3:

$$-\frac{p_1}{p_2} = -\alpha$$



$\Rightarrow$ The supply of the firm:

$$\{(-z, q) \mid z \geq 0 \text{ and } q = \alpha z\}$$