Macroeconomics: Economic Growth (Licence 3)
Lesson 4: The Solow Model (Part 3)

Maria Bas

University of Paris 1 Pantheon-Sorbonne

Acknowledges: some slides and figures are taken or adapted from the supplemental ressources to the textbook "Introduction to Economic Growth" by Charles I. Jones and Dietrich Vollrath, Third Edition, Norton W.W. Company Inc.

Economic Growth

Assumptions of the Solow Model with exogenous technological progress

- Perfect competition

- (1) **A production function:** The Cobb-Douglas with exogenous technological progress A

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{1}$$

- Exogenous constant growth of A: $\frac{\dot{A}}{A} = \gamma$
- (2) **A capital accumulation equation:**

$$\dot{K} = sY - \delta K \tag{2}$$

- The growth rate of capital is:

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta. \tag{3}$$

- **Other assumptions:** The growth rate of the number of workers, $\dot{L}/L$ is constant and equal to n
- **Two main diagrams**

Two main figures of the Solow model

- (1) **The Solow Diagram:** determines the level of capital per unit of effective labor in the long run steady state equilibrium
  - based on the dynamic equation of the model: the accumulation of capital per unit of effective labor determines
- (1) **The Transition Dynamics:**
  - based on the growth of capital per unit of effective labor

Economic Growth

- Defining $\hat{k} = \frac{K}{AL}$ and taking the log and derivative with respect to time we obtain **the accumulation of capital per unit of effective labor**

$$ln\hat{k} = lnK - lnA - lnL \tag{4}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \tag{5}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\frac{Y}{K} - \delta - \gamma - n \tag{6}$$

Dividing and multiplying by AL and then replacing $\hat{y} = \hat{k}^{\alpha}$ we get:

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\frac{\hat{k}^{\alpha}}{\hat{k}} - (\delta + \gamma + n) \tag{7}$$

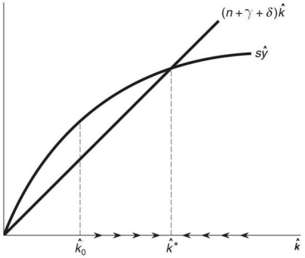
We get the **dynamic equation of capital (per effective units of labor) accumulation in the Solow model**

$$\dot{\hat{k}} = s\hat{k}^{\alpha} - (\delta + \gamma + n)\hat{k} \tag{8}$$

In the long run equilibrium in the steady state, Balance Growth Path (BGP), the $\dot{\hat{k}} = 0$

The Solow Diagram

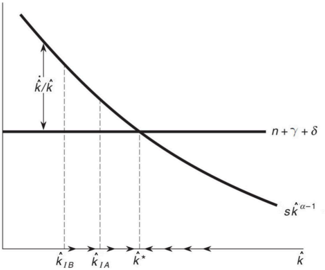
The Solow diagram with technological progress: expressed in **per unit of effective labor (in terms of AL)** $\dot{\hat{k}} = s\hat{k}^{\alpha} - (\delta + \gamma + n)\hat{k}$



- In the long run equilibrium in the steady state, Balance Growth Path (BGP), the $\dot{\hat{k}} = 0 \rightarrow s\hat{k}^{\alpha} = (\delta + \gamma + n)\hat{k}$
- Capital deepening capital per effective worker increases over time when **the amount of investment per effective labor exceeds the amount to keep capital-technology ratio constant** when $s\hat{k}^{\alpha} > (\delta + n + \gamma)\hat{k}$
- Capital widening capital per effective worker declines when $s\hat{k}^{\alpha} < (\delta + n + \gamma)\hat{k}$
- **Transition dynamic plot:** growth rate of capital in effective units of labor

Transition dynamics

Transition dynamics based on growth rate of capital in effective units of labor:
$\frac{\dot{\hat{k}}}{\hat{k}} = s\hat{k}^{\alpha-1} - (\delta + \gamma + n)$



- The first curve is $s\hat{k}^{\alpha-1}$: the higher the level of capital per effective units of labor, the lower the average product of capital (diminishing returns to capital since $\alpha < 1$)
- The second term $(\delta + \gamma + n)$ does not depend on capital so it is a horizontal line.
- The difference between the two lines is the growth rate of capital stock $\frac{\dot{\hat{k}}}{\hat{k}}$.

Long run levels: variables per unit of effective labor and per capita

- Solving the model we find **the steady state equilibrium levels of capital in effective units of labor**
- $\hat{k}^*$ is the value of $\hat{k}$ such that $\dot{\hat{k}} = 0$. So

$$\dot{\hat{k}} = s\hat{k}^{\alpha} - (\delta + \gamma + n)\hat{k} = 0 \quad (9)$$

This solves to

$$(\delta + \gamma + n)\hat{k} = s\hat{k}^{\alpha} \quad (10)$$

$$\hat{k}^{1-\alpha} = \frac{s}{\delta + \gamma + n} \quad (11)$$

$$\hat{k} = \left(\frac{s}{\delta + \gamma + n}\right)^{1/(1-\alpha)} \quad (12)$$

which is the capital per worker at steady state, or

$$\hat{k}^* = \left(\frac{s}{\delta + \gamma + n}\right)^{1/(1-\alpha)} \quad (13)$$

Long run levels: variables per unit of effective labor and per capita

- Solving the model we find **the steady state equilibrium levels of capital and income in effective units of labor**
- which depends now also **on the growth rate of the exogenous technological progress (γ)**

$$\hat{k}^* = \frac{k^*}{A} = (\frac{s}{\delta + \gamma + n})^{\frac{1}{1-\alpha}} \quad (14)$$

$$\hat{y}^* = \frac{y^*}{A} = \frac{k^{*\alpha}}{A} = (\frac{s}{\delta + \gamma + n})^{\frac{\alpha}{1-\alpha}} \quad (15)$$

- We have seen that in the long run $\dot{\hat{k}} = 0$, thus:

$$\frac{\dot{\hat{k}}}{\hat{k}} = 0 \quad (16)$$

- Since $\frac{\dot{\hat{y}}}{\hat{y}} = \alpha \frac{\dot{\hat{k}}}{\hat{k}}$

$$\frac{\dot{\hat{y}}}{\hat{y}} = 0 \quad (17)$$

Long run growth

- **Main difference with the Solow model without technological progress is in the long run growth of per worker variables:**
- We have defined: $\hat{k} = \frac{k}{A} \rightarrow \frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{k}}{k} - \frac{\dot{A}}{A}$
- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{A}}{A} = \gamma$
- The growth rate of capital per worker and income per worker in the long run are different.

Long run growth (per worker variables)

- In the long run (BGP), the growth rate of capital per worker and income per worker are determined by the trend of growth rate of exogenous technological change:

$$\frac{\dot{k}}{k} = \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma \tag{18}$$

- $\dot{y}/y = \gamma$
- Long run growth (aggregate variables): $\frac{\dot{K}}{K}$ and $\frac{\dot{Y}}{Y}$

Long run growth (aggregate variables)

- Since $\hat{k} = \frac{K}{AL}$ thus:

$$\frac{\dot{K}}{K} = \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{L}}{L} + \frac{\dot{\hat{A}}}{\hat{A}} = n + \gamma \tag{19}$$

- Since $\hat{y} = \frac{Y}{AL}$ thus:

$$\frac{\dot{Y}}{Y} = \frac{\dot{\hat{y}}}{\hat{y}} + \frac{\dot{L}}{L} + \frac{\dot{\hat{A}}}{\hat{A}} = n + \gamma \tag{20}$$

Lesson 4:

- **Comparative statics of Solow model with exogenous technological progress**
- Analyze the response of the model to changes in parameters
  - (1) The effects of a change in the population (n) on economic growth
  - (2) The effects of a change in investment (savings, s) on economic growth
- **Results are similar to the model without technological progress**
- (1) and (2) have **ONLY temporary** effects on the GROWTH RATES during the transition dynamics and **permanent** effects on the long run LEVELS

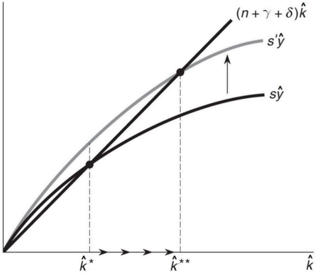
Comparative statics: a change in investment or saving s

- Suppose the economy is at the steady-state, what happens if s changes?
- No impact on long-run growth
- Since $g_y = g_k = \gamma$
- $\rightarrow$ A change in s has NO impact on the long-run growth of y and k
- Since $g_Y = g_K = n + \gamma$
- $\rightarrow$ A change in s has NO impact on the long-run growth of Y and K

Comparative statics: a change in investment or saving s

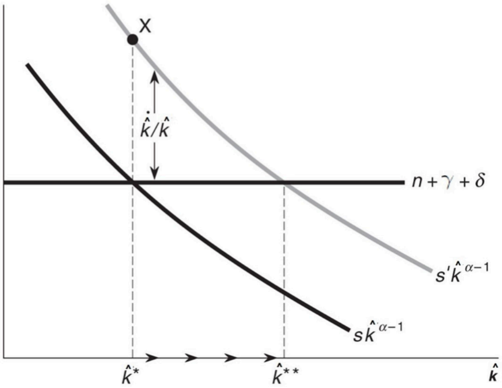
- Suppose the economy is at the steady-state, what happens if s changes?
- Only impacts on levels of y, k
- Since $k^* = A(\frac{s}{\delta+\gamma+n})^{\frac{1}{1-\alpha}}$ and $y^* = A(\frac{s}{\delta+\gamma+n})^{\frac{\alpha}{1-\alpha}}$
- $\rightarrow$ A change in s has an impact on the long-run levels of y and k
- Higher s increases k^* and y^* and $\hat{k}^*$ and $\hat{y}^*$

Comparative statics: a change in investment or saving s

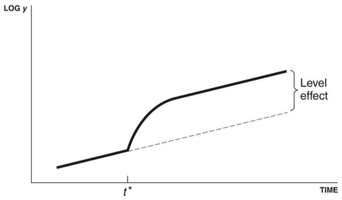


- Suppose the economy is at the steady-state, what happens if s changes?
- Since $\frac{\dot{\hat{k}}}{\hat{k}} = s\hat{k}^{\alpha-1} - (n+\delta+\gamma)$, $\frac{\dot{k}}{k} = \frac{\dot{\hat{k}}}{\hat{k}} + \gamma$ and $\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \gamma$
- $\rightarrow$ A change in s has an impact on the growth rates during transition dynamics of y and k
- A change in s has only temporary effects on the growth rates of k and y and permanent effects on the long run levels of k and y

Comparative statics: a change in investment or saving s



Comparative statics: a change in investment or saving s



- Only impacts on levels of y, k
- Since $k^* = A(\frac{s}{\delta+\gamma+n})^{\frac{1}{1-\alpha}}$ and $y^* = A(\frac{s}{\delta+\gamma+n})^{\frac{\alpha}{1-\alpha}}$
- $\rightarrow$ A change in s has an impact on the long-run levels of y and k
- Higher s increases k^* and y^*

Comparative statics: a change in investment or saving s

- Despite in this version of the Solow model we have the presence of technological change and permanent growth of per capita output,

- changes in investment rates have permanent effects on the levels of per capita output only and temporary effects on the transition growth (no effect on the long run growth)

- **This implies that y grows after the increase in s but then converges on a higher trajectory that is parallel to the previous one.**

Empirical evidence of the Solow model

- Evaluating the Solow model

- How the Solow model answers key questions of economic growth and development?

Empirical evidence of the Solow model

- Evaluating the Solow model
- (1) Why some countries are so rich and other so poor?
- The Solow model predicts that countries that are rich are those that invest more and have lower population growth
- $\rightarrow$ Allowing rich countries to accumulate more capital per worker
- and increase labor productivity.
- $\rightarrow$ It is supported by empirical analysis

Empirical evidence of the Solow model

- Evaluating the Solow model
- How the Solow model answers key questions of economic growth and development?
- (2) Why economies exhibit sustained growth in the Solow model?

Empirical evidence of the Solow model

- Evaluating the Solow model
- (2) Why economies exhibit sustained growth in the Solow model?
- Due to technological progress
- Technological progress offset the tendency for the marginal product of capital to fall (due to diminishing return to capital accumulation)
- **In the long run countries exhibit per capita growth at the rate of technological progress**

Empirical evidence of the Solow model

- Evaluating the Solow model
- (3) How does the Solow model account for differences in growth rates across countries?

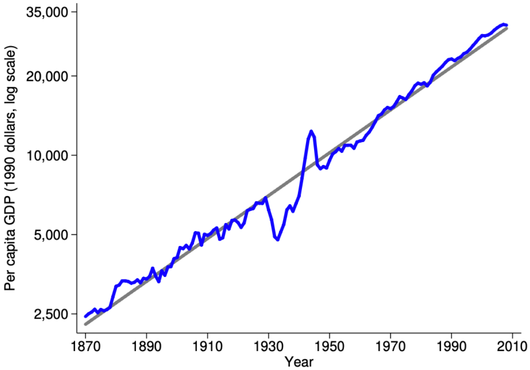
Empirical evidence of the Solow model

- Evaluating the Solow model
- (3) How does the Solow model account for differences in growth rates across countries?
- Based on the transition dynamics of the model
- **countries can grow at rates different from their long run growth rates**
- Example: an economy capital-technology ratio below its long run level will grow rapidly until the capital technology ratio reaches its steady-sate level.
- This explain why the rapid growth of Germany and Japan after the WWII when they had lower levels of capital stocks .

Empirical evidence of the Solow model

- The capital to output ratio is roughly constant over long periods of time
- Shares of capital and labor in national income are roughly stable over long periods of time
- But growth rates of productivity vary among countries.

Economic Growth



Evaluating the Solow model

- Two ways to test empirically the Solow model and compute TFP growth:
  - (1) Estimation of a production function
  - (2) Growth accounting decomposition of TFP growth from growth of the traditional factors
- → Both methods show that the way in which TFP is computed is a "residual", a difference between growth of income and growth of production factors.

TFP estimation

- **Production function estimation**
  - ◇ Firm productivity
  - ◇ TFP estimation
  - ◇
  - "'**Solow residual**"': TFP is calculated as the residual from the production function estimation
  - ◇ Revenue vs. Quantity TFP (efficiency)

Production function estimation

$$Y_{it} = AK_{it}^{\alpha} L_{it}^{\beta} M_{it}^{\lambda} \tag{21}$$

- Taking natural logarithms, we estimate the Cobb-Douglas production function:

$$lnY_{it} = \alpha lnK_{it} + \beta lnL_{it} + \lambda lnM_{it} + \eta_{it} \tag{22}$$

- All variables are expressed in natural logs.
- Y_{it} is the total value of revenues of firm i at time t,
- L_{it} is labor,
- M_{it} is materials,
- L_{it} stands for capital stock

- The residual η_{it} is TFP_R (ln A)

$$TFP_R = \eta_{it} = lnY_{it} - \alpha lnK_{it} - \beta lnL_{it} - \lambda lnM_{it} \tag{23}$$

Production function estimation

- TFP-R vs. TFP-Q
- Using the value of production as dependent variable includes changes in quantity and prices :
- The error term captures changes in prices (and markups) not related to changes in efficiency
- Ideally, we need to use physical quantity as dependent variable (empirical counterpart of Y in the Solow model):
- So we can get quantity based TFP (TFP_Q)

$$lnQ_{it} = \alpha lnK_{it} + \beta lnL_{it} + \lambda lnM_{it} + \eta_{it} \tag{24}$$

$$TFP_Q = \eta_{it} = lnQ_{it} - \alpha lnK_{it} - \beta lnL_{it} - \lambda lnM_{it} \tag{25}$$

Growth accounting

- The second method to compute TFP is based on a simple accounting of the contribution to output growth due to:
  - Factor accumulation (capital and labor) vs.
  - Technological change.

Growth accounting

- Solow published a second paper in 1957 "Technical Change and the Aggregate Production Function"
- He did a simple accounting exercise breaking down the growth of output into
- (1) Growth in capital
- (2) Growth in labor
- (3) Growth in technological change

Growth accounting

- Consider a Cobb-Douglas production function:

$$Y = BK^{\alpha}L^{1-\alpha}$$

- Where *B* is the Hicks-neutral Multi-factor Productivity

- Log-differentiating the above function, yields to:

$$\frac{\dot{Y}}{Y} = \frac{\dot{B}}{B} + \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}$$

- $\rightarrow$ Output growth is equal to a weighted average of capital and labor growth plus the growth of TFP.

- This equation has been used to understand the sources of growth in output.

Growth accounting

- Usually the interest is on output per worker rather than total output, thus dividing by L all variables we get:

$$\frac{\dot{y}}{y} = \frac{\dot{B}}{B} + \alpha \frac{\dot{k}}{k}$$

- $\rightarrow$ The growth rate of output per worker is decomposed into the contribution of physical capital per worker and TFP growth

Growth accounting

- In the empirical exercise, Solow adapts this equation: $\frac{\dot{y}}{y} = \frac{\dot{B}}{B} + \alpha \frac{\dot{k}}{k}$ in different ways to study the the growth accounting decomposition for the US from 1948 to 2010 done by the US Bureau of Labor Statistics (BLS).
- (1) Labor input is computed as total number of hours rather than total employees.
- (2) They add an additional term to that equation: the changing in labor composition that takes into account differences in skill composition in labor force over time.
- (3) For the US α is found to be around $1/3$ thus $1 - \alpha$ is around $2/3$ (can be measured from national accounts data looking at wL/Y)
- Where wL is total income for wage and salary earners and thus the complement to capital income.

Growth accounting

- From $\frac{\dot{y}}{y} = \frac{\dot{B}}{B} + \alpha\frac{\dot{k}}{k}$ they can computed growth in TFP (technological change $\frac{\dot{B}}{B}$) as:
- $\frac{\dot{B}}{B} = \frac{\dot{y}}{y} - \alpha\frac{\dot{k}}{k}$
- They adjust the previous equation to take into account changes in labor composition λ and compute TFP growth as a residual:
- $TFP = \frac{\dot{B}}{B} = \frac{\dot{y}}{y} - \alpha\frac{\dot{k}}{k} - \lambda$

Economic Growth

Annual average growth rates

	1948–2010	1948–73	1973–95	1995–2000	2000–2010
Output per hour	2.6	3.3	1.5	2.9	2.7
Contributions from:					
Capital per hour worked	1.0	1.0	0.7	1.2	1.2
Information technology	0.2	0.1	0.4	0.9	0.5
Other capital services	0.8	0.9	0.3	0.3	0.7
Labor composition	0.2	0.2	0.2	0.2	0.3
Multifactor productivity	1.4	2.1	0.6	1.5	1.3

- Output per hour grew at average annual rate of 2.6 between 1948-2010.
- The contribution from capital per hour was 1 p.p.
- The contribution of changing labor composition was 0.2 p.p
- The difference between output growth and traditional factors is attributed to the contribution of technical progress in TFP: 1.4 p.p. $(= 2.6 - 1.2)$

- The table also shows that GDP growth and its determinants (capital, labor and TFP growth) have change over the periods in the US
- Output growth slowdown determined by productivity growth slowdown in the period 1973-1995
- A rise in output growth as well as productivity growth in the period 1995-2010.
- How can we explain this growth?

Economic Growth

- How can we explain this growth?
- (1) Oil price shocks in 73 and 79 $\rightarrow$ main issue of this explanation is that in real terms oil prices were lower in 80' than before the shock
- (2) Reallocation of ressources across sectors: from manufacturing (with high labor productivity) towards services (with low labor productivity) in 70s and then recover in manufacturing in 80s
- (3) Growth slowdown in the 70s might be related to) slowdown of expenditures in RD in the late 60s

- (4) Some economists highlight the role of information technology
- Information-technology revolution (adoption of computers) can explain both the productivity slowdown as well as recent productivity growth:
- Growth slow temporarily while the factories were adapting to new production techniques and workers learnt to use the new information technology
- The recent productivity boom is then associated with successful widespread adoption of this new technology

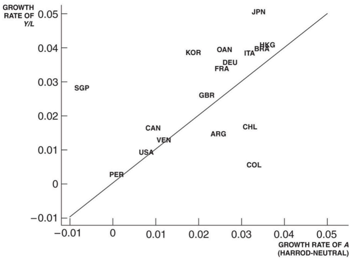
Economic Growth

- Because of the way in which TFP is computed it has been called "the Solow residual"
- It captures the growth in output that can not be explained by capital and labor growth
- Ina. sense it is a black box
- But what determines TFP growth and technological progress?

Growth accounting

- Figure in the next slide uses the growth accounting framework to identify the relative role of:
- factor accumulation vs technical change across countries over the period 1960-1990.
- Labor input is computed as total number of hours rather than total employees.
- Assume that $B = A^{1-\alpha}$
- In the graph, on a BGP output per worker and labor-augmenting technical change would grow at the same rate $(g_y = g_A)$
- and thus observations would lie on the 45 degree line

Economic Growth



- If the actual observation lies above the line
- it means that factor accumulation contributed to growth beyond technical progress
- Example Singapore (SGP) growth in output per worker is much higher than growth in TFP

Growth accounting

- In many fast growing Asian countries: factor accumulation contributed to growth beyond technical progress
- Singapore (SGP) is an extreme example, as growth of A is negative
- Which are the main determinants of growth in South Asian countries?

Growth accounting

- Which are the main determinants of growth in South Asian countries?
- Growth accounting has been applied to the NICs (New Industrialized countries) such as South Korea, Hong Kong, Singapore and Taiwan
- Young (1995) showed that growth in those countries since 60s is related to factor accumulation:
- increases in investment in physical capital and education, labor force participation
- and also reallocation of ressources from agricultural (low productivity) to manufacturing sector (high productivity)