Economic analysis of financial market
S1 2023-2024

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Exchange economy with time and uncertainty

Outline

1. Time and uncertainty
2. Consumers
3. Contingent commodity equilibrium

Contents of the course

1) Exchange economy with time and uncertainty
2) Contingent commodities and Arrow securities
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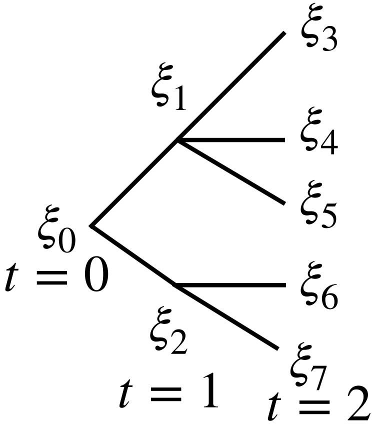
A date - event tree

The uncertainty is described by a date-event tree $\mathbb{D}$ of length $T+1$. The set $\mathbb{D}_t$ is the set of nodes (also called date-events) that could occur at date t and the family $(\mathbb{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathbb{D}$; for each $\xi \in \mathbb{D}$, we denote by $t(\xi)$ the unique date $t \in \mathcal{T}$ such that $\xi \in \mathbb{D}_t$.


ξ_3
ξ_1
ξ_4
ξ_5
ξ_0
$t = 0$
ξ_6
ξ_2
ξ_7
$t = 1$
$t = 2$

At date $t = 0$, there is a unique node ξ_0, that is $\mathbb{D}_0 = \{\xi_0\}$. As $\mathbb{D}$ is a tree, each node ξ in $\mathbb{D} \setminus \{\xi_0\}$ has a unique immediate predecessor denoted *pr*(ξ) or ξ^-. The mapping *pr* maps $\mathbb{D}_t$ to $\mathbb{D}_{t-1}$. Each node $\xi \in \mathbb{D} \setminus \mathbb{D}_T$ has a set of immediate successors defined by $\xi^+ = \{\bar{\xi} \in \mathbb{D} : \xi = \bar{\xi}^-\}$.

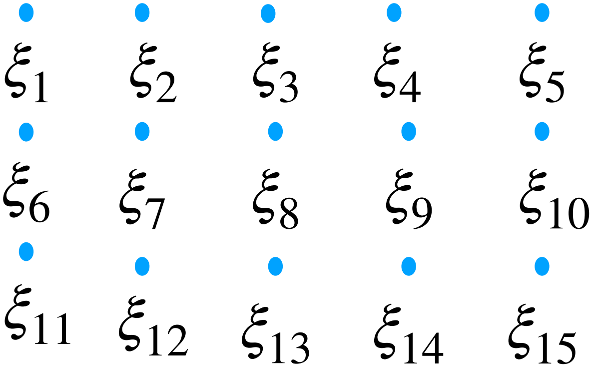
At each period $t \geq 1$, only one node prevails among the immediate successors of the node, which prevailed at the period $t-1$. So, the sequential revelation of uncertainty is represented by a path joining the initial node ξ_0 to a terminal node ξ_T, in such a way that $\xi_{t+1} \in \xi^+(t)$. Hence, at a given period t, the possible future states are only the one of the sub-tree with initial node $\xi(t)$.

$\mathbb{D}^{+}(\xi)$ sub-tree starting at ξ.

$\mathbb{D}^{-}(\xi)$ unique path joining ξ_0 to ξ

Note that $\xi' \in \mathbb{D}^{+}(\xi)$ if and only if $\xi \in \mathbb{D}^{-}(\xi')$ and similarly $\xi' \in \xi^{+}$ if and only if $\xi = (\xi')^{-}$.

Tree and filtration by partitions



Contingent commodities

At each node $\xi \in \mathbb{D}$, there is a finite set of ℓ divisible physical goods available. We assume that each good does not last for more than one period. So a commodity is a couple $(h, \xi))$ of a physical good h and a node $\xi \in \mathbb{D}$ at which it will be available. Hence the commodity space is $\mathbb{R}^{\mathbb{L}}$, where $\mathbb{L} = \ell \times \sharp\mathbb{D}$.

At the initial date 0, a commodity (h, ξ) for the state $\xi \in \mathbb{D}_t$ for $t \geq 1$ is a contingent commodity since it will be available only if ξ prevails at date t.

A consumption or consumption plan is an element $x = (x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$, where the components of $x(\xi)$ are the quantities of the ℓ contingent commodities in the state ξ. For example, to be sure to have one unit of commodity h tomorrow, you need to have one unit of the contingent commodities $(h, \xi)_{\xi \in \mathbb{D}_1}$. So, the consumption plan is such that $x_h(\xi) = 1$ for all $\xi \in \mathbb{D}_1$.

Prices

A spot price vector $p = (p(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ specifies the prices of the commodities in each states.

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The economic agents, called consumers or traders, are in finite number and they are represented by the index i, $i \in \mathcal{I}$. Each agent has a consumption set $X_i \subset \mathbb{R}^{\mathbb{L}}$ and her preferences are represented by a utility function $u_i : X_i \to \mathbb{R}$.
Each agent has also an initial endowments $e_i \in \mathbb{R}^{\mathbb{L}}$, which is a basket of contingent commodities. So the endowments $e_{ih}(\xi)$ for the contingent commodity h at the node ξ is received by the consumer i only if the date-event ξ prevails.

Basic Assumptions

Assumption C. For all $i \in \mathcal{I}$,

a) X_i is nonempty, convex, closed and bounded from below ;

b) u_i is continuous and quasi-concave on X_i.

Assumption S. *(Survival Assumption)* For all $i \in \mathcal{I}$, $e_i \in \mathrm{int} X_i$.

Non satiation at every states

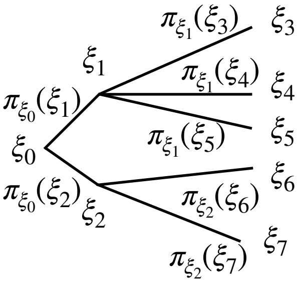
Assumption NSS. *For all* $i \in \mathcal{I}$, *for all* $x_i \in X_i$, *for every* $\xi \in \mathbb{D}$, *there exists a sequence* (x_i^ν) *of* X_i *such that* $x_i(\xi') = x_i^\nu(\xi')$ *for all* $\xi' \neq \xi$, $u_i(x_i^\nu) > u_i(x_i)$ *and* $\lim_{\nu \to \infty} x_i^\nu = x_i$.

Remark

Note that the above assumptions are satisfied when $X_i = \mathbb{R}_+^{\mathbb{L}}$, $e_i \gg 0$ *and the utility functions are strictly increasing, quasi-concave and continuous.*

About probabilities on the tree $\mathbb{D}$

To relate this presentation with the one in Finance where we have a probability space as a primary concept, we can assume that, at each node ξ, there is a probability π_ξ on the set of successors of ξ, ξ^+.



From these transition probability, one can define a probability π_t on $\mathbb{D}_t$, the states at period t, bu the following recursive formula :

$$\pi_t(\xi) = \pi_{pr(\xi)}(\xi)\pi_{t-1}(pr(\xi))$$

and $\pi_0(\xi_0) = 1$.

We have in particular a probability π_T on the terminal node. We can reconstruct the probabilities (π_t) backward, starting from π_T, as follows :

if the probability π_{t+1} on $\mathbb{D}_{t+1}$ is known, for all $\xi \in \mathbb{D}_t$, we define $\pi_t(\xi) = \sum_{\xi' \in \xi^+} \pi_{t+1}(\xi')$.

Note that we can show by induction that these probability is the conditional probability of π_T on the partition $\mathcal{P}_t = \{S_\xi \mid \xi \in \mathbb{D}_t\}$ with $S_\xi = \mathbb{D}^+(\xi) \cap \mathbb{D}_T$, the set of terminal nodes which are successors of ξ.

It is also true that this probability is the conditional probability of π_{t+1} on the set $\mathbb{D}_{t+1}$ on the partition $\mathcal{P}_t^{t+1} = \{\xi^+(\xi) \mid \xi \in \mathbb{D}_t\}$, the set of immediate successors of ξ.

Discounted expected welfare

With this probability, we can define a discounted expected utility u starting from a Bernouilli utility function v defined on a subset Ξ of $\mathbb{R}^\ell$ as follows :

$$u(x) = \sum_{t=0}^{T} \beta^t \sum_{\xi \in \mathbb{D}_t} \pi_t(\xi) v(x(\xi))$$

where $\beta \in]0,1[$ is the discount factor or $\beta = \frac{1}{1+r}$ with r being the interest rate. The global welfare of the agent is measured as the discounted sum of the expected welfare at each period.

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Contingent commodity equilibrium

There is a unique market at state ξ_0 on which all current commodities available at date 0 are exchanged, which is a spot market, but also all contingent commodities (h, ξ) for all commodities and all nodes of the tree $\mathbb{D}$. So, there is $\mathbb{L}$ commodities traded on this market according to a price vector $p \in \mathbb{R}^{\mathbb{L}}$.

Each consumer comes to the market with her current endowments $e_i(\xi_0)$ and their contingent endowments $(e_i(\xi))_{\xi \in \mathbb{D}_1}$. The exchanges take place for the current commodity and contracts are signed for the contingent commodity promising the delivery of a given quantity of a given commodity at node ξ if this node prevails in the future and nothing if this node does not prevail. So, after the market, each consumer has an allocation $x_i^* \in \mathbb{R}^{\mathbb{L}}$.

Contingent commodity equilibrium continued

The exchanges take place according to the market price p^*. The price for the initial node ξ_0 are ordinary prices whereas the prices for the future nodes are future prices with an irrevocable paiement now for a contingent delivery of one unit of the given commodity at the given node in the future. The consumers have access only to the financially affordable consumption, which means those which are in the budget set :

$$B_i^W(p, p \cdot e_i) = \{x_i \in X_i \mid p \cdot x_i \leq p \cdot e_i\}$$

Finally, the market clearing condition imposes that the sum of the final allocations are equal to the sum of the endowments.

Formal definition of a CC equilibrium

Definition

A Contingent Commodity equilibrium of the private ownership exchange economy $\mathcal{E} = (\mathbb{D}, \mathbb{R}^{\mathbb{L}}, (X_i, u_i, e_i)_{i \in \mathcal{I}})$ is an element $((x_i^*), p^*)$ of $(\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}}$ such that
(a) [Preference maximization] for every $i \in \mathcal{I}$,
x_i^* is a "maximal" element of u_i in the budget set $B_i^W(p^*, p^* \cdot e_i)$ in the sense $x_i^* \in B_i^W(p^*, p^* \cdot e_i)$ and
$B_i^W(p^*, p^* \cdot e_i) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;
(b) [attainability]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

Existence of a CC equilibrium

Theorem

Under Assumptions C, S and NSS, there exists a Contingent Commodity Equilibrium.

Optimality of a CC equilibrium

Proposition

(Optimality of the equilibrium allocations) Under Assumptions C and NSS, if $((x_i^*), p^*)$ *is a Contingent Commodity Equilibrium, then* (x_i^*) *is a Pareto optimal allocation, which means that it does not exist an allocation* $(x_i') \in \prod_{i \in \mathcal{I}} X_i$ *such that* $\sum_{i \in \mathcal{I}} x_i' = \sum_{i \in \mathcal{I}} e_i$, $u_i(x_i') \geq u_i(x_i^*)$ *for all* $i \in \mathcal{I}$ *with a strict inequality for at least one consumer.*

Decentralisation of optimal allocation

Proposition

(Decentralisation of optimal allocations) Under Assumptions C and NSS, let $(\bar{x}_i)$ *be a Pareto optimal allocation. Then, there exists a non zero price* $\bar{p}$ *such that (Cost minimisation) for all* $i \in \mathcal{I}$*, for all* $x_i \in X_i$ *such that* $u_i(x_i) \geq u_i(\bar{x}_i)$*,* $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot x_i$*. Furthermore, if there exists a consumption* $\underline{x}_i \in X_i$ *such that* $\bar{p} \cdot \underline{x}_i < \bar{p} \cdot \bar{x}_i$*, then* $\bar{x}_i$ *is a maximal element for* u_i *in the budget set* $B_i^W(\bar{p}, \bar{p} \cdot \bar{x}_i)$*.*

Reopening of the markets

The preferences of the consumers are represented by a discounted average welfare function, that is :

$$u_i(x_i) = \sum_{t=0}^{T} \beta^t \sum_{\xi \in \mathbb{D}_t} \pi_t(\xi) v_i(x_i(\xi))$$

Let us assume that $((x_i^*), p^*)$ is a contingent commodity equilibrium. At date 1, one state $\xi_1 \in \mathbb{D}_1$ prevails. If we reopen the market at this date, the consumers will only consider the subtree $\mathbb{D}(\xi_1)$ of the successors of ξ_1 and the preferences on this new commodity space are represented by :

$$u_i^{\xi_1}(x_i) = \sum_{t=1}^{T} \beta^{t-1} \sum_{\xi \in \mathbb{D}_t \cap \mathbb{D}(\xi_1)} \pi_t(\xi) v_i(x_i(\xi))$$

Reopening of the markets continued

The reopening of the markets does not cancel previous contracts. So, the new initial endowments of the consumers are given by $(\epsilon_i(\xi))_{\xi \in \mathbb{D}(\xi_1)} = (x_i^*(\xi))_{\xi \in \mathbb{D}(\xi_1)}$, which is the sum of the initial endowments and the net trades during the market at date 0.
So, the truncated allocation $(x_i^{*\xi_1})$ where we only keep the consumption for the sub-tree $\mathbb{D}(\xi_1)$ and the corresponding truncated price $p^{*\xi_1}$ are a contingent equilibrium.