Economic analysis of financial market
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General financial structures and financial equilibrium

Outline

1. General financial structures and absence of arbitrage
   - Financial economy and financial equilibrium
   - Different types of assets

Outline

Definition of an asset

The date-event tree is denoted $\mathbb{D}$ over T périods.

Definition

An asset j is issued at a node of issuance $\xi(j)$. It is a contract which promises to deliver a payoff in each state ξ, which is a successor of its issuance node. . The paiement takes place only if state ξ prevails. The payoff may depend on the spot price vector $p \in \mathbb{R}^{\mathbb{L}}$ at the node ξ. The payoff of asset j at node ξ is denoted $v_j(p, \xi)$.

To simplify the notation, we consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and we assume that it is equal to zero if ξ is not a successor of the issuance node $\xi(j)$.

Definition

The vector $V_j(p) = (v_j(p, \xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\sharp \mathbb{D}}$ is called the payoff vector of asset j.

The asset is then represented by a mapping, a random variable, defined on $\mathbb{D}$, $\xi \to v_j(p, \xi)$ depending on the spot prices.

Financial structure

A financial structure is a finite collection $\mathcal{J}$ of assets.

A financial structure is then represented by a mapping $p \to V(p)$ from $\mathbb{R}^{\mathbb{L}}$ to the set of $\sharp\mathbb{D} \times \mathcal{J}$-matrices :

$$V(p) = (v_j(p, \xi))_{\xi \in \mathbb{D}, j \in \mathcal{J}}$$

The entry on the column j and the row ξ, $v_j(p, \xi)$ is the payoff of Asset j at the state ξ if the spot price is p, which takes place only if state ξ prevails. For a given price p, $V(p)$ is called the payoff matrix of the financial structure.

Note that the first row of the matrix V has only 0 entries. A portfolio z is an element in $\mathbb{R}^{\mathcal{J}}$.

Example of a payoff matrix with 3 assets

Let $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}\}$ with $\xi_0^+ = \{\xi_1, \xi_2\}$, $\xi_1^+ = \{\xi_{11}\}$ and $\xi_2^+ = \{\xi_{21}\}$.
We consider the financial structure with three assets $\mathcal{J} = \{j^1, j^2, j^3\}$, $\xi(j^1) = \xi_0$, $\xi(j^2) = \xi_1$ and $\xi(j^3) = \xi_2$. The payoff matrix is

$$\mathbf{V} = \begin{bmatrix} \mathbf{0} & 0 & 0 \\ 1 & \mathbf{0} & 0 \\ -1 & 0 & \mathbf{0} \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{21} \end{matrix}$$

Financial market

The assets are traded on a financial market only at the issuance node and the asset price vector in $\mathbb{R}^{\mathcal{J}}$ is denoted q. A portfolio is a vector $z = (z_j)_{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$. The payoff of a portfolio z at state $\xi \in \mathbb{D}$ is $\sum_{j\in\mathcal{J}} z_j v_j(p,\xi)$. Globally, the return of a portfolio is the vector $(\sum_{j\in\mathcal{J}} z_j v_j(p,\xi))_{\xi\in\mathbb{D}} \in \mathbb{R}^{\mathbb{D}}$. One checks that it is equal to $V(p)z$, the image of the portfolio z by the payoff matrix $V(p)$.

Full payoff matrix

The cost for buying a quantity z_j of asset j is $q_j z_j$ paid at the node of issuance $\xi(j)$.

For a given spot price p and a given asset price q, the full payoff matrix $W(p,q)$ is the $\mathbb{D} \times \mathcal{J}$-matrix defined by

$$W_\xi^j(q) \;:= v^j(p,\xi) - \delta_{\xi,\xi(j)} q_j,$$

where $\delta_{\xi,\xi'} = 1$ if $\xi = \xi'$ and 0 otherwise.

Marketable payoffs

For a given spot price p and a given asset price q, an element r of $\mathbb{R}^{\mathbb{D}}$, which is affordable by the financial structure, that is, for which there exists $z \in \mathbb{R}^{\mathcal{J}}$ such that $r = W(p, q)z$, is called a marketable payoff. The set of marketable payoffs is nothing else than the range of the full payoff matrix. It is a linear subspace of $\mathbb{R}^{\mathbb{D}}$.

Remark on re-trading and derivative

The re-trading of an asset is equivalent to the issuance of a new asset at the node of re-trading. The payoff of the new asset is the payoff of the initial asset restricted to the successors of the node of re-trading.

The derivative of an asset j is a new asset. The payoff of the derivative is a function of the payoff of the initial asset j.

Example : A call option is a contract between a buyer and a seller to purchase a certain stock at a certain price up until a defined expiration date.

Financial economy

A financial economy is the combination of an exchange economy $(X_i, u_i, e_i)_{i \in \mathcal{I}}$ *with a financial structure represented by its payoff matrix* V*.*

Nevertheless we add an additional component to take into account the fact that the economic agents may not be allowed to own all portfolios in $\mathbb{R}^{\mathcal{J}}$*. So, we denote by* Z_i *the portfolio set of agent* i *which is a subset of* $\mathbb{R}^{\mathcal{J}}$*.*

Definition

When $Z_i = \mathbb{R}^{\mathcal{J}}$ for all i, we say that the financial structure is unconstrained.

Basic assumption on the financial structure

Assumption F : *for each* $j \in \mathcal{J}$ *and* $\xi \in \mathbb{D}$, $v_j(\cdot, \xi)$ *is a continuous function from* $\mathbb{R}^{\mathbb{L}}$ *to* $\mathbb{R}$ *and there is no trivial asset such that* $V_j(p) = 0$ *for all* p.

Budget constraints

In presence of spot markets at each node with the price $p \in \mathbb{R}^{\mathbb{L}}$ and financial markets at the issuance nodes with the price q, the affordable consumptions of Consumer i are the elements of the budget set $B_i^{\mathcal{F}}(p, q)$ defined as :

$$\left\{ x_i \in X_i \,\middle|\, \begin{array}{l} \exists z_i \in Z_i, \forall \xi \in \mathbb{D} \\ p(\xi) \cdot x_i(\xi) + \sum_{j | \xi(j) = \xi} q_j z_{ij} \leq p(\xi) \cdot e_i(\xi) + V(p, \xi) \cdot z_i \end{array} \right\}$$

Compact presentation of budget constraints

$$\begin{pmatrix} p(\xi_0)\cdot(x_i(\xi_0)-e_i(\xi_0)) \\ (p(\xi)\cdot(x_i(\xi)-e_i(\xi)))_{\xi\in\mathbb{D}_1} \end{pmatrix} \leq W(p,q)z_i$$

To simplify the notation, if x and p are vectors of $\mathbb{R}^{\ell\sharp\mathbb{D}}$, the box-product is defined by :

$$p \,\square\, x = (p(\xi)\cdot x(\xi))_{\xi\in\mathbb{D}} \in \mathbb{R}^{\sharp\mathbb{D}}$$

So, the budget constraints are :

$$p \,\square\, (x_i - e_i) \leq W(p,q)z_i$$

Financially affordable consumptions

Definition

For a pair of spot-asset price vectors (p, q), we say that the consumption x_i is financially affordable by the portfolio $z_i \in \mathbb{R}^{\mathcal{J}}$ if $p \,\square\, x_i \leq W(p, q) z_i$.

Financial equilibrium

Definition

Let us consider a financial economy $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$.
A financial equilibrium of $\mathcal{E}_{\mathcal{F}}$ is an element $((x_i^*, z_i^*), p^*, q^*)$ of $(\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ such that
(a) [Preference maximization] for every $i \in \mathcal{I}$,
x_i^* is a “maximal” element of u_i in the budget set $B_i^{\mathcal{F}}(p^*, q^*)$ in the sense that $z_i^* \in Z_i$, for all $\xi \in \mathbb{D}$,

$$p^*(\xi) \cdot x_i^*(\xi) + \sum_{j \mid \xi(j) = \xi} q_j^* z_{ij}^* \leq p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot z_i^*$$

and $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;

Definition continued

(b) [Market clearing condition on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

(c) [Market clearing condition on the financial markets]

$$\sum_{i \in \mathcal{I}} z_i^* = 0.$$

First property

For all $i \in \mathcal{I}$ and for all $\xi \in \mathbb{D}$, the budget constraints are binding, that is,

$$p^*(\xi) \cdot x_i^*(\xi) + \sum_{j|\xi(j)=\xi} q_j^* z_{ij}^* = p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot z_i^*$$

Contingent commodities and Arrow securities

The contingent commodity (h, ξ) and the Arrow security j^ξ associated to node ξ presented in the previous chapters are particular cases of an asset. Indeed, the issuance node is the initial node ξ_0.

The payoff of the contingent commodity (h, ξ) is given by the following vector in $\mathbb{R}^{\mathbb{D}_1}$:
$v_{(h,\xi)}(p, \xi') = p_h(\xi)$ if $\xi = \xi'$ and 0 otherwise.
The payoff of the Arrow security associated to node ξ is given by the following vector in $\mathbb{R}^{\mathbb{D}_1}$:
$v_{j^\xi}(p, \xi') = 1$ if $\xi = \xi'$ and 0 otherwise.
The payoff vectors of the Arrow securities are the elements of the canonical basis of $\mathbb{R}^{\mathbb{D}}$ except the first one associated to the initial node ξ_0.

Properties of the payoff matrices

We can check that the payoff matrices of the two financial structures composed of all contingent commodities or to all Arrow securities have the same range when the spot prices are non zero in each state. Actually, the range is equal to

$$\{r \in \mathbb{R}^{\sharp \mathbb{D}} \mid r_{\xi_0} = 0\}$$

Real assets

A real asset j issued at the node $\xi(j)$ is described by a basket of commodities $r_j(\xi) \in \mathbb{R}^\ell$ at each node $\xi \in \xi(j)^+$. which can be gathered in a $\sharp\mathbb{D}^{\times}\ell$ real return matrix R_j. A row for $\xi \notin \xi(j)^+$ is equal to 0_ℓ.

Then the return of this real asset in state ξ is the value of the basket of commodity $r_j(\xi)$ for the spot price $p(\xi)$, that is $V_j(p,\xi) = p(\xi) \cdot r_j(\xi)$ or $V_j(p) = (p(\xi) \cdot r_j(\xi))_{\xi \in \mathbb{D}}$.

Examples

Two real assets associated to the two commodities in a two period, two states, two commodities model.

Note that a contingent commodity (h, ξ) *is a real asset where* $r_{(h,\xi)}(\xi')$ *is the h-th vector of the canonical basis of* $\mathbb{R}^\ell$ *when* $\xi' = \xi$ *and* 0 *otherwise.*

Future contract

A future contract j^h for a commodity $h \in \mathbb{R}^\ell$ is a real asset issued at a node $\xi(j^h)$ at date t, which promises to deliver the value of one unit of commodity h in each state of nature $\xi \in \mathbb{D}_\tau \cap \mathbb{D}^+(\xi(j^h))$, $\tau > t$. So, it is defined by $r_{j^h}(\xi')$ is the h-th vector of the canonical basis of $\mathbb{R}^\ell$ for all $\xi' \in \mathbb{D}_\tau \cap \mathbb{D}^+(\xi(j^h))$.

Numeraire asset

$\nu \in \mathbb{R}^\ell$ is the numéraire basket of commodities.

A numéraire asset j is a real asset issued at a node $\xi(j^h)$, which promises to deliver the value of $\rho_j(\xi)$ units of the numéraire ν in each state of nature $\xi \in \mathbb{D}^+(\xi(j^h))$, that is, the payoff at note ξ is $\rho_j(\xi)p(\xi) \cdot \nu$.

The return is the value of $\rho_j(\xi)$ unit of the numéraire at the spot price. So, the numéraire asset j is defined by the matrix $r_j(\xi') = \rho_j(\xi')\nu$ for all $\xi' \in \mathbb{D}^+(\xi(j))$.

Nominal asset

The payoffs of a nominal asset are expressed in terms of a unit of account in each state. This means that the vector of payoffs is a constant vector V_j which does not depend on the spot price p.

A bond issued at node ξ_0, which promises to deliver one unit of account in each state at a future date $t > 0$ is a typical example of a nominal asset. In this case V_j is the vector of $\mathbb{R}^{\mathbb{D}}$, which coordinates are all equal to 1 for the node $\xi \in \mathbb{D}_t$ and 0 for the other nodes.