# Master MMMEF, 2023-2024
# Lectures notes on:
# General Equilibrium Theory:
# Economic analysis of financial markets[1]

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# Chapter 4

# General financial structures

In this section, we present the concept of a financial structure, which is closer to what we observe on the financial market. To maintain a reasonable level of complexity for this first course on economic analysis of financial markets, we assume from now on that the time horizon is limited to two dates, which means that the date-event tree is defined for the dates $t = 0$ and 1 and the tree is limited to the initial node $\xi_0$ and its successors $\xi \in \mathbb{D}_1$.

## 4.1 Definition of a financial asset

A financial structure is a finite collection $\mathcal{J}$ of assets.

**Définition 3** An asset $j$, $j \in \mathcal{J}$, is a contract which promises to deliver a payoff in each state $\xi$ of the period $t = 1$. The paiement takes place only if state $\xi$ prevails. The payoff may depend on the spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. It is denoted $v_j(p, \xi)$. The vector $V_j(p) = (v_j(p, \xi))_{\xi \in \mathbb{D}_1} \in \mathbb{R}^{\sharp \mathbb{D}_1}$ is called the payoff vector of Asset $j$.

The asset is then represented by a mapping, a random variable, defined on $\mathbb{D}_1$, $\xi \to v_j(p, \xi)$ depending on the spot prices.

A financial structure is then represented by a mapping $p \to V(p)$ from $\mathbb{R}^{\mathbb{L}}$ to the set of $\sharp \mathbb{D}_1 \times \mathcal{J}$-matrices:

$$V(p) = (v_j(p, \xi))_{\xi \in \mathbb{D}_1, j \in \mathcal{J}}$$

The entry on the column $j$ and the row $\xi$, $v_j(p, \xi)$ is the payoff of Asset $j$ at the state $\xi$ if the spot price is $p$, which takes place only if state $\xi$ prevails. For a given price $p$, $V(p)$ is called the payoff matrix of the financial structure.

The assets are traded on a financial market at the initial node $\xi_0$ and the asset price vector in $\mathbb{R}^{\mathcal{J}}$ is denoted $q$. A portfolio is a vector $z = (z_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$. The payoff of a portfolio $z$ at state $\xi \in \mathbb{D}_1$ is $\sum_{j \in \mathcal{J}} z_j v_j(p, \xi)$. Globally, the return of a portfolio is the vector $(\sum_{j \in \mathcal{J}} z_j v_j(p, \xi))_{\xi \in \mathbb{D}_1} \in \mathbb{R}^{\mathbb{D}_1}$. One checks that it is equal to $V(p)z$, the image of the portfolio $z$ by the payoff matrix $V(p)$.

The cost of a portfolio is simply $\sum_{j \in \mathcal{J}} z_j q_j = q \cdot z$, which is paid at date 0.

For a given spot price $p$ and a given asset price $q$, the full payoff matrix $W(p, q)$ is the $\mathbb{D} \times \mathcal{J}$-matrix defined by

$$W(p, q) = \begin{pmatrix} -q \\ V(p) \end{pmatrix}$$

For a given spot price $p$ and a given asset price $q$, an element $r$ of $\mathbb{R}^{\mathbb{D}}$, which is affordable by the financial structure, that is, for which there exists $z \in \mathbb{R}^{\mathcal{J}}$ such that $r = W(p, q)z$, is called a marketable payoff. The set of marketable payoffs is nothing else than the range of the full payoff matrix. It is a linear subspace of $\mathbb{R}^{\mathbb{D}}$.

## 4.2 Financial economy and financial equilibrium

A financial economy is the combination of an exchange economy $(X_i, u_i, e_i)_{i \in \mathcal{I}}$ with a financial structure represented by its payoff matrix $V$. Nevertheless we add an additional component to take into account the fact that the economic agents may not be allowed to own all portfolios in $\mathbb{R}^{\mathcal{J}}$. So, we denote by $Z_i$ the portfolio set of agent $i$ which is a subset of $\mathbb{R}^{\mathcal{J}}$. When $Z_i = \mathbb{R}^{\mathcal{J}}$ for all $i$, we say that the financial structure is unconstrained. Nevertheless since the very beginning of this theory as in Radner [18], some authors consider portfolio constraints like short sale constraints, which means that $z_j$ is bounded from below.

So a financial economy is a collection

$$\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$$

We posit a simple basic assumption on $V$:

**Assumption F:** for each $j \in \mathcal{J}$ and $\xi \in \mathbb{D}_1$, $v_j(\cdot, \xi)$ is a continuous function from $\mathbb{R}^{\mathbb{L}}$ to $\mathbb{R}$ and there is no trivial asset such that $V_j(p) = 0$ for all $p$.

In presence of spot markets at each node with the price $p \in \mathbb{R}^{\mathbb{L}}$ and a financial market at node $\xi_0$ with the price $q$, the affordable consumptions of Consumer $i$ are the elements of the budget set $B_i^{\mathcal{F}}(p, q)$ defined as:

$$\left\{ x_i \in X_i \,\middle|\, \exists z_i \in Z_i \begin{array}{l} p(\xi_0) \cdot x_i(\xi_0) + q \cdot z_i \leq p(\xi_0) \cdot e_i(\xi_0) \\ p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi) + V(p, \xi) \cdot z_i, \quad \forall \xi \in \mathbb{D}_1 \end{array} \right\}$$

As in the model with Arrow securities or with pure spot markets, the consumers face $\sharp\mathbb{D}$ budget constraints, one for each date-event. These inequalities can be summarised using the full payoff matrix $W(p, q)$ as follows:

$$\begin{pmatrix} p(\xi_0) \cdot (x_i(\xi_0) - e_i(\xi_0)) \\ (p(\xi) \cdot (x_i(\xi) - e_i(\xi)))_{\xi \in \mathbb{D}_1} \end{pmatrix} \leq W(p, q) z_i$$

To simplify the notation, if $x$ and $p$ are vectors of $\mathbb{R}^{\mathbb{L}}$, the box-product is defined by:

$$p \,\square\, x = (p(\xi) \cdot x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\sharp \mathbb{D}}$$

So, the budget constraints are:

$$p \,\square\, (x_i - e_i) \leq W(p, q) z_i$$

**Définition 4** For a pair of spot-asset price vectors $(p, q)$, we say that the consumption $x_i$ is financially affordable by the portfolio $z_i \in \mathbb{R}^{\mathcal{J}}$ if $p \,\square\, x_i \leq W(p, q) z_i$.

As in the usual definition of a competitive equilibrium, the consumers are assuming to take the price as given and to maximise the utility function over the financial budget set. So, we get the following definition of a financial equilibrium.

**Définition 5** Let us consider a financial economy $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$. A financial equilibrium of $\mathcal{E}_{\mathcal{F}}$ is an element $((x_i^*, z_i^*), p^*, q^*)$ of $(\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ such that

(a) [Preference maximization] for every $i \in \mathcal{I}$,

$x_i^*$ is a "maximal" element of $u_i$ in the budget set $B_i^{\mathcal{F}}(p^*, q^*)$ in the sense that $z_i^* \in Z_i$ and

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot z_i^* \leq p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \leq p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot z_i^*, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

and $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;

(b) [Market clearing condition on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

(c) [Market clearing condition on the financial markets]

$$\sum_{i \in \mathcal{I}} z_i^* = 0.$$

Note that the market clearing condition on the spot markets integrates the fact that there is no storage technology or durable commodities. The market clearing condition on financial markets means that there is no net supply. From these two conditions, we can check that for all $i \in \mathcal{I}$ and for all $\xi \in \mathbb{D}$, the budget constraints are binding, that is,

$$\begin{cases} p(\xi_0) \cdot x_i(\xi_0) + q \cdot z_i = p(\xi_0) \cdot e_i(\xi_0) \\ p(\xi) \cdot x_i(\xi) = p(\xi) \cdot e_i(\xi) + V(p, \xi) \cdot z_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

By summing the budget constraints over the set of consumers, we get for all $\xi \in \mathbb{D}_1$,

$$\begin{aligned} 0 \quad & \geq \textstyle\sum_{i \in \mathcal{I}} p(\xi) \cdot (x_i(\xi) - e_i(\xi)) - V(p, \xi) \cdot z_i \\ & = p(\xi) \cdot \left(\textstyle\sum_{i \in \mathcal{I}} (x_i(\xi) - e_i(\xi))\right) - V(p, \xi) \cdot \textstyle\sum_{i \in \mathcal{I}} z_i \\ & = 0 \end{aligned}$$

So, all non-positive terms are equal to 0, which means that the budget constraints are binding. The argument is the same for the initial node $\xi_0$.

## 4.3 Different types of assets

Note first that the contingent commodity $(h,\xi)$ and the Arrow security $j^\xi$ associated to node $\xi$ presented above are particular cases of an asset. Indeed, the payoff of the contingent commodity $(h,\xi)$ is given by the following vector in $\mathbb{R}^{\mathbb{D}_1}$: $v_{(h,\xi)}(p,\xi') = p_h(\xi)$ if $\xi = \xi'$ and 0 otherwise. The payoff of the Arrow security associated to node $\xi$ is given by the following vector in $\mathbb{R}^{\mathbb{D}_1}$: $v_{j^\xi}(p,\xi') = 1$ if $\xi = \xi'$ and 0 otherwise. The payoff vectors of the Arrow securities are the element of the canonical basis of $\mathbb{R}^{\mathbb{D}_1}$.

We can check that the payoff matrices of the two financial structures composed of all contingent commodities or to all Arrow securities have the same range when the spot prices are non zero in each state. Actually, the range is equal to $\mathbb{R}^{\sharp\mathbb{D}_1}$.

Let us now consider a first category of assets called *real assets*. A real asset $j$ is described by a basket of commodities $r_j(\xi) \in \mathbb{R}^\ell$ at each node, which can be gathered in a $\sharp\mathbb{D}_1 \times \ell$ real return matrix $R_j$. Then the return of this real asset in state $\xi$ is the value of the basket of commodity $r_j(\xi)$ for the spot price $p(\xi)$, that is $V_j(p,\xi) = p(\xi) \cdot r_j(\xi)$ or $V_j(p) = (p(\xi) \cdot r_j(\xi))_{\xi \in \mathbb{D}_1}$.

Note that a contingent commodity $(h,\xi)$ is a real asset where $r_{(h,\xi)}(\xi')$ is the $h$-th vector of the canonical basis of $\mathbb{R}^\ell$ when $\xi' = \xi$ and 0 otherwise.

A future contract $j^h$ for a commodity $h \in \mathbb{R}^\ell$ is a real asset which promises to deliver the value of one unit of commodity $h$ in each state of nature $\xi \in \mathbb{D}_1$. So, it is defined by $r_{j^h}(\xi')$ is the $h$-th vector of the canonical basis of $\mathbb{R}^\ell$ for all $\xi' \in \mathbb{D}^1$.

A *numéraire asset* is a generalisation of the two previous examples. It is a special kind of a real asset. We take a given vector $\nu \in \mathbb{R}^\ell$ which is the numéraire basket of commodities. Then, the returns of a numéraire asset $j$ is determined by a vector $\rho_j \in \mathbb{R}^{\mathbb{D}_1}$ and, at state $\xi$, it is equal to $\rho_j(\xi)p(\xi) \cdot \nu$. The return is the value of $\rho_j(\xi)$ unit of the numéraire at the spot price. So, the numéraire asset $j$ is defined by $r_j(\xi') = \rho_j(\xi')\nu$ for all $\xi' \in \mathbb{D}^1$.

A *Nominal asset* is in some sense the most simple type of asset since the payoff are expressed in terms of a unit of account in each state. This means that the vector of payoffs is a constant vector $V_j$ which does not depend on the spot price $p$. A bond which promises to deliver one unit of account in each state is a typical example of a nominal asset. In this case $V_j$ is the vector of $\mathbb{R}^{\mathbb{D}_1}$ which coordinates are all equal to 1.

# Chapter 5

# Arbitrage

The absence of arbitrage opportunities is a fundamental property of the asset prices on a financial market. It comes from the fact that there is no free lunch on the markets which means that we cannot have a non-zero non-negative returns in each state of nature including the current date. In other words, a portfolio with non-zero non-negative returns in each state tomorrow has a positive cost today. This remark comes from the fact that free lunch is incompatible with an equilibrium since demand for an arbitrage portfolio would be infinite. We will precise these statements below.

## 5.1 Characterisation of arbitrage free financial structures

We consider an unconstrained financial structure represented by the return matrix $V$ and portfolios sets $Z_i = \mathbb{R}^{\mathcal{J}}$ for every $i$. Let $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ be a pair of spot and asset price vectors.

**Définition 6** The financial structure is arbitrage free at $(p, q)$ if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p, q)z \in \mathbb{R}^{\mathbb{D}}_{+} \setminus \{0\}$.

If we decompose the above formula, it means that it does note exist an arbitrage portfolio $z$ such that $q \cdot z \leq 0$, $\sum_{j \in \mathcal{J}} v_j(p, \xi) z_j \geq 0$ for all $\xi \in \mathbb{D}_1$ with at least one strict inequality.

The following result formalises the fact that at equilibrium, the financial structure is arbitrage free.

**Proposition 5** *Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$ be an unconstrained financial structure ($Z_i = \mathbb{R}^{\mathcal{J}}$ for all $i \in \mathcal{I}$) satisfying Assumption NSS. For a commodity-asset price pair $(p, q)$, if there exists a consumer $i$ and $x_i \in X_i$, which is optimal in the budget set $B_i^{\mathcal{F}}(p, q)$, then the financial structure is arbitrage free at $(p, q)$.*

*Consequently, if $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$, then the financial structure is arbitrage free at $(p^*, q^*)$.*

**Proof.** If it is not true, there exists $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p,q)z \in \mathbb{R}^{\mathbb{D}}_{+} \setminus \{0\}$. Since $x_i \in X_i$ is optimal in the budget set $B_i^{\mathcal{F}}(p,q)$, there exists $z_i \in \mathbb{R}^{\mathcal{J}}$ such that $(x_i, z_i)$ satisfies all budget constraints. So the consumption-portfolio pair $(x_i, z_i + z)$ satisfies all budget constraints and at least one is not binding. Thanks to Assumption NSS, modifying the consumptions in the states where the budget constraint is not binding, there exists $x'_i \in X_i$ such that $(x'_i, z_i + z)$ satisfies all budget constraints and $u_i(x_i) > u_i(x_i)$ which contradicts the fact that $x_i$ is optimal in the budget set.

The second part of the proof is obvious. □

As a consequence of the above proposition, we can simplify the definition of an equilibrium for unconstrained financial economies satisfying Assumption NSS.

**Proposition 6** *Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$ be an unconstrained financial structure ($Z_i = \mathbb{R}^{\mathcal{J}}$ for all $i \in \mathcal{I}$) satisfying Assumption NSS. Let $((x_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ such that*

*(a) [Preference maximization] for every $i \in \mathcal{I}$,*

*$x_i^*$ is a "maximal" element of $u_i$ in the budget set $B_i^{\mathcal{F}}(p^*, q^*)$ in the sense that there exists $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ such that*

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot \tilde{z}_i \le p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \le p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

*and $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x'_i \in X_i \mid u_i(x'_i) > u_i(x_i^*)\} = \emptyset$;*

*(b) [Market clearing condition on the spot markets]*

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

*Then, there exists $(z_i^*) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$,*

Note that in the above definition, we have no more a market clearing condition for the financial market at $\xi_0$. Actually, as shown below in the proof, we can redistribute the excess demand of assets to a consumer without changing the optimality of her consumption, since, thanks to the no-arbitrage condition, the returns of this excess demand of assets is 0 in every states.

**Proof.** Note first that the previous proposition implies that the financial structure is arbitrage free at $(p^*, q^*)$. Then, since for all $i$, there exists $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ such that:

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot \tilde{z}_i \le p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \le p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

Summing these inequalities state by state, we get

$$\begin{cases} p^*(\xi_0) \cdot (\sum_{i \in \mathcal{I}} x_i^*(\xi_0)) + q^* \cdot (\sum_{i \in \mathcal{I}} \tilde{z}_i) \le p^*(\xi_0) \cdot (\sum_{i \in \mathcal{I}} e_i(\xi_0)) \\ p^*(\xi) \cdot (\sum_{i \in \mathcal{I}} x_i^*(\xi)) \le p^*(\xi) \cdot (\sum_{i \in \mathcal{I}} e_i(\xi)) + V(p^*, \xi) \cdot (\sum_{i \in \mathcal{I}} \tilde{z}_i), \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

From the market clearing condition on the spot commodity markets, we now that $p(\xi)\cdot(\sum_{i\in\mathcal{I}} x_i(\xi)) = p(\xi)\cdot(\sum_{i\in\mathcal{I}} e_i(\xi))$ for all $\xi \in \mathbb{D}$, so we get:

$$\begin{cases} 0 \le -q\cdot(\sum_{i\in\mathcal{I}} \tilde{z}_i) \\ 0 \le V(p,\xi)\cdot(\sum_{i\in\mathcal{I}} \tilde{z}_i), \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

or in a compact form $0 \le W(p^*,q^*)(\sum_{i\in\mathcal{I}} \tilde{z}_i)$ Since the financial structure is arbitrage free at $(p^*,q^*)$, we deduce that $0 = W(p^*,q^*)(\sum_{i\in\mathcal{I}} \tilde{z}_i)$. So, the returns of the excess demand of assets are 0 in all states $\xi \in \mathbb{D}$.

Now we choose an arbitrary consumer $i_0$ and we define the portfolios $(z_i^*)$ as follows: for all $i \neq i_0$, $z_i^* = \tilde{z}_i$ and $z_{i_0}^* = \tilde{z}_{i_0} - (\sum_{i\in\mathcal{I}} \tilde{z}_i) = -\sum_{i\in\mathcal{I}, i\neq i_0} \tilde{z}_i$. We check that $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium since the market clearing condition are now satisfied also for the asset market, nothing change for all consumers but $i_0$. For this consumer, we remark that the budget constraints are the same with $\tilde{z}_{i_0}$ and with $z_{i_0}^*$, since $W(p^*,q^*)(\tilde{z}_{i_0} - z_{i_0}^*) = W(p^*,q^*)(\sum_{i\in\mathcal{I}} \tilde{z}_i) = 0$. So, $x_i^*$ is still an optimal consumption in the budget set. □

**Remark 5** We borrow this example from lectures notes of B. Cornet. Without Assumption NSS, the market clearing condition on the financial market is not redundant. Let us consider an economy with one commodity, four states at Period 1, $\mathbb{D}_1 = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ and two consumers $\mathcal{I} = \{1, 2\}$. The consumptions sets are $\mathbb{R}^5_+$ and the initial endowments are $e_1 = e_2 = (1,1,1,1,1)$. The utility functions are

$$u_1(x_1) = x_{11} - x_{14} + \min\{1, x_{12}\} + \min\{1, x_{13}\}$$

and

$$u_2(x_2) = -x_{21} + x_{24} + \min\{1, x_{22}\} + \min\{1, x_{23}\}$$

The financial structure is composed of two nominal assets given by the returns $V_1 = (1,1,0,-1)$ and $V_2 = (-1,0,1,1)$. The asset price is $q = (0,0)$, which is not arbitrage free. Indeed,

$$W(q)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix}$$

Let $(\bar{x}_1, \bar{x}_2, \bar{z}_1, \bar{z}_2, \bar{p}, q)$ satisfying the equilibrium conditions but not necessarily the market clearing condition on the asset markets $(\bar{z}_1 + \bar{z}_2 = 0)$.

Then, by the arbitrage possibility and feasibility we must have $\bar{x}_{12} = \bar{x}_{13} = \bar{x}_{22} = \bar{x}_{23} = 1$. Indeed if one of the consumptions is strictly lower than 1, the consumer can buy at no cost one unit of the arbitrage portfolio $(1,1)$ and increase her income in the corresponding state, so increasing the consumption of the commodity and the global welfare, which is in contradiction with the utility maximisation.

Moreover, $\bar{p}_1$ and $\bar{p}_4$ are positive, otherwise, Consumer 1 or Consumer 2 could increase her welfare by buying positive quantity of Commodity 1 or Commodity

4. $\bar{x}_{14} = 0$ and $\bar{x}_{21} = 0$ again from the maximisation of the utility under the budget constraint. So, from the market clearing condition, one deduces that $\bar{x}_{11} = \bar{x}_{24} = 2$. Thus, the returns of the portfolio $\bar{z}_1$ is positive in the state $\xi_1$ and non negative in the states $\xi_2$ and $\xi_3$ and the returns of the portfolio $\bar{z}_2$ is positive in the state $\xi_4$ and non negative in the states $\xi_2$ and $\xi_3$.

If the market clearing condition holds on the financial market, from the payoff matrix $V$, one deduces that $\bar{z}_{11} \geq 0$, $\bar{z}_{12} \geq 0$, $\bar{z}_{21} = -\bar{z}_{11} \geq 0$ and $\bar{z}_{22} = -\bar{z}_{12} \geq 0$, so $\bar{z}_1 = \bar{z}_2 = (0,0)$ which contradicts the fact that the return of $\bar{z}_1$ is positive at the state $\xi_1$. So, a financial equilibrium does not exist.

Nevertheless, without the market clearing condition on the financial market, $\bar{x}_1 = (1,2,1,1,0)$, $\bar{z}_1 = (1,0)$, $\bar{x}_2 = (0,1,1,2)$, $\bar{z}_2 = (0,1)$, $\bar{p} = (1,1,1,1,1)$ and $\bar{q} = (0,0)$, we have an equilibrium on the commodity market with utility maximisation, an excess demand for both assets and an arbitrage opportunity.

In the following proposition, we characterise the arbitrage free asset prices.

**Proposition 7** *The financial structure $V$ is arbitrage free at $(p,q)$ if and only if there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V(p,\xi)$.*

This proposition means that the price of an asset is a positive linear combination of the future payoffs and the coefficients of the linear combination are the same for all assets.

If $V(p)^t$ denotes the transpose of the matrix $V(p)$, the non-arbitrage condition says that the asset price vector $q$ is in the range of the matrix $V(p)^t$ with a positive pre-image. So, for a given spot price $p$, the set of no-arbitrage asset prices is the convex cone $Q(p) = V(p)^t \mathbb{R}^{\mathbb{D}_1}_{++}$. If $V(p)$ is of rank $\sharp \mathcal{J}$, then $V(p)^t$ is onto and then $Q(p)$ is open in $\mathbb{R}^{\mathcal{J}}$.

**Proof.** If there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V(p,\xi)$, for all $z \in \mathbb{R}^{\mathcal{J}}$, we remark that $q \cdot z = \left( \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V(p,\xi) \right) \cdot z$ which implies $-q \cdot z + \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V(p,\xi) \cdot z = 0$. So, if $W(p,q)z \in \mathbb{R}^{\mathbb{D}}_+$, all the terms of the above sum are non negative and the sum of them is equal to 0, which implies that all of them are actually equal to 0. So, it does not exist $z$ such that $W(p,q)z \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$.

Conversely, let $A$ denote the range of $W(p,q)$ in $\mathbb{R}^{\mathbb{D}}$ and $\Delta$ be the simplexe of $\mathbb{R}^{\mathbb{D}}$, that is,

$$\Delta = \{\delta \in \mathbb{R}^{\mathbb{D}}_+ \mid \sum_{\xi \in \mathbb{D}} \delta_\xi = 1\}$$

Since $A$ is a linear subspace and by hypothesis $A \cap \mathbb{R}^{\mathbb{D}}_+ = \{0\}$, then $A \cap \Delta = \emptyset$. So, applying a strict Separation Theorem between the convex compact subset $\Delta$ and the closed convex subset $A$, there exists $\mu \in \mathbb{R}^{\mathbb{D}}$ such that

$$\sup\{\mu \cdot a \mid a \in A\} < \min\{\mu \cdot \delta \mid \delta \in \Delta\}$$

Since $A$ is a linear subspace and the linear mapping $a \to \mu \cdot a$ is bounded above on $A$, one deduces that $\mu$ belongs to the orthogonal complement $A^\perp$ of $A$ and

$\sup\{\mu \cdot a \mid a \in A\} = 0$. Now, since the vectors of the canonical basis of $\mathbb{R}^{\mathbb{D}}$ belongs to $\Delta$ and $0 < \min\{\mu \cdot \delta \mid \delta \in \Delta\}$, one concludes that all components of $\mu$ are positive. If we define $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ by $\lambda_\xi = \frac{\mu_\xi}{\mu_{\xi_0}}$, we get that for all $a \in A$, $\lambda \cdot a = 0$. So, in particular, the columns of the matrix $W(p,q)$ belong to the range of the matrix $W(p,q)$. Applying the result to them give $-q_j + \sum_{\xi \in \mathbb{D}_1} \lambda_\xi v_j(p,\xi) = 0$ for all $j \in \mathcal{J}$, or equivalently $q = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V(p,\xi)$. □

The above proposition shows the existence of a vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ which allows to compute the price of an asset from its payoffs in the future. But the question of the unicity of this vector is also important. We will come back later on it when we will study the completeness of the market but we state now this simple remark which is a direct consequence of a linear algebra result.

**Proposition 8** *Let $V$ be a financial structure, which is arbitrage free for the pair $(p,q)$. Let $\bar{\lambda} \in \Lambda = \{\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++} \mid q = V(p)^t \lambda\}$ . Then*

$$\Lambda = \left(\{\bar{\lambda}\} + \mathrm{Ker} V(p)^t\right) \cap \mathbb{R}^{\mathbb{D}_1}_{++}$$

*So, $\Lambda$ is a singleton if and only if $\mathrm{Ker} V(p)^t = \{0\}$ or, equivalently, $V(p)$ is onto.*

In other words, the set $\Lambda$ is the intersection of an affine space, the direction of which is the kernel of the transpose of $V(p)$, and the strictly positive orthant of $\mathbb{R}^{\mathbb{D}}$.

**Remark 6** In the finance literature, $\lambda_\xi$ is called the *present value* at date 0 of one unit of account in state $\xi$ and the vector $\lambda$ is called the *present value vector* across states. Indeed if, among the assets, we have an Arrow security associated to the state $\xi$, then, according to the no-arbitrage characterisation, the price of this Arrow security is equal to $\lambda_\xi$. As already mentioned above, the price of this Arrow security is the price to be paid at the initial node $\xi_0$ to receive one additional unit of account in state $\xi$ and nothing in the other states. Following this remark, we now state a proposition showing that, in absence of opportunity of arbitrage, the financial budget set is included in the Walras budget set for prices which are discounted according to the present value vector $\lambda$.

**Proposition 9** *Let us consider a financial structure $V$ and an exchange economy. If $V$ is arbitrage free at $(p,q)$ and $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ is a present value vector associated to $q$ then*

$$B_i^{\mathcal{F}}(p,q) \subset B_i^W(\pi, \pi \cdot e_i)$$

*where $\pi$ is defined by $\pi(\xi_0) = p(\xi_0)$ and $\pi(\xi) = \lambda_\xi p(\xi)$ for all $\xi \in \mathbb{D}_1$.*

**Proof.** Let $x_i \in X_i$ such that there exists $z_i \in Z_i$ satisfying the budget equations:

$$\left\{ \begin{array}{l} p(\xi_0) \cdot x_i(\xi_0) + q \cdot z_i \leq p(\xi_0) \cdot e_i(\xi_0) \\ p(\xi) \cdot x_i(\xi) \leq p(\xi) \cdot e_i(\xi) + V(p,\xi) \cdot z_i, \quad \forall \xi \in \mathbb{D}_1 \end{array} \right.$$

Then, multiplying the second group of inequalities by $\lambda_\xi$ and summing all inequalities, we get

$$\begin{array}{rl} p(\xi_0)\cdot x_i(\xi_0) + \sum_{\xi\in\mathbb{D}_1} \lambda_\xi p(\xi)\cdot x_i(\xi) + q\cdot z_i \leq & p(\xi_0)\cdot e_i(\xi_0) \\ & + \sum_{\xi\in\mathbb{D}_1} \lambda_\xi p(\xi)\cdot e_i(\xi) \\ & + \left(\sum_{\xi\in\mathbb{D}_1} \lambda_\xi V(p,\xi)\right)\cdot z_i \end{array}$$

Since $q = \sum_{\xi\in\mathbb{D}_1} \lambda_\xi V(p,\xi)$, we get:

$$p(\xi_0)\cdot x_i(\xi_0) + \sum_{\xi\in\mathbb{D}_1} \lambda_\xi p(\xi)\cdot x_i(\xi) \leq p(\xi_0)\cdot e_i(\xi_0) + \sum_{\xi\in\mathbb{D}_1} \lambda_\xi p(\xi)\cdot e_i(\xi)$$

that is $\pi\cdot x_i \leq \pi\cdot e_i$, so $x_i$ belongs to $B_i^W(\pi, \pi\cdot e_i)$. □

**Exercise 3** *Let us consider a financial structure $V$ and an exchange economy. We assume that $V$ is arbitrage free at $(p,q)$ and $\Lambda = \{\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++} \mid q = V(p)^t\lambda\}$ is the set of present value vectors associated to $q$. For all $\lambda\in\Lambda$, $\pi^\lambda$ is defined by $\pi^\lambda(\xi_0) = p(\xi_0)$ and $\pi^\lambda(\xi) = \lambda_\xi p(\xi)$ for all $\xi\in\mathbb{D}_1$. Show that*

$$B_i^{\mathcal{F}}(p,q) = \cap_{\lambda\in\Lambda} B_i^W(\pi^\lambda, \pi^\lambda\cdot e_i)$$

*Hint: show that one inclusion is a direct consequence of the previous proposition. For the converse, consider $x_i \notin B_i^{\mathcal{F}}(p,q)$, show that in the space $\mathbb{R}^{\mathbb{L}}\times\mathbb{R}^{\mathcal{J}}$, $\{x_i - e_i\}\times\mathbb{R}^{\mathcal{J}}$ does not intersect the cone $C = \{(\zeta,z)\in\mathbb{R}^{\mathbb{L}}\times\mathbb{R}^{\mathcal{J}} \mid p(\xi_0)\cdot\zeta(\xi_0) + q\cdot z \leq 0, p(\xi)\cdot\zeta(\xi)\leq V(p,\xi)\cdot z, \forall\xi\in\mathbb{D}_1\}$, apply a separation theorem and conclude.*

**Remark 7** If we consider a complete set of contingent commodities, the no-arbitrage condition tells us that the price at node $\xi_0$ of the contingent commodities contracts of node $\xi$ is positively proportional to the spot price at this node. The present value vector is just this coefficient of proportionality.

If we consider a complete set of Arrow security, the no-arbitrage characterisation holds true if and only if all Arrow security prices at node $\xi_0$ are positive. The components of the present value vector is just the price of the Arrow securities.

**Remark 8** To do the link with the literature in finance, we assume that the bond is among the asset. Its payoffs are equal to 1 in all states of $\mathbb{D}_1$, so the price of this bond is $\bar{\lambda} = \sum_{\xi\in\mathbb{D}_1}\lambda_\xi$. This is the price to be paid today to be sure to have one additional unit of account in each state of nature tomorrow. So, in terms of interest rate $r$ between the current date and tomorrow, $\bar{\lambda} = \frac{1}{1+r}$. So, the discounted present value vector $\mu = (1+r)\lambda$ is a probability measure on the state tomorrow called the risk neutral probability measure.

This terminology can be justified by the following remark. Let us assume that we have just one commodity per state (pure financial model where only the wealth matters) with normalised spot prices at 1 and a risk-neutral agent having a subjective probability $\chi$ on the $\mathbb{D}_1$. Her utility function is:

$$u_i(x_i) = x_i(\xi_0) + \frac{1}{1+r} \sum_{\xi \in \mathbb{D}_1} \chi(\xi) x_i(\xi)$$

If this agent is maximising her utility at an interior solution in the Walras budget set $B_i^W(\pi)$ associated to the discounted prices, we get $\chi(\xi) = \mu(\xi)$ for all $\xi \in \mathbb{D}_1$. So, a risk-neutral agent can maximise her preferences at an interior solution over the Walras budget set only if her subjective probabilities are the same as the risk-neutral probability given by the present value vector.

Let us now consider the question of the optimality of a financial equilibrium in an unconstrained economy $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i \in \mathcal{I}}, V)$. We consider an equilibrium $((x_i^*, z_i^*), p^*, q^*)$ and we assume for simplicity that the utility functions are differentiable on the interior of the consumption sets, that all partial derivatives are positive and that the equilibrium consumptions $(x_i^*)$ belong to the interior of the consumption sets. Then the allocation is Pareto optimal if and only if all gradient vectors $(\nabla u_i(x_i^*))_{i \in \mathcal{I}}$ are colinear. If we write the first order optimality conditions for the utility maximisation problem over the budget set, we get that there exists a multiplier $\mu_i \in \mathbb{R}_+^{\mathbb{D}}$ such that:

$$\left\{ \begin{array}{l} \nabla u_i(x_i^*) = (\mu_{i\xi} p(\xi))_{\xi \in \mathbb{D}} \\ \mu_{i\xi_0} q^* = V(p^*) \mu_i \end{array} \right.$$

Since $\nabla u_i(x_i^*) \in \mathbb{R}_{++}^{\mathbb{L}}$, we get that $\mu_{i\xi} > 0$ for all $\xi$ and the vector $\lambda_i \in \mathbb{R}_{++}^{\mathbb{D}_1}$ defined by $\lambda_{i\xi} = \frac{\mu_{i\xi}}{\mu_{i\xi_0}}$ is a present value vector associated to the no arbitrage equilibrium asset price $q^*$. So, if there exists a unique present value vector, that is, if $V(p)$ is onto, then one concludes that all gradient vectors $(\nabla u_i(x_i^*))_{i \in \mathcal{I}}$ are colinear and the equilibrium allocation $(x_i^*)_{i \in \mathcal{I}}$ is Pareto optimal. Otherwise we cannot conclude and generically, the equilibrium allocation is not Pareto optimal. Actually, we can remark that $x_i^*$ is an optimal consumption in the Walras budget set associated to the personalised discounted price $\pi_i$ defined by $\pi_i(\xi_0) = p(\xi_0)$ and $\pi_i(\xi) = \lambda_{i\xi} p(\xi)$ for all $\xi \in \mathbb{D}_1$. So, each agent maximises her welfare but not according to the same prices. So, the price signal is not sufficient for a coordination of the agents and then, it leads to a non optimal allocation of resources. As usual, missing markets is the source of an imperfect functioning of the market mechanism.

Using the usual differentiability assumptions on the utility functions, one can prove that, generically at the competitive equilibrium, the individual transfers $(p^*(\xi) \cdot (x_i^*(\xi) - e_i(\xi))_{\xi \in \mathbb{D}})$ generate a subspace of dimension $\min\{\sharp\mathcal{I}, \sharp\mathbb{D}_1\}$. So, if the number of agents is greater than the number of states of nature, it is impossible to reach a competitive allocation with an incomplete financial structure since, then, the transfers belong to the marketable space, which has a dimension strictly smaller than $\sharp\mathbb{D}_1$.

The above remark showing that the agents choose an optimal consumption on a Walras budget set according to personalised present value vectors, leads to the following result which is very useful in the proof of the existence of a financial equilibrium for a nominal asset structure. It is called "Cass trick" in the literature since it was introduced by David Cass in [7].

**Proposition 10** *Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i \in \mathcal{I}}, V)$ be a financial economy satisfying Assumption NSS. Let $((x_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ such that:*

*(a) $q^*$ is a no arbitrage asset price associated to a present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$*

*(b) there exists an agent $i_0 \in \mathcal{I}$ such that $x_{i_0}^*$ is a "maximal" element of $u_{i_0}$ in the budget set $B_{i_0}^W(\pi^*, \pi^* \cdot e_{i_0})$ where $\pi^*$ is defined by $\pi^*(\xi_0) = p^*(\xi_0)$ and $\pi^*(\xi) = \lambda_\xi p^*(\xi)$ for all $\xi \in \mathbb{D}_1$.*

*(c) for every $i \in \mathcal{I}$, $i \neq i_0$,*

*$x_i^*$ is a "maximal" element of $u_i$ in the budget set $B_i^{\mathcal{F}}(p^*, q^*)$ in the sense that there exists $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ such that*

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot \tilde{z}_i \leq p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \leq p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

*and $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;*

*(d) [Market clearing condition on the spot markets]*

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

*Then, there exists $(z_i^*) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$,*

In the above proposition, Agent $i_0$ is maximising over the Walras budget set which is larger than the financial budget set. Nevertheless, we will prove that her allocation is affordable for the suitable portfolio $-\sum_{i \neq i_0} \tilde{z}_i$. So, her allocation belongs to the financial budget set and it is obviously optimal in it. We can conclude that $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$ with $z_i^* = \tilde{z}_i$ for $i \neq i_0$ and $z_{i_0}^* = -\sum_{i \neq i_0} \tilde{z}_i$.

**Proof.** From Assumption NSS, for all $i \neq i_0$, $p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot \tilde{z}_i = p^*(\xi_0) \cdot e_i(\xi_0)$ and $p^*(\xi) \cdot x_i^*(\xi) = p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i$, for all $\xi \in \mathbb{D}_1$. Adding these equalities and using the market clearing condition on the commodity markets, we get $p^*(\xi_0) \cdot x_{i_0}^*(\xi_0) + q^* \cdot (-\sum_{i \neq i_0} \tilde{z}_i) = p^*(\xi_0) \cdot e_{i_0}(\xi_0)$ and $p^*(\xi) \cdot x_{i_0}^*(\xi) = p^*(\xi) \cdot e_{i_0}(\xi) + V(p^*, \xi) \cdot (-\sum_{i \neq i_0} \tilde{z}_i)$, for all $\xi \in \mathbb{D}_1$. So $x_{i_0}^*$ is affordable by the portfolio $-\sum_{i \neq i_0} \tilde{z}_i$. Thus, since the financial budget set is included in the Walrasian budget set, $x_{i_0}^*$ is maximising the preferences of Agent $i_0$ in the financial budget set. As for the market clearing condition on the financial market, it is obviously satisfied thanks to the choice of the portfolio $\tilde{z}_{i_0} = -\sum_{i \neq i_0} \tilde{z}_i$. □

**Exercise 4** *We consider a date-event tree with three states at the second period. For the following nominal financial structure with two assets, draw on a two dimensional space the set of no arbitrage asset prices:*

*1)* $V = \begin{pmatrix} 1/2 & 2 \\ 1 & 1 \\ 2 & 1/2 \end{pmatrix}$.

*2)* $V = \begin{pmatrix} 0 & 2 \\ 1/2 & 1 \\ 1 & 0 \end{pmatrix}$.

*3)* $V = \begin{pmatrix} 1/2 & 2 \\ -1 & -1 \\ 2 & 1/2 \end{pmatrix}$.

*4)* $V = \begin{pmatrix} -1 & 2 \\ 0 & 0 \\ 1 & -1/2 \end{pmatrix}$.

**Exercise 5** *We consider a financial structure represented by its payoff matrix $V$. Let $p$ be a spot price vector. Let $Q(p)$ be the set of no-arbitrage asset prices. Show that $Q(p) \neq \mathbb{R}^{\mathcal{J}}$ if and only if there exists a portfolio $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(p)z \geq 0$, that is a non-zero portfolio with non negative returns in every states.*

**Exercise 6** *We consider a financial structure represented by its payoff matrix $V$. Let $p$ be a spot price vector such that $V(p)$ is one to one. Let $Q(p)$ be the set of no-arbitrage asset prices. Show that $0 \notin Q(p)$ if and only if there exists a portfolio $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(p)z \geq 0$, that is a non-zero portfolio with non negative returns in every states.*

*Hint: use the fact that $V(p)^t$ is onto and then $Q(p)$ open.*

## 5.2 Redundant asset

In this subsection, we deal with the question to know whether or not a financial structure is minimal in the sense we cannot withdraw an asset without reducing the transfer capacities of the agents. So, the basic concept is the one of redundant asset and useless portfolio.

We recall that we maintain our basic hypothesis of an unconstrained financial economy, that is $Z_i = \mathbb{R}^{\mathcal{J}}$ for all agents.

**Définition 7** Let a financial structure represented by its payoff matrix $V$. Given a spot price $p$, an asset $j$ is redundant if the payoff vector $V_j(p)$ is a linear combination of the payoff vectors of the other assets $(V_k(p))_{k \in \mathcal{J}, k \neq j}$. A portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is useless if the payoff in each state is equal to 0, that is $V(p)z = 0$.

**Remark 9** If $q$ is a no-arbitrage asset price for the spot price vector $p$, then there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V(p)^t \lambda$. So, if portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is useless, then $q \cdot z = V(p)^t \lambda \cdot z = \lambda \cdot V(p)z = \lambda \cdot 0 = 0$. So the value of a useless portfolio is equal to 0 for all no-arbitrage asset price $q$. In other words, the kernel of the payoff matrix $V(p)$ and the one of the full payoff matrix $W(p, q)$ coincide for all no-arbitrage asset price $q$.

We now characterise an asset structure without redundant asset.

**Proposition 11** *Let a financial structure represented by its payoff matrix $V$ and $p$ be a spot price. Then there is no redundant asset if and only if one of the two following condition is satisfied:*

*a) $V(p)$ is one-to-one or equivalently the rank of $V(p)$ is equal to $\sharp\mathcal{J}$;*

*b) the unique useless portfolio is $0$.*

We now remark that, under Assumption NSS, an optimal portfolio in the budget set $B_i^{\mathcal{F}}(p,q)$, is affordable by a unique portfolio $z_i$ when there is no redundant asset.

**Proposition 12** *Let a financial structure represented by its payoff matrix $V$ and $(p,q)$ be a spot - asset price pair such that $V$ is arbitrage free at $(p,q)$.Let $\bar{x}_i$ be optimal for $u_i$ in the budget set $B_i^{\mathcal{F}}(p^*,q^*)$. Let $z_i$ and $z'_i$ to portfolios, which finance $\bar{x}_i$. Then, if Assumption NSS holds, $z_i - z'_i$ is a useless portfolio.*

*Consequently, if there is no redundant asset for $V(p)$, then $\bar{x}_i$ is affordable for a unique portfolio in $\mathbb{R}^{\mathcal{J}}$.*

The proof is a direct consequence of the fact that the budget constraints are binding under Assumption NSS.

Using the above result, we can get a slightly more precise result than Proposition 6 for the financial structure without redundant asset.

**Proposition 13** *Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i\in\mathcal{I}}, V)$ be an unconstrained ($Z_i = \mathbb{R}^{\mathcal{J}}$ for all $i \in \mathcal{I}$) financial economy satisfying Assumption NSS and with no redundant asset. Let $((x_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ such that*

*(a) [Preference maximization] for every $i \in \mathcal{I}$,*

*$x_i^*$ is a "maximal" element of $u_i$ in the budget set $B_i^{\mathcal{F}}(p^*,q^*)$ in the sense that there exists $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ such that*

$$\begin{cases} p^*(\xi_0)\cdot x_i^*(\xi_0) + q^*\cdot\tilde{z}_i \le p^*(\xi_0)\cdot e_i(\xi_0) \\ p^*(\xi)\cdot x_i^*(\xi) \le p^*(\xi)\cdot e_i(\xi) + V(p^*,\xi)\cdot\tilde{z}_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

*and $B_i^{\mathcal{F}}(p^*,q^*) \cap \{x'_i \in X_i \mid u_i(x'_i) > u_i(x_i^*)\} = \emptyset$;*

*(b) [Market clearing condition on the spot markets]*

$$\sum_{i\in\mathcal{I}} x_i^* = \sum_{i\in\mathcal{I}} e_i.$$

*Then, $\sum_{i\in\mathcal{I}} \tilde{z}_i = 0$ and $((x_i^*, \tilde{z}_i), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$.*

**Proof.** The proof is the same as the one of Proposition 6, where it is shown that $\sum_{i\in\mathcal{I}} \tilde{z}_i$ is a useless portfolio, from which one concludes that it is equal to 0. □

**Exercise 7** *Let a financial structure represented by its payoff matrix $V$ and $p$ be a spot price. Show that there is no redundant asset if and only if $Q(p)$ is open.*

We conclude this subsection by showing that in the unconstrained case, we can easily come back to a financial structure with no-redundant asset by deleting the redundant assets. This operation is innocuous for the consumers since it does not change the budget set and the equilibrium allocations.

Let $V$ be a financial structure and $p$ be a spot price vector. We know that $(V_j(p))_{j\in\mathcal{J}}$ is a spanning family of the range of $V(p)$ and we can find a maximal sub-family $\tilde{\mathcal{J}} \subset \mathcal{J}$ such that $(V_j(p))_{j\in\tilde{\mathcal{J}}}$ is still spanning the range of $V(p)$ and is linearly independent. It means that for $j \in \mathcal{J} \setminus \tilde{\mathcal{J}}$, the asset $j$ is redundant in the sense that $V_j(p) = \sum_{k\in\tilde{\mathcal{J}}} \mu^j_k V_k(p)$ for some $\mu^j \in \mathbb{R}^{\tilde{\mathcal{J}}}$.

We can define a substructure $\tilde{V}$ by keeping only the asset in $\tilde{\mathcal{J}}$. This new structure $\tilde{V}$ has no redundant asset for the spot price $p$ since $(V_j(p))_{j\in\tilde{\mathcal{J}}}$ is linearly independent. Furthermore, the budget sets of the consumer for no-arbitrage asset prices are the same for the two financial structure if we keep the asset prices unchanged. Indeed, since $\tilde{V}$ is a substructure of $V$, the budget set associated to $\tilde{V}$ is a priori smaller than the one associated to $V$. Now, let $x_i \in X_i$ and assume that there exists $z_i \in \mathbb{R}^{\mathcal{J}}$ such that $p \,\square\, (x_i - e_i) \le W(p,q)z_i$. Then

$$W(p,q)z_i = \begin{pmatrix} -\sum_{j\in\tilde{\mathcal{J}}} q_j z_j - \sum_{j\notin\tilde{\mathcal{J}}} q_j z_j \\ \sum_{j\in\tilde{\mathcal{J}}} z_j V_j(p) + \sum_{j\notin\tilde{\mathcal{J}}} z_j V_j(p) \end{pmatrix}$$

But

$$\sum_{j\notin\tilde{\mathcal{J}}} z_j V_j(p) = \sum_{j\notin\tilde{\mathcal{J}}} z_j \sum_{k\in\tilde{\mathcal{J}}} \mu^j_k V_k(p) = \sum_{k\in\tilde{\mathcal{J}}} (\sum_{j\notin\tilde{\mathcal{J}}} z_j \mu^j_k) V_k(p)$$

Furthermore, since $q$ is a no-arbitrage price, we get that for all $j \notin \tilde{\mathcal{J}}$, $q_j = \sum_{k\in\tilde{\mathcal{J}}} \mu^j_k q_k$. Consequently,

$$W(p,q)z_i = \begin{pmatrix} -\sum_{k\in\tilde{\mathcal{J}}} (z_k + (\sum_{j\notin\tilde{\mathcal{J}}} z_j \mu^j_k)) q_k \\ \sum_{k\in\tilde{\mathcal{J}}} (z_k + (\sum_{j\notin\tilde{\mathcal{J}}} z_j \mu^j_k)) V_k(p) \end{pmatrix}$$

Hence $x_i$ is affordable in the substructure $\tilde{V}$ by the portfolio $\left(z_k + (\sum_{j\notin\tilde{\mathcal{J}}} z_j \mu^j_k)\right)_{k\in\tilde{\mathcal{J}}}$, so its belongs to the budget set associated to this substructure.

We can summarise this remark in the following proposition.

**Proposition 14** *Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i\in\mathcal{I}}, V)$ be an unconstrained financial economy satisfying Assumption NSS. Then there exists a substructure $\tilde{V}$ composed by a subset $\tilde{\mathcal{J}}$ of the assets of $V$ such that*

*a) $\tilde{V}$ has no redundant asset;*

*b) If $((x^*_i, z^*_i), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ is a financial equilibrium for the structure $V$, then there exists $(\zeta^*_i) \in (\mathbb{R}^{\tilde{\mathcal{J}}})^{\mathcal{I}}$ such that $((x^*_i, \zeta^*_i), p^*, \tilde{q}^*)$ is a financial equilibrium for the structure $\tilde{V}$, where the price $\tilde{q}^*$ is the standard projection of $q^*$ on $\mathbb{R}^{\tilde{\mathcal{J}}}$.*

*c) If $((x_i^*, \tilde{z}_i^*), p^*, \tilde{q}^*) \in (\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ is a financial equilibrium for the structure $\tilde{V}$, then there exists $(z_i^*) \in (\mathbb{R}^{\tilde{\mathcal{J}}})^{\mathcal{I}}$ such that $((x_i^*, z_i^*), p^*, \tilde{q}^*)$ is a financial equilibrium for the structure $V$, where the price $q^*$ is computed for the asset $j \in \mathcal{J} \setminus \{\tilde{\mathcal{J}}\}$ according to the present value vector associated to $\tilde{q}^*$ and $z_i^*$ is the natural embedding of $\tilde{z}_i^*$ in $\mathbb{R}^{\mathcal{J}}$ by adding 0 for the additional components.*

,

**Exercise 8** *Check for the following financial structure if there exists redundant assets and, if yes, provide an equivalent substructure without redundant asset.*

*1)* $V = \begin{pmatrix} 1/2 & 2 & 5 \\ 1 & 1 & 4 \\ 2 & 1/2 & 5 \end{pmatrix}$.

*2)* $V = \begin{pmatrix} 0 & 2 & 0 \\ 1/2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

*3)* $V = \begin{pmatrix} 1/2 & 2 & -1 \\ -1 & -1 & -1 \\ 2 & 1/2 & 7/2 \end{pmatrix}$.

*4)* $V = \begin{pmatrix} -1 & 2 \\ 0 & 0 \\ 1 & -2 \end{pmatrix}$.

**Exercise 9** *We consider a financial structure represented by its payoff matrix $V$. Let $p$ be a spot price vector. Let $Q(p)$ be the set of no-arbitrage asset prices. Show that $0 \notin Q(p)$ if and only if there exists a portfolio $z \in \mathbb{R}^J \setminus \{0\}$ such that $V(p)z \geq 0$, that is a non-zero portfolio with non negative returns in every states.*

*Hint: use the equivalent substructure without redundant assets and the previous exercise on the same topic.*

## 5.3 Pricing by arbitrage

Let us now come back to the consequence of the absence of arbitrage opportunity on the pricing of a redundant asset. Let $V$ be a financial structure and $p$ be a spot price vector. Let $j_0 \in \mathcal{J}$ be a redundant asset. Then there exists $\mu \in \mathbb{R}^{\mathcal{J}\setminus\{j_0\}}$ such that $V_{j_0}(p) = \sum_{j \in \mathcal{J}, j \neq j_0} \mu_j V_j(p)$. Now, let $q$ be a no-arbitrage asset price. Then, there exists there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V(p)^t \lambda$. Hence, $q_{j_0} = V_j(p)^t \lambda = \sum_{j \in \mathcal{J}, j \neq j_0} \mu_j V_j(p)^t \lambda = \sum_{j \in \mathcal{J}, j \neq j_0} \mu_j q_j$. So, the price of the asset $j_0$ is a linear combination of the prices of the other assets with the coefficient given by the fact that the payoff vector of asset $j_0$ is a linear combination of the payoff vectors of the other assets. This remark is the basis of the pricing by arbitrage.

Let us now assume that an asset $k$ is a redundant asset of the extended financial structure obtained by adding this new asset to the collection $\mathcal{J}$. Let $p$ be a spot price vector and $q$ be a no-arbitrage asset price of the financial structure.

Thanks to the no-arbitrage characterisation, we can compute the unique price of this new asset for which the extended financial structure is arbitrage free without computing the equivalent portfolio $\mu$ if we know the present value vector $\lambda$ associated to the asset price $q$. Note that this present value vector is obtained from the interest rate and the risk-neutral probability on $\mathbb{D}_1$. So, $q_k = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V_k(\xi)$. This is the formula of the pricing by arbitrage, that is the discounted expected return according to the risk-neutral probability.

Note that even if we have several present value vectors associated to the asset price $q$, the pricing by arbitrage of the new asset is well defined. Indeed, if $\lambda'$ is another present value vector, then $\lambda - \lambda' \in \mathrm{Ker} V^t = (\mathrm{Im} V)^\perp$. So, since $V_k \in \mathrm{Im} V$, we get that $(\lambda - \lambda') \cdot V_k = 0$, and so the price $q_k = \lambda \cdot V_k$ is equal to $\lambda' \cdot V_k$.

## 5.4 Over hedging pricing

In this subsection, we study which information we can derive on the pricing of an asset which is not spanned by the existing assets. Once again, we assume that the market is not perturbed by the introduction of this new asset and that the present value vector is constant. This assumption is much more demanding than the one for an additional redundant asset. The main idea is that the price of a portfolio, which provides a smaller payoffs than another portfolio, must be smaller that the price of the second portfolio. So, in other words, if $V_k \leq V_j$ for two assets, then $q_k \leq q_j$. In this case, the asset $j$ is over hedging the asset $k$ since if a consumer buys one unit of asset $j$ and sell one unit of asset $k$, in all states of nature at date 1, she is able to cover the payoffs due from the selling of asset $k$ by the returns coming for the payoffs of asset $j$.

So, the over hedging pricing consists in computing an upper bound for the price of a portfolio as the minimum of the prices of the portfolios which deliver an higher payoffs in each state.

Formally, let $V$ be a financial structure, $p$ be a spot price and $q$ an arbitrage free asset price associated to the present value vector $\lambda$ and $k$ an asset represented by its payoff vector $v \in \mathbb{R}^{\mathbb{D}_1}$. Then, the over hedging price of $k$ is the value of the following minimisation problem.

$$\begin{cases} \text{Minimise } \sum_{J \in \mathcal{J}} q_j z_j \\ V(p) z \geq v \\ z \ \in \mathbb{R}^{\mathcal{J}} \end{cases}$$

We remark that this value may be $+\infty$ if there is no portfolio $z$ such that $V(p)z \geq c$. Nevertheless, if the bond is among the existing portfolio, or, more generally, if there exists a portfolio $\underline{z}$ such that $V(p)\underline{z} \gg 0$, we are sure that the value is finite for every $v \in \mathbb{R}^{\mathbb{D}_1}$. Actually, this is also a necessary condition.

For some authors, this pricing is called the cost of the financial structure $V$.

**Proposition 15** *Let $V$ be a financial structure and $(p, q)$ be a spot - asset price vector such that $V$ is arbitrage free at $(p, q)$. Let us assume that there exists a* *portfolio $\underline{z}$ such that $V(p)\underline{z} \gg 0$. Then the over hedging price function $q^+$ satisfies the following properties:*

*a) $q^+$ is a positively homogeneous convex function on $\mathbb{R}^{\mathbb{D}_1}$, so it is Lipschitz continuous.*

*b) If $\lambda$ is a present value vector associated to $q$, then $q^+(v) \geq \lambda \cdot v$.*

*c) the restriction of $q^+$ to the range of $V(p)$ is the linear mapping $\lambda \cdot v$.*

*d) $q^+(v) = \max\{\lambda \cdot v \mid \lambda \in \mathbb{R}_+^{\mathbb{D}_1}, V(p)^t\lambda = q\}$.*

Note that the set $\{\lambda \in \mathbb{R}_+^{\mathbb{D}_1} \mid V(p)^t\lambda = q\}$ is the closure of the set of present value vectors associated to $q$.

**Proof.** a) Let $v$ and $v'$ in $\mathbb{R}^{\mathbb{D}_1}$ and $z$ and $z'$ in $\mathbb{R}^{\mathcal{J}}$ such that $V(p)z \geq v$ and $V(p)z' \geq v'$. Then, for all $t \in [0,1]$, $V(p)(tz + (1-t)z') \geq tv + (1-t)v'$. Consequently, $q^+(tv+(1-t)v') \leq q\cdot(tz+(1-t)z') = tq\cdot z+(1-t)q\cdot z'$. Taken the infimum over $z$ and then over $z'$, we get $q^+(tv+(1-t)v') \leq tq^+(v)+(1-t)q^+(v')$, so $q^+$ is convex. With the same kind of reasoning, we easily show that $q^+(\tau v) = \tau q^+(v)$ for all $\tau \geq 0$. Since $q^+$ is convex and finite on $\mathbb{R}^{\mathbb{D}_1}$, a general result tells us that $q^+$ is locally Lipschitz continuous. The homogeneity of $q^+$ implies that it is actually Lipschitz continuous.

b) Let $v$ in $\mathbb{R}^{\mathbb{D}_1}$ and $z$ in $\mathbb{R}^{\mathcal{J}}$ such that $V(p)z \geq v$. Then $\lambda\cdot V(p)z = V(p)^t\lambda\cdot z = q \cdot z \geq \lambda \cdot v$. So $q^+(v) \geq \lambda \cdot v$.

c) If, $v$ belongs to the range of $V(p)$, there exists $z \in \mathbb{R}^{\mathcal{J}}$ such that $v = V(p)z$. Then, with the above computation, we deduce that $q \cdot z = \lambda \cdot v \leq q^+(v)$, so $q^+(v) = \lambda \cdot v$.

d) From (b), one deduces that $q^+(v) \geq \max\{\lambda \cdot v \mid \lambda \in \mathbb{R}_+^{\mathbb{D}_1}, V(p)^t\lambda = q\}$ since the set $\{\lambda \in \mathbb{R}_+^{\mathbb{D}_1} \mid V(p)^t\lambda = q\}$ is the closure of the set of present value vectors associated to $q$. From a general result on linear programming, since the value of the cost problem is finite, we know that there exists a solution $\bar{z}$. Writing the first order necessary and sufficient condition at $\bar{z}$, we get that there exists a vector of multipliers $\mu \in \mathbb{R}_+^{\mathbb{D}_1}$ such that

$$q = V(p)^t\mu \text{ and } \mu \cdot (V(p)\bar{z} - v) = 0$$

From which, one deduces that $\mu\cdot v = q\cdot\bar{z}$, so $q^+(v) \leq \max\{\lambda\cdot v \mid \lambda \in \mathbb{R}_+^{\mathbb{D}_1}, V(p)^t\lambda = q\}$. □

**Exercise 10** *Show that $q^+$ is linear on $\mathbb{R}^{\mathbb{D}_1}$ if and only if $V(p)$ is onto.*

**Exercise 11** *Let $V$ be the nominal financial structure defined by:*

$$V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

*Let $q = (1, 2)$.*
*1) Show that $q$ is a no-arbitrage price and compute the set of present value vectors associated to $q$.*
*2) Compute the over hedging price of the following assets:*
$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

**Exercise 12** *We consider a financial structure with a unique asset, the bond, which is a nominal asset delivering one unit of the unit of account in each state. Compute the over hedging price for all assets in $\mathbb{R}^{\mathbb{D}_1}$.*

We have another interpretation of the over hedging price which comes from the following remark. If we introduce a new asset with the payoff $v$, then, if the price of this asset is greater or equal to the over hedging price $q^+(v)$, then this asset is useless for the consumers. Indeed, the consumer can reach with a suitable portfolio of the existing assets a payoff in each state of nature at least as good as the one provided by the new asset at a price smaller than the price of the new asset. So, even if the market is open for this new asset, no transaction will take place. Hence, the over hedging price is a threshold above which no transaction takes place on the financial market for this asset.

## 5.5 Arbitrage with short sale constraints

In this subsection, we illustrate on one particular example the effect of constraints on the portfolio sets on the no-arbitrage condition. Let us assume that the portfolio sets are no more $\mathbb{R}^{\mathcal{J}}$ but $\{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+$, with $\underline{z}_i \leq 0$, which means that the consumers face a short sale constraints in the sense that they cannot sell on the market a quantity greater than $|\underline{z}_{ij}|$ of asset $j$.

In this case, we have the following condition to guarantee that a consumer can find an optimal consumption in her budget set.

**Proposition 16** *Let us assume that Assumption NSS is satisfied by the financial economy $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i = \{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+)_{i \in \mathcal{I}}, V)$. For a commodity-asset price pair $(p, q)$, if there exists a consumer $i$ and $x_i \in X_i$, which is optimal in the budget set $B_i^{\mathcal{F}}(p, q)$, then it does not exists $\zeta \in \mathbb{R}^{\mathcal{J}}_+$ such that $W(p, q)\zeta \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$.*

**Proof.** Let $z_i \in Z_i$ be a portfolio for which $x_i$ is affordable. If there exists $\zeta \in \mathbb{R}^{\mathcal{J}}_+$ such that $W(p, q)\zeta \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$, then $x_i$ is affordable for the portfolio $z_i + \zeta$ and $z_i + \zeta \in Z_i$. Furthermore, at least one budget constraint is not binding. So, thanks to Assumption NSS, the consumer can modify her consumption in this state, increasing her welfare and remaining affordable for the portfolio $z_i + \zeta$. So, we get a contradiction with the fact that $x_i$ is optimal. □

In this case, we can modify the definition of an arbitrage free financial structure as follows.

**Définition 8** The financial structure $V$ with the portfolio sets $(Z_i = \{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+)_{i\in\mathcal{I}}$ is arbitrage free for a commodity-asset price pair $(p, q)$, if it does not exists $\zeta \in \mathbb{R}^{\mathcal{J}}_+$ such that $W(p, q)\zeta \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$.

We can now characterise the arbitrage free financial structure using a separation theorem as above for the unconstrained case.

**Proposition 17** *The financial structure $V$ with the portfolio sets $(Z_i = \{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+)_{i\in\mathcal{I}}$ is arbitrage free at $(p, q)$ if and only if there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q \geq \sum_{\xi\in\mathbb{D}_1} \lambda_\xi V(p, \xi) = V(p)^t\lambda$.*

This proposition means that the price of an asset is greater than a positive linear combination of the future returns and the coefficients of the linear combination are the same for all assets.

If $V(p)^t$ denotes the transpose of the matrix $V(p)$, the non-arbitrage condition says that the asset price vector $q$ is greater than an element in the range of the matrix $V(p)^t$ with a positive pre-image. So, for a given spot price $p$, the set of no-arbitrage asset prices is the convex cone $Q(p) = V(p)^t\mathbb{R}^{\mathbb{D}_1}_{++} + \mathbb{R}^{\mathcal{J}}_+$.

**Proof.** If there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q \geq \sum_{\xi\in\mathbb{D}_1} \lambda_\xi V(p, \xi)$, for all $\zeta \in \mathbb{R}^{\mathcal{J}}_+$, we remark that $q \cdot \zeta \geq V(p)^t\lambda \cdot \zeta$ which implies $0 \geq -q \cdot \zeta + V(p)^t\lambda \cdot \zeta$. So, if $W(p, q)\zeta \in \mathbb{R}^{\mathbb{D}}_+$, all the terms of the above sum are non negative and the sum of them is non positive, which implies that all of them are actually equal to 0. So, it does not exist $\zeta \in \mathbb{R}^{\mathcal{J}}_+$ such that $W(p, q)\zeta \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$.

Conversely, let $A$ denote the image of $\mathbb{R}^{\mathcal{J}}_+$ by $W(p, q)$ in $\mathbb{R}^{\mathbb{D}}$ and $\Delta$ be the simplexe of $\mathbb{R}^{\mathbb{D}}$, that is,

$$\Delta = \{\delta \in \mathbb{R}^{\mathbb{D}}_+ \mid \sum_{\xi\in\mathbb{D}} \delta_\xi = 1\}$$

Since $\mathbb{R}^{\mathcal{J}}_+$ is finitely generated, $A$ is a closed convex cone. By hypothesis $A \cap \mathbb{R}^{\mathbb{D}}_+ = \{0\}$, then $A \cap \Delta = 0$. So, applying a strict Separation Theorem between the convex compact subset $\Delta$ and the closed convex subset $A$, there exists $\mu \in \mathbb{R}^{\mathbb{D}}$ such that

$$\sup\{\mu \cdot a \mid a \in A\} < \min\{\mu \cdot \delta \mid \delta \in \Delta\}$$

Since $A$ is a convex cone and the linear mapping $a \to \mu \cdot a$ is bounded above on $A$, one deduces that $\mu$ belongs to the negative polar cone $A^\circ$ of $A$ and $\sup\{\mu \cdot a \mid a \in A\} = 0$. Now, since the vectors of the canonical basis of $\mathbb{R}^{\mathbb{D}}$ belongs to $\Delta$ and $0 < \min\{\mu \cdot \delta \mid \delta \in \Delta\}$, one concludes that all components of $\mu$ are positive.

Let us define $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ by $\lambda_\xi = \frac{\mu_\xi}{\mu_{\xi_0}}$. $(1, \lambda) \in A^\circ$. So, for all $\zeta \in \mathbb{R}^{\mathcal{J}}_+$, $(1, \lambda) \cdot W(p, q)\zeta = W(p, q)^t(1, \lambda) \cdot \zeta = (-q + V(p)^t\lambda) \cdot \zeta \leq 0$. This implies that $(-q + V(p)^t\lambda) \leq 0$ in $\mathbb{R}^J$, or $q \geq V(p)^t\lambda$. □

We remark that the short sale constraints on the portfolios enlarges the set of arbitrage free asset prices. Indeed, even if an arbitrage portfolio $\zeta$ exists, if it is not non negative, then the consumers cannot exploit it by selling an unbounded quantity of it to get an higher and higher payoff since the short sale constraint is binding. So, the restriction on the possible asset prices are less demanding. By adapting the above result, we can check that if we have short sale constraints on some assets and not on the remaining assets, then the characterisation must be modified as follows:

there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q_j \geq \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V_j(p,\xi)$ for the assets $j$ with a short sales constraints and $q_j = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V_j(p,\xi)$ for the other.

This adaptation is left as an exercise.

**Exercise 13** *We consider a date-event tree with three states at the second period. For the following nominal financial structure with two assets, draw on a two dimensional space the set of no arbitrage asset prices with short sale constraints.*

*1)* $V = \begin{pmatrix} 1/2 & 2 \\ 1 & 1 \\ 2 & 1/2 \end{pmatrix}$.

*2)* $V = \begin{pmatrix} 0 & 2 \\ 1/2 & 1 \\ 1 & 0 \end{pmatrix}$.

*3)* $V = \begin{pmatrix} 1/2 & 2 \\ -1 & -1 \\ 2 & 1/2 \end{pmatrix}$.

*4)* $V = \begin{pmatrix} -1 & 2 \\ 0 & 0 \\ 1 & -1/2 \end{pmatrix}$.

## 5.6 Complete financial structures

In this subsection, we are considering unconstrained financial structure, that is $Z_i = \mathbb{R}^{\mathcal{J}}$ for all $i$. As mentioned at the beginning of the course, the contingent commodity market is the benchmark as it leads to Pareto optimal equilibrium allocation. As shown previously, with a general financial structure, for an arbitrage free price pair $(p,q)$, the financially affordable consumptions are always financially affordable for the contingent commodity market for the suitable discounted prices according to the present value vector associated to $q$. So, we say that a financial structure is complete if the possibilities of transfer among time and among states of nature are the same as the one offer by the contingent commodity market. The consumer can freely transfer facing only a global constraint given by the present value of her future endowments.

So, we have the following formal definition:

**Définition 9** The unconstrained financial structure $V$ is complete at the price

$p$, if for every present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, the two following sets are equal:

$$\mathcal{B}^W(\pi) = \{x \in \mathbb{R}^{\mathbb{L}} \mid \pi \cdot x \leq 0\}$$

and

$$\mathcal{B}^{\mathcal{F}}(p,q) = \left\{ x \in \mathbb{R}^{\mathbb{L}} \middle| \exists z \in \mathbb{R}^{\mathcal{J}} \begin{array}{l} p(\xi_0) \cdot x(\xi_0) + q \cdot z \leq 0 \\ p(\xi) \cdot x(\xi) \leq V(p,\xi) \cdot z, \quad \forall \xi \in \mathbb{D}_1 \end{array} \right\}$$

where $q = V(p)^t \lambda$ and $\pi = (p(\xi_0), (\lambda_\xi p(\xi))_{\xi \in \mathbb{D}_1})$.

**Remark 10** The financial structure associated to the full set of Arrow securities is complete as it has been shown in Proposition 4.

We remark that if the financial structure $V$ is complete, then for a consumer having a consumption set $X_i$ and initial endowments $e_i$, then the two budget sets $B_i^W(\pi, \pi \cdot e_i)$ and $B_i^{\mathcal{F}}(p,q)$ are equal

$$B_i^W(\pi, \pi \cdot e_i) = \{x_i \in X_i \mid \exists \chi_i \in \mathcal{B}^W(\pi), x_i = e_i + \chi_i\}$$

and

$$B_i^{\mathcal{F}}(p,q) = \{x_i \in X_i \mid \exists \chi_i \in \mathcal{B}^{\mathcal{F}}(p,q), x_i = e_i + \chi_i\}$$

So, a complete financial structure offers the same range of possibilities as a contingent commodity market. Hence, one deduces immediately the following correspondence between the equilibrium allocations for the two market structure when the non satiation state by state holds true.

**Proposition 18** *Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i \in \mathcal{I}}, V)$ be an unconstrained financial economy, which satisfies Assumption NSS. We assume that the financial structure is complete at a spot price $\bar{p}$. Let $(\bar{x}_i) \in \prod_{i \in \mathcal{I}} X_i$ and $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$. Then, the two conditions are equivalent:*

*a) There exists $(\bar{z}_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $((\bar{x}_i, \bar{z}_i), \bar{p}, \bar{q})$ is a financial equilibrium with $\bar{q} = V(\bar{p})^t \lambda$.*

*b) $((\bar{x}_i), \bar{\pi})$ is a contigent commodity equilibrium with $\bar{\pi} = (\bar{p}(\xi_0), (\lambda_\xi \bar{p}(\xi))_{\xi \in \mathbb{D}_1})$.*

We now provide a characterisation of a complete financial structure.

**Proposition 19** *Let us consider an unconstrained financial structure $V$ and a spot price vector $p \in \mathbb{R}^{\mathbb{L}} \setminus \{0\}$. Then:*

*a) If the rank of $V(p)$ is equal to $\sharp \mathbb{D}_1$, that is $V(p)$ is onto, then $V$ is complete at $p$.*

*b) If $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$ and $V$ is complete at $p$, then the rank of $V(p)$ is equal to $\sharp \mathbb{D}_1$.*

**Proof.** We consider a present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$.

a) It suffices to prove that for all $\mathcal{B}^W(\pi) \subset \mathcal{B}^{\mathcal{F}}(p,q)$. Let $x \in \mathcal{B}^W(\pi)$. Since $V(p)$ is onto, there exists $z \in \mathbb{R}^{\mathcal{J}}$ such that $(p(\xi)\cdot x(\xi))_{\xi\in\mathbb{D}_1} = V(p)z$. Then,

$$\begin{aligned} p(\xi_0)\cdot x(\xi_0) &= \pi(\xi_0)\cdot x(\xi_0) \le -\textstyle\sum_{\xi\in\mathbb{D}_1}\pi(\xi)\cdot x(\xi) \\ &= -\textstyle\sum_{\xi\in\mathbb{D}_1}\lambda_\xi p(\xi)\cdot x(\xi) \\ &= -\lambda\cdot V(p)z = -V(p)^t\lambda\cdot z \\ &= -q\cdot z \end{aligned}$$

so $p(\xi_0)\cdot x(\xi_0) + q\cdot z \le 0$ and $x \in \mathcal{B}^{\mathcal{F}}(p,q)$.

b) Let $v \in \mathbb{R}^{\mathbb{D}_1}$. We define $x \in \mathbb{R}^{\mathbb{L}}$ as follows: for all $\xi \in \mathbb{D}_1$, $x(\xi) = \frac{1}{\|p(\xi)\|^2}v(\xi)p(\xi)$ and $x(\xi_0) = \frac{1}{\|p(\xi_0)\|^2}(-\lambda\cdot v)\,p(\xi_0)$. We can do it since by assumption $p(\xi)\neq 0$ for all $\xi\in\mathbb{D}$.

We remark that $\pi\cdot x = -\lambda\cdot v + \sum_{\xi\in\mathbb{D}_1}\lambda_\xi v(\xi) = 0$ since $\pi(\xi_0) = p(\xi_0)$ and $\pi(\xi) = \lambda_\xi p(\xi)$ for all $\xi\in\mathbb{D}_1$. So $x\in\mathcal{B}^W(\pi)$. Since $V$ is complete, $x\in\mathcal{B}^{\mathcal{F}}(p,q)$ and there exists $z\in\mathbb{R}^{\mathcal{J}}$ such that $p(\xi_0)\cdot x(\xi_0) + q\cdot z \le 0$ and $(p(\xi)\cdot x(\xi))_{\xi\in\mathbb{D}_1} \le V(p)z$. We remark that $p(\xi_0)\cdot x(\xi_0) = -\lambda\cdot v$ and $p(\xi)\cdot x(\xi) = v(\xi)$. So, we get $v \le V(p)z$ and $q\cdot z\le\lambda\cdot v$. Doing the inner product with $\lambda$ for the first inequality, we keep the inequality since $\lambda$ is positive and so $\lambda\cdot v \le \lambda\cdot V(p)z = V(p)^t\lambda\cdot z = q\cdot z$. So $\lambda\cdot v = q\cdot z$ and all inequalities are actually equalities, that is $v = V(p)z$. Hence $v$ is in the range of $V$, which shows that $V$ is onto. □

**Example.** Let us take a simple date-event tree $\mathbb{D}$ with two nodes at period 1, $\xi_1$ and $\xi_2$. There is a unique nominal asset $j$ with the payoff vector $V_j = (1,0)$, so the payoff matrix $V$ is not onto. Nevertheless, if we consider the spot price $p = (1,1,0)$ and the present value vector $\lambda = (1,1)$ and the associated asset price $q = 1$, we check that $\mathcal{B}^W(\pi) = \{x\in\mathbb{R}^3 \mid x(\xi_0) + x(\xi_1) \le 0\}$ and $\mathcal{B}^{\mathcal{F}}(p,q) = \{x \in \mathbb{R}^3 \mid \exists z\in\mathbb{R}, x(\xi_0) + z \le 0, x(\xi_1)\le z\}$ are equal. So, when a spot price vector is vanishing at a node, the asset structure can be complete even if the return matrix $V$ is not onto.

We leave as an exercise to prove that the asset structure is complete at $p$ if and only if the range of $V(p)$ contains the linear subspace of $\mathbb{R}^{\mathbb{D}_1}$ defined as $\{v\in\mathbb{R}^{\mathbb{D}_1} \mid v(\xi) = 0 \text{ if } p(\xi) = 0\}$.

**Remark 11** If the payoff matrix is onto at $p$, then for each no arbitrage asset price $q$, there is a unique present value vector $\lambda\in\mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V(p)^t\lambda$ since $V(p)^t$ is one to one. So, some authors define a complete market structure as a market structure with a unique present value vector.

We also remark that when the payoff matrix is onto at $p$, so the market is complete, then the cost function or over hedging price $q^+$ is linear on $\mathbb{R}^{\mathbb{D}_1}$ and just equal to $\lambda\cdot v$ for the unique present value vector $\lambda$ associated to $v$.

Finally, we also remark that in the absence of redundant asset, $V(p)$ is onto if and only if $V(p)$ is regular. So, in many articles, the authors simply assume that the return matrix is regular, which implies that the markets are complete and that there exists a unique portfolio associated to any wealth transfer $v$ on $\mathbb{D}_1$.

**Exercise 14** *We consider a date-event tree with three states at the second period. For the following nominal financial structures, check which ones are complete for all non-zero spot prices.*

*1)* $V = \begin{pmatrix} 1/2 & 2 & 5 \\ 1 & 1 & 4 \\ 2 & 1/2 & 5 \end{pmatrix}.$

*2)* $V = \begin{pmatrix} 0 & 2 \\ 1/2 & 1 \\ 1 & 0 \end{pmatrix}.$

*3)* $V = \begin{pmatrix} 1/2 & 2 & 0 \\ -1 & -1 & 1 \\ 2 & 1/2 & 3 \end{pmatrix}.$

*4)* $V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$

**Remark 12** For the nominal asset structures or for the numéraire asset structures, the rank of the payoff matrix does not depend on the spot price as long as the value of the numéraire is not vanishing. So, the market is complete for all spot prices. For real asset structure, the rank of the matrix depends on the spot price and the rank can drop, which means that the market is no more complete for some spot prices whereas it is complete for almost all spot prices.

**Exercise 15** *We consider a date-event tree with two states at the second period, two perishable commodities and a financial structure with two real assets. The first asset* 1 *delivers the value of the basket* $(1,0)$ *in each state and the second asset delivers the value of the basket* $(0,1)$ *in each state.*

*Determine the set of spot price vectors for which the market structure is complete.*

We conclude this subsection by explaining what could be the role of option to complete a financial structure. Let us assume that we have a unique asset, which discriminates the state of the world tomorrow, in the sense that if $(\xi_1, \xi_2, \ldots, \xi_k)$ are the $k$ states at period 1, $v(\xi_1) < v(\xi_2) < \ldots < v(\xi_k)$. Then, an option at the strike price $\sigma$ is a financial asset which promises to deliver $v(\xi)$ for the states such that $v(\xi) \geq \sigma$ and 0 otherwise. We introduce $k-1$ options at the strike price $v(\xi_\kappa)$ for $\kappa = 2, \ldots, k$. Show that the financial structure build with the initial asset and the $k-1$ options is complete and without redundant assets.

## 5.7 Equivalent financial structures

In this subsection, we consider a given exchange economy with the uncertainty described by the tree $\mathbb{D}$ over two periods $\mathcal{E} = ((X_i, u_i, e_i)_{i \in \mathcal{I}})$ and we compare financial structures, which are represented by their payoff matrix $V$ which may depend on the spot price $p$.

We remark that, at equilibrium, what matters for the consumer is not the asset themselves but the possible transfer of wealth among period and states, that is the range of the full payoff matrix $W(p,q)$. So, different financial structures may have the same outcome in terms of equilibrium consumptions. In that case, we say that these financial structures are equivalent.

More precisely:

**Définition 10** Let $p$ be a spot price vector and $V_1$ and $V_2$ be two financial structures. $V_1$ is equivalent to $V_2$ at the price $p$ if for all $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, $\mathcal{B}^{\mathcal{F}}_{V_1}(p, V_1(p)^t\lambda) = \mathcal{B}^{\mathcal{F}}_{V_2}(p, V_2(p)^t\lambda)$.

The following proposition shows that two financial structures have the same equilibrium consumptions if they are equivalent for the equilibrium spot price.

**Proposition 20** *Let $V_1$ and $V_2$ two financial structures. If $\mathcal{E}$ satisfies Assumption NSS, then, if $((\bar{x}_i, \bar{z}_i), \bar{p}, \bar{q})$ is a financial equilibrium for the financial structure $V_1$ and $V_1$ and $V_2$ are equivalent at $\bar{p}$, then there exists $\bar{\zeta} \in (\mathbb{R}^{\mathcal{J}_2})^{\mathcal{I}}$ and an asset price vector $\bar{\chi} \in \mathbb{R}^{\mathcal{J}_2}$ such that $((\bar{x}_i, \bar{\zeta}_i), \bar{p}, \bar{\chi})$ is a financial equilibrium for the financial structure $V_2$.*

**Proof.** The proof is a direct consequence of the fact that, thanks to Assumption NSS, the market clearing condition on the asset market is redundant. Furthermore, again from Assumption NSS and the characterisation of the non arbitrage asset price, there exists a present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V_1(\bar{p})^t\lambda$ So, if we let $\bar{\chi} = V_2(\bar{p})^t\lambda$, since the two markets structures are equivalent:

$$\begin{array}{ll} B^{\mathcal{F}}_{V_1 i}(\bar{p}, \bar{q}) & = B^{\mathcal{F}}_{V_1 i}(\bar{p}, V_1(\bar{p})^t\lambda) \\ & = \{x_i \in X_i \mid \exists x'_i \in \mathcal{B}^{\mathcal{F}}_{V_1}(\bar{p}, V_1(\bar{p})^t\lambda), x_i = e_i + x'_i\} \\ & = \{x_i \in X_i \mid \exists x'_i \in \mathcal{B}^{\mathcal{F}}_{V_2}(\bar{p}, V_2(\bar{p})^t\lambda), x_i = e_i + x'_i\} \\ & = B^{\mathcal{F}}_{V_2 i}(\bar{p}, V_2(\bar{p})^t\lambda) \\ & = B^{\mathcal{F}}_{V_2 i}(\bar{p}, \bar{\chi}) \end{array}$$

□

**Remark 13** According to the previous sub-section and the definition of equivalent financial structures, we remark that a financial structure is complete if and only if it is equivalent to the financial structure with all contingent commodities or to the structure with all Arrow securities.

We now characterise equivalent structures by their ranges.

**Proposition 21** *Let $V_1$ and $V_2$ two financial structures. For a spot price $p$ such that $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$, $V_1$ and $V_2$ are equivalent at $p$, if and only if the range of $V_1(p)$ is equal to the range of $V_2(p)$.*

**Proof.** Let $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$. Let $q = V_1(p)^t\lambda$ and $\chi = V_2(p)^t\lambda$. Let us assume that $V_1$ and $V_2$ are equivalent at $p$. Let $v$ in the range of $V_1(p)$ and let $z \in \mathbb{R}^{\mathcal{J}_1}$ such that $v = V_1(p)z$. We define $x \in \mathbb{R}^{\mathbb{L}}$ as follows: for all $\xi \in \mathbb{D}_1$, $x(\xi) = \frac{1}{\|p(\xi)\|^2}v(\xi)p(\xi)$ and $x(\xi_0) = \frac{1}{\|p(\xi_0)\|^2}(-\lambda \cdot v)\, p(\xi_0)$. We can do it since by assumption $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$. So, $x \in \mathcal{B}^{\mathcal{F}}_{V_1}(p,q)$ since $p(\xi) \cdot x(\xi) = v(\xi) = V_1(p,\xi) \cdot z$ for all $\xi \in \mathbb{D}_1$ and $p(\xi_0) \cdot x(\xi_0) = -\lambda \cdot v = -\lambda \cdot V_1(p)z = -V_1(p)^t\lambda \cdot z = -q \cdot z$. So, since $V_1$ and $V_2$ are equivalent at $p$, $x$ belongs to $\mathcal{B}^{\mathcal{F}}_{V_2}(p,\chi)$. Hence there exists $\zeta \in \mathbb{R}^{\mathcal{J}_2}$ such that $p(\xi_0) \cdot x(\xi_0) + \chi \cdot \zeta \leq 0$ and $(p(\xi) \cdot x(\xi))_{\xi \in \mathbb{D}_1} \leq V_2(p)z$. We remark that $p(\xi_0) \cdot x(\xi_0) = -\lambda \cdot v$ and $p(\xi) \cdot x(\xi) = v(\xi)$. So, we get $v \leq V(p)\zeta$ and $\chi \cdot \zeta \leq \lambda \cdot v$. Doing the inner product with $\lambda$ for the first inequality, we keep the inequality since $\lambda$ is positive and so $\lambda \cdot v \leq \lambda \cdot V_2(p)\zeta = V_2(p)^t\lambda \cdot \zeta = \chi \cdot z$. So $\lambda \cdot v = \chi \cdot \zeta$ and all inequalities are actually equalities, that is $v = V_2(p)\zeta$. Hence $v$ is in the range of $V_2(p)$. We prove in the same way that the range of $V_2(p)$ is included in the range of $V_1(p)$, so they are equal.

Conversely, if the ranges coincide, then let $x \in \mathcal{B}^{\mathcal{F}}_{V_1}(p,q)$. Then, there exists $z \in \mathbb{R}^{\mathcal{J}_1}$ such that $p(\xi) \cdot x(\xi) \leq V_1(p,\xi) \cdot z$ for all $\xi \in \mathbb{D}_1$ and $p(\xi_0) \cdot x(\xi_0) \leq -q \cdot z$. Let $\zeta \in \mathbb{R}^{\mathcal{J}_2}$ such that $V_1(p)z = V_2(p)\zeta$, which exists from the equality of the range. So, $p(\xi) \cdot x(\xi) \leq V_2(p,\xi) \cdot \zeta$ for all $\xi \in \mathbb{D}_1$. Furthermore, $-\chi \cdot \zeta = -V_2(p)^t\lambda \cdot \zeta = -\lambda \cdot V_2(p)\zeta = -\lambda \cdot V_1(p)z = -V_1(p)^t\lambda \cdot z = -q \cdot z$. So, $x$ belongs to $\mathcal{B}^{\mathcal{F}}_{V_2}(p,\chi)$ and the converse inclusion is proved in the same way. □

**Remark 14** Note that we do not use the assumption $p(\xi) \neq 0$ for all $\xi$ in the second part of the proof. So, the equality of the ranges implies the equivalence for all spot prices.

In the subsection about redundant assets, the substructure obtained by deleting the redundant assets is actually a process to build a structure, which is equivalent to the initial one and without redundant asset.

**Proposition 22** *In the unconstrained case, each structure is equivalent to a structure without redundant asset.*

In many papers, the authors assume that the assets have non negative payoffs and the bond among the financial structure. The exercise below show that if a structure has a portfolio with positive payoffs at each state, then it is equivalent to a structure where all portfolio have positive payoffs at each state.

**Exercise 16** *Let $V$ be a nominal financial structure. We assume that there exists a portfolio $z^+$ such that $Vz^+ \gg 0$. Show that there exists an equivalent structure $\tilde{V}$ such that for $V_j \gg 0$ for all $j \in \tilde{\mathcal{J}}$. Hint: take a basis $(v_1, v_2, \ldots, v_k)$ of the orthogonal complement of $Vz^+$ in the range of $V$, take an antecedent $z_k$ by $V$ for all $v_k$, show that there exists $t > 0$ small enough such that $V(z^+ + tz_k) \gg 0$ for all $k$. Define the structure $\tilde{V}$ with $k+1$ assets by $V_\kappa = V(z^+ + tz_\kappa)$ for $\kappa = 1, \ldots k$ and $V_{k+1} = Vz^+$. Show that it is equivalent to $V$ by the equality of the ranges.*

**Exercise 17** *Let $V$ be a nominal financial structure. Let $\tilde{V}$ be the financial* *structure obtained by including one additional asset $k$ to the existing assets. Show that $\tilde{V}$ is equivalent to $V$ if and only if $k$ is redundant in the structure $\tilde{V}$.*

**Exercise 18** *Let $V$ be a complete nominal financial structure without redundant asset. Let $\hat{V}$ be the financial structure obtained by deleting one asset $j_0$ from the existing assets. Show that $\hat{V}$ is not complete.*