Mathematics of Insurance and Risk
Tutorial classes

Exercise sheet for Market Risk Measures

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Exercise 1: Minimum variance portfolio with a twist (Exam 2019)

Fix $d \in \mathbb{N}$ and consider d-risky assets $(S^1, \ldots, S^d)$ such that their risky excess returns are assumed to follow a multivariate Gaussian $\mathcal{N}(m, \Sigma)$, with mean vector $m \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{S}^d_+$. An investor seeks to solve the following optimization problem to find the optimal vector of weights $w = (w_1, \ldots, w_d)^\top$ invested in the stocks $(S^1, \ldots, S^d)$:

$$\begin{aligned} &\min_{w \in \mathbb{R}^d} && \frac{1}{2} w^\top \Sigma w \\ &\text{subject to} && e^\top w = 1. \end{aligned} \tag{1}$$

Here e is the vector of ones in $\mathbb{R}^d$, i.e. all of its components are equal to one and $\top$ denotes the transpose operation.

Classical Minimum Variance Portfolio: For questions 1 to 5, we assume that Σ is invertible.

1. Justify the appellation 'minimum variance portfolio'?
2. Justify that problem (1) is equivalent to the following problem
$$\max_{\beta \in \mathbb{R}} \min_{w \in \mathbb{R}^d} \frac{1}{2} w^\top \Sigma w - \beta(e^\top w - 1). \tag{2}$$
3. Solve the optimization problem and show that the minimum variance portfolio w_{MV} is given by
$$w_{\text{MV}} = \frac{\Sigma^{-1} e}{e^\top \Sigma^{-1} e}.$$
4. What is the Sharpe ratio of w_{MV}?
5. Is the invertibility assumption of Σ satisfied in practice? Justify.

Minimum Variance Portfolio with an l^2-twist: From now on we no longer assume that Σ is invertible and we consider the previous optimization problem but under an additional l^2-constraint on the weights:

$$\begin{aligned} &\min_{w \in \mathbb{R}^d} && \frac{1}{2} w^\top \Sigma w \\ &\text{subject to} && e^\top w = 1 \quad \text{and} \quad w^\top w \leq c, \end{aligned} \tag{3}$$

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where $c > 0$ is a given constant.

6. Prove that the problem does not admit a solution if $c < 1/d$.

7. Justify that problem (3) is equivalent to the following maximization problem:

$$\max_{\gamma \in \mathbb{R}_+} \max_{\beta \in \mathbb{R}} \min_{w \in \mathbb{R}^d} \frac{1}{2} w^\top \Sigma w - \beta(e^\top w - 1) + \gamma(w^\top w - c) \tag{4}$$

8. Keeping $\gamma > 0$ fixed, show that the solution to the inner minimization problem is given by

$$\widetilde{w}_{\mathrm{MV}}(\gamma) = \frac{\Sigma(\gamma)^{-1} e}{e^\top \Sigma(\gamma)^{-1} e}$$

where $\Sigma(\gamma)$ is a $d \times d$-matrix to be determined in terms of Σ and γ. Justify that $\Sigma(\gamma)$ is invertible.

9. What is the advantage of introducing the l^2-constraint?

Exercise 2: Pareto distributions and VaR

Let X, Y be two independent random variables following a Pareto distribution $(1, 1)$, meaning that the density is given by

$$f(x) = \mathbf{1}_{x \geq 0} \frac{1}{(1+x)^2}, \quad x \in \mathbb{R}.$$

1. Verify that f is indeed a density function and that

$$\mathbb{P}(X \geq t) = \frac{1}{1+t}, \quad t \geq 0.$$

2. Compute $\mathrm{VaR}_\alpha(X)$ for $\alpha \in (0, 1)$.

3. Compute $\mathbb{P}(X + Y \geq t)$, for $t \geq 0$.

4. Compare $\mathrm{VaR}_\alpha(X + Y)$ and $\mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$, for any $\alpha \in (0, 1)$.

5. Comment.

Exercise 3: On spherical distributions (Exam 2019)

Fix $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $d \in \mathbb{N}$. We will denote by $\top$ the transpose operation and by $\|t\| = \sqrt{t^\top t} = \sqrt{t_1^2 + t_2^2 + \ldots + t_d^2}$ the euclidean norm of a vector $t = (t_1, \ldots, t_d)^\top \in \mathbb{R}^d$. For a d-dimensional vector-valued random variable $X = (X_1, \ldots, X_d)^\top$ we denote by ϕ_X its characteristic function, that is

$$\phi_X(t) = \mathbb{E}\left[\exp(it^\top X)\right], \quad t \in \mathbb{R}^d.$$

We say that the d-dimensional vector $X = (X_1, \ldots, X_d)^\top$ has a spherical distribution if there exists a function $\psi : \mathbb{R} \to \mathbb{R}$ such that its characteristic function satisfies

$$\phi_X(t) = \psi(t^\top t) = \psi(t_1^2 + t_2^2 + \ldots + t_d^2).$$

We will write $X \sim \mathcal{S}_d(\psi)$ to denote that X has a spherical distribution with characteristic function $\psi(t^\top t)$. Throughout this exercise we fix $X \sim \mathcal{S}_d(\psi)$ for some function ψ and we define the $\mathbb{R}^d$-valued random variable

$$Y = \mu + CX, \tag{5}$$

where $\mu \in \mathbb{R}^d$ and $C \in \mathbb{R}^{d\times d}$.

1. Let $Z \sim \mathcal{N}(0, I_d)$, where I_d is the identity matrix. Show that $Z \sim S_d(\psi_0)$ for a function ψ_0 to be determined.

2. Fix $a \in \mathbb{R}^d$. Show that

$$a^\top X \overset{d}{=} \|a\| X_1,$$

where $\overset{d}{=}$ stands for the equality in distribution and we recall that X_1 is the first component of the vector X.

3. Deduce that

$$a^\top Y \overset{d}{=} a^\top \mu + \|C^\top a\| X_1,$$

for all $a \in \mathbb{R}^d$.

Part 1. Value-at-Risk. Fix $\alpha \in (0, 1)$ and set $d = 2$.

4. Justify that $\mathrm{VaR}_\alpha(U) = \mathrm{VaR}_\alpha(V)$, for any two random variables U and V such that $U \overset{d}{=} V$.

5. Deduce that

$$\mathrm{VaR}_\alpha(a^\top Y) = a^\top \mu + \|C^\top a\| \mathrm{VaR}_\alpha(X_1),$$

for all $a \in \mathbb{R}^d$.

6. Using the above, show that $\mathrm{VaR}_\alpha(Y_1 + Y_2) \leq \mathrm{VaR}_\alpha(Y_1) + \mathrm{VaR}_\alpha(Y_2)$.

7. What is the financial interpretation of the previous inequality? Does it hold for more general distributions Y?

Part 2. Optimization problem. More generally, let $d \geq 2$ and and consider d-risky assets $(S^1, \ldots, S^d)$ such that their risky excess returns are assumed to follow the distribution Y as in (5). We seek to find the optimal vector of weights $w = (w_1, \ldots, w_d)^\top$ invested in the stocks $(S^1, \ldots, S^d)$ mimimzing the Value-at-Risk of the portfolio:

$$\begin{aligned} \min_{w\in\mathbb{R}^d} \quad & \frac{1}{2}\mathrm{VaR}_\alpha(w^\top Y) \\ \text{subject to} \quad & e^\top w = 1 \quad \mu^\top w = r \end{aligned} \tag{6}$$

for a fixed level of returns $r > 0$, and $e = (1, \ldots, 1)^\top$ the vector of ones in $\mathbb{R}^d$.

8. Show that the minimization problem (6) is equivalent to

$$\begin{aligned} \min_{w\in\mathbb{R}^d} \quad & \frac{1}{2} w^\top \Sigma w \\ \text{subject to} \quad & e^\top w = 1. \end{aligned}$$

where Σ is a $d \times d$-matrix to be determined.

9. What problem do you recognize? Find the optimal vector of weights w^*.

10. Can we replace VaR_α in (6) by more general risk measures ρ? What properties should ρ satisfy to obtain the same conclusions? Does it work for the expected shortfall?

Exercise 4: Value-at-Risk (Exam 2019)

Let X denote a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which corresponds to the losses of a portfolio. Recall that the Value-at-Risk $\mathrm{VaR}_\alpha(X)$ of the portfolio X for the threshold α is defined by

$$\mathrm{VaR}_\alpha(X) = F_X^-(\alpha),$$

where F_X^- is given by

$$F_X^-(y) = \inf\{x \in \mathbb{R} : F_X(x) \geq y\}, \quad y \in (0,1),$$

and F_X is the cumulative distribution function of X.

1. Show that F^- is non-decreasing and deduce that for all $\alpha_1 \leq \alpha_2$, $\mathrm{VaR}_{\alpha_1}(X) \leq \mathrm{VaR}_{\alpha_2}(X)$.
2. Show how this result can be deduced from a graph on the Value-at-Risk, under a suitable assumption to be specified.

From now on, we assume that $X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}$. Fix a threshold $\alpha \in (0,1)$, $n \in \mathbb{N}$ and let $X_1, \ldots, X_n$ denote n independent observations of X, i.e. $X_i \sim \mathcal{N}(\mu, \sigma^2)$, $i = 1 \ldots, n$. We denote by $L_n(\mu, \sigma^2)$ the likelihood function of $(X_1, \ldots, X_n)$ and we set $l_n(\mu, \sigma^2) := \log L_n(\mu, \sigma^2)$.

3. Show that
$$\mathrm{VaR}_\alpha(X) = \mu + \sigma z_\alpha,$$
where $z_\alpha = F_{\mathcal{N}(0,1)}^-(\alpha)$ is the α-quantile of a standard Gaussian $\mathcal{N}(0,1)$.
4. Show that
$$l_n(\mu, \sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2.$$
5. Derive the maximum likelihood estimators $(\hat{\mu}, \hat{\sigma}^2)$ for (μ, σ^2) and deduce an estimator for $\mathrm{VaR}_\alpha(X)$.
6. What are the advantages and disadvantages of this method? Explain alternative methods correcting these issues.

Exercise 5: Wang risk measures (Exam 2019)

Let X be a non-negative random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{G}$ the following set of functions

$$\mathcal{G} = \{g : [0,1] \to [0,1] \text{ non-decreasing and right-continuous such that } g(0) = 0 \text{ and } g(1) = 1\}.$$

For any $g \in \mathcal{G}$, we define ρ_g by

$$\rho_g(X) = \int_0^\infty g(1 - F_X(x))dx,$$

where F_X is the cumulative distribution function of X.

1. Justify that ρ_g can be considered as a risk measure.
2. Verify that $\mathrm{id} : y \mapsto y$ belongs to $\mathcal{G}$ and show that $\rho_{\mathrm{id}}(X) = \mathbb{E}[X]$.

3. Fix $\alpha \in (0,1)$. Verify that $g_\alpha : y \mapsto \mathbf{1}_{\{y \geq 1-\alpha\}}$ belongs to $\mathcal{G}$ and compute $\rho_{g_\alpha}(X)$.

4. We recall that for any non-decreasing and right-continuous function we can define its Stieltjes measure dg given by $dg((s,t]) = g(t) - g(s)$ so that we have $g(t) = g(s) + \int_s^t dg(u)$, for all $s \leq t$. Show that, for all $g \in \mathcal{G}$,

$$\rho_g(X) = \int_0^1 \mathrm{VaR}_{1-\alpha}(X) dg(\alpha).$$

5. Fix $g \in \mathcal{G}$. Show that ρ_g is invariant by translation, positive homogeneous and monotone.

6. Is ρ_g sub-additive for all $g \in \mathcal{G}$? Justify.

7. Let $g \in \mathcal{G}$ twice differentiable with continuous first and second derivatives. Assume that F_X is continuous.

   (a) Recall the definition of the expected shortfall $\mathrm{ES}_\alpha(X)$ and show that

   $$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_p(X) dp, \quad \alpha \in (0,1)$$

   (b) Show that

   $$\rho_g(X) = -\int_0^1 \mathrm{ES}_{1-\xi}(X) \xi g''(\xi) d\xi + g'(1)\mathbb{E}[X].$$

   (c) Deduce that ρ_g is sub-additive if g is concave.

   (d) What can be said on ρ_g when g is concave?