Economic analysis of financial market
S1 2023-2024

J.-M. Bonnisseau

Arbitrage free asset prices

Outline

Outline

Arbitrage free financial structures

A $\sharp\mathbb{D} \times \mathcal{J}$ return matrix V, $Z_i = \mathbb{R}^{\mathcal{J}}$ for all i.
$(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ a pair of spot and asset price vectors.

Definition

The financial structure is arbitrage free at (p, q) if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p, q)z \in \mathbb{R}^{\mathbb{D}}_{+} \setminus \{0\}$.

A portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p, q)z \in \mathbb{R}^{\mathbb{D}}_{+} \setminus \{0\}$ is an arbitrage opportunity.

Arbitrage opportunity or free lunch

z arbitrage opportunity, then

a) $\sum_{j|\xi(j)=\xi_0} q_j zj \leq 0$,

b) $\sum_{j\in\mathcal{J}} v_j(p,\xi)z_j - \sum_{j|\xi(j)=\xi} q_j zj \geq 0$

for all $\xi \in \mathbb{D}^+(\xi_0)$

with at least one strict inequality.

Utility maximisation and absence of arbitrage

Proposition

Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$ *be an unconstrained financial structure (*$Z_i = \mathbb{R}^{\mathcal{J}}$ *for all* $i \in \mathcal{I}$*) satisfying Assumption NSS. For a commodity-asset price pair* (p, q)*, if there exists a consumer i and* $x_i \in X_i$*, which is optimal in the budget set* $B_i^{\mathcal{F}}(p, q)$*, then the financial structure is arbitrage free at* (p, q)*.*

Absence of arbitrage at equilibrium

Proposition

If $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$, then the financial structure is arbitrage free at (p^*, q^*).

About the market clearing condition on the financial market

Proposition

Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$ *be an unconstrained financial structure* ($Z_i = \mathbb{R}^{\mathcal{J}}$ *for all* $i \in \mathcal{I}$) *satisfying Assumption NSS. Let* $((x_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ *such that*
(a) [Preference maximization] for every $i \in \mathcal{I}$,
x_i^* *is a "maximal" element of* u_i *in the budget set* $B_i^{\mathcal{F}}(p^*, q^*)$ *in the sense that there exists* $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ *such that*

$$p^*(\xi) \cdot x_i^*(\xi) + \sum_{j | \xi(j) = \xi} q_j^* \tilde{z}_{ij} \leq p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i$$

for all $\xi \in \mathbb{D}$ *and* $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;

Proposition continued

(b) [Market clearing condition on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

Then, there exists $(z_i^*) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ *such that* $((x_i^*, z_i^*), p^*, q^*)$ *is a financial equilibrium of* $\mathcal{E}^{\mathcal{F}}$*,*

An example without Assumption NSS

- *An economy with two periods and one commodity per state ;*
- $\mathbb{D}_1 = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ *;*
- *two consumers* $\mathcal{I} = \{1, 2\}$*, consumptions sets* $\mathbb{R}^5_+$*, initial endowments are* $e_1 = e_2 = (1, 1, 1, 1, 1)$ *;*
- *utility functions :* $u_1(x_1) = x_{11} - x_{14} + \min\{1, x_{12}\} + \min\{1, x_{13}\}$ *and* $u_2(x_2) = -x_{21} + x_{24} + \min\{1, x_{22}\} + \min\{1, x_{23}\}$ *;*
- *Financial structure, two nominal assets with* $V = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$
- $q = (0, 0)$ *not arbitrage free (To be checked).*

Charactersation of arbitrage free asset prices

Proposition

The financial structure V is arbitrage free at (p, q) *if and only if there exists* $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ *such that*

$$\lambda_{\xi(j)} q = \sum_{\xi \in \mathbb{D}} \lambda_\xi v_j(p, \xi) = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi v_j(p, \xi)$$

Note that we can normalize λ so that $\lambda_{\xi_0} = 1$.

The set of arbitrage free asset prices : $Q(p)$

Interpretation of the characterisation

Remark

λ_ξ is called the present value at date 0 of one unit of account in state ξ and the vector λ is called the present value vector across states.
j^ξ Arrow security associated to the state ξ, then, according to the no-arbitrage characterisation, the price of this Arrow security is equal to λ_ξ.

Relation between the Walras and the financial budget sets

Proposition

Let us consider a financial structure V and an exchange economy. If V is arbitrage free at (p, q) *and* $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ *is a present value vector associated to q then*

$$B_i^{\mathcal{F}}(p, q) \subset B_i^{W}(\pi, \pi \cdot e_i)$$

where π *is defined by* $\pi(\xi_0) = p(\xi_0)$ *and* $\pi(\xi) = \lambda_\xi p(\xi)$ *for all* $\xi \in \mathbb{D}_1$.

No-arbitrage and contingent commodities

Remark

If we consider a complete set of contingent commodities, the no-arbitrage condition tells us that the price at node ξ_0 *of the contingent commodities contracts of node* ξ *is positively proportional to the spot price at this node. The present value vector is just this coefficient of proportionality.*

No-arbitrage and Arrow securities

Remark

If we consider a complete set of Arrow security, the no-arbitrage characterisation holds true if and only if all Arrow security prices at node ξ_0 *are positive. The components of the present value vector is just the price of the Arrow securities.*

Utility maximisation over the financial budget set

An optimal consumption $((x_i^*, z_i^*)$ with the prices (p^*, q^*) with differentiable utility functions and interior solution : then first order necessary condition :

$$\begin{cases} \nabla u_i(x_i^*) = (\mu_{i\xi} p(\xi))_{\xi \in \mathbb{D}} \\ \text{For all } j,\ \mu_{i\xi(j)} q_j^* = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi v_j(p^*, \xi) \end{cases}$$

Then $\lambda_{i\xi} = \frac{\mu_{i\xi}}{\mu_{i\xi_0}}$ is a present value vector associated to the no arbitrage equilibrium asset price q^*.

Outline

A simpler formula

With two periods, the unique node of issuance is ξ_0, so

$$Q(p) = V_{-\xi_0}(p)^t \mathbb{R}^{\mathbb{D}1}_{++}$$

Proposition

$V(p)$ is of rank $\sharp\mathcal{J}$ if and only if $Q(p)$ is open.

Uniqueness of the present value vector

Proposition

Let V be a financial structure, which is arbitrage free for the pair (p, q). Let $\bar{\lambda} \in \Lambda = \{\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++} \mid q = V(p)^t \lambda\}$. Then

$$\Lambda = \left(\{\bar{\lambda}\} + \mathrm{Ker}\, V(p)^t\right) \cap \mathbb{R}^{\mathbb{D}_1}_{++}$$

So, Λ is a singleton if and only if $\mathrm{Ker}\, V(p)^t = \{0\}$ or, equivalently, $V(p)$ is onto.

Link with the finance literature

Interest rate

the bond is among the asset. Its payoffs are equal to 1 *in all states of* $\mathbb{D}_1$*, so the price of this bond is* $\bar{\lambda} = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi$*. This is the price to be paid today to be sure to have one additional unit of account in each state of nature tomorrow. So, in terms of interest rate* r *between the current date and tomorrow,* $\bar{\lambda} = \frac{1}{1+r}$*.*

Risk neutral probability measure

The discounted present value vector $\mu = (1 + r)\lambda$ *is a probability measure on the state tomorrow called the risk neutral probability measure.*

Utility maximisation of a risk neutral agent

One commodity per state with normalised spot prices at 1 ;
a risk-neutral agent having a subjective probability χ on the $\mathbb{D}_1$.
Her utility function is :

$$u_i(x_i) = x_i(\xi_0) + \frac{1}{1+r} \sum_{\xi \in \mathbb{D}_1} \chi(\xi) x_i(\xi)$$

Maximisation of the utility at an interior solution in the Walras budget set $B_i^W(\pi)$ associated to the discounted prices, gives

$$\chi(\xi) = \lambda(\xi) \quad \forall \xi \in \mathbb{D}_1$$

Optimality continued

Remark

if there exists a unique present value vector, that is, if $V(p)$ is onto, then one concludes that all gradient vectors $(\nabla u_i(x_i^*))_{i\in\mathcal{I}}$ are colinear and the equilibrium allocation $(x_i^*)_{i\in\mathcal{I}}$ is Pareto optimal.

Remark

If they are several present value vectors, we cannot conclude and generically, the equilibrium allocation is not Pareto optimal. x_i^* is an optimal consumption in the Walras budget set associated to the personalised discounted price π_i defined by $\pi_i(\xi_0) = p(\xi_0)$ and $\pi_i(\xi) = \lambda_{i\xi}p(\xi)$ for all $\xi \in \mathbb{D}_1$. So, each agent maximises her welfare but not according to the same prices.

Optimality continued

Using the usual differentiability assumptions on the utility functions, one can prove that, generically at the competitive equilibrium, the individual transfers $(p^*(\xi) \cdot (x_i^*(\xi) - e_i(\xi))_{\xi \in \mathbb{D}})$ generate a subspace of dimension $\min\{\sharp\mathcal{I}, \sharp\mathbb{D}\}$. So, if the number of agents is greater than the number of states of nature, it is impossible to reach a competitive allocation if the rank of $V(p)$ is strictly smaller than $\sharp\mathbb{D}_1$ since, then, the transfers belong to the marketable space, which has a dimension strictly smaller than $\sharp\mathbb{D}_1$.

About the Cass trick

We can modify the definition of a financial equilibrium by assuming that one agent is maximising over a Walras budget set instead of maximising over the financial budget set.

Proposition

Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i\in\mathcal{I}}, V)$ *be a financial economy satisfying Assumption NSS. Let* $((x_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ *such that :*
(a) q^* *is a no arbitrage asset price associated to a present value vector* $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$
(b) there exists an agent $i_0 \in \mathcal{I}$ *such that* $x_{i_0}^*$ *is a "maximal" element of* u_{i_0} *in the budget set* $B^W_{i_0}(\pi^*, \pi^* \cdot e_{i_0})$ *where* π^* *is defined by* $\pi^*(\xi_0) = p^*(\xi_0)$ *and* $\pi^*(\xi) = \lambda_\xi p^*(\xi)$ *for all* $\xi \in \mathbb{D}_1$*.*

Proposition continued

(c) for every $i \in \mathcal{I}$, $i \neq i_0$,
x_i^* is a "maximal" element of u_i in the budget set $B_i^{\mathcal{F}}(p^*, q^*)$ in the sense that there exists $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ such that

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot \tilde{z}_i \leq p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \leq p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

and $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;
(d) [Market clearing condition on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

Then, there exists $(z_i^*) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $((x_i^*, z_i^*), p^*, q^*)$ is a financial equilibrium of $\mathcal{E}^{\mathcal{F}}$,