Economic analysis of financial market
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Pricing by arbitrage

Outline

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Redundant assets and useless portfolios

Definition

Let a financial structure represented by its payoff matrix V. Given a spot price p, an asset j is redundant if the payoff vector $V_j(p)$ is a linear combination of the payoff vectors of the other assets $(V_k(p))_{k \in \mathcal{J}, k \neq j}$.

A portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is useless if the payoff in each state is equal to 0, that is $V(p)z = 0$.

Value of a useless portfolios

If q is a no-arbitrage asset price for the spot price vector p, then there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ *such that* $q = V(p)^t\lambda$. *So, if portfolio* $z \in \mathbb{R}^{\mathcal{J}}$ *is useless, then* $q \cdot z = V(p)^t\lambda \cdot z = \lambda \cdot V(p)z = \lambda \cdot 0 = 0$. *So the value of a useless portfolio is equal to* 0 *for all no-arbitrage asset price q.*

In other words, the kernel of the payoff matrix $V(p)$ *and the one of the full payoff matrix* $W(p,q)$ *coincide for all no-arbitrage asset price q.*

Characterisation of an asset structure without redundant asset

Proposition

Let a financial structure represented by its payoff matrix V and p be a spot price. Then there is no redundant asset if and only if one of the two following condition is satisfied :

a) $V(p)$ is one-to-one or equivalently the rank of $V(p)$ is equal to $\sharp\mathcal{J}$;

b) the unique useless portfolio is 0.

Uniqueness of the supporting portfolio for an optimal consumption

Proposition

Let a financial structure represented by its payoff matrix V and (p, q) be a spot - asset price pair such that V is arbitrage free at (p, q).Let $\bar{x}_i$ be optimal for u_i in the budget set $B_i^{\mathcal{F}}(p, q)$. Let z_i and z_i' to portfolios, which finance $\bar{x}_i$. Then, if Assumption NSS holds, $z_i - z_i'$ is a useless portfolio.

Consequently, if there is no redundant asset for $V(p)$, then $\bar{x}_i$ is affordable for a unique portfolio in $\mathbb{R}^{\mathcal{J}}$.

On the asset market clearing condition for a financial structure without redundant asset

Proposition

Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i)_{i \in \mathcal{I}}, V)$ *be an unconstrained* ($Z_i = \mathbb{R}^{\mathcal{J}}$ *for all* $i \in \mathcal{I}$) *financial economy satisfying Assumption NSS and with no redundant asset. Let* $((x_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ *such that*

(a) [Preference maximization] for every $i \in \mathcal{I}$,
x_i^* *is a "maximal" element of* u_i *in the budget set* $B_i^{\mathcal{F}}(p^*, q^*)$ *in the sense that there exists* $\tilde{z}_i \in \mathbb{R}^{\mathcal{J}}$ *such that*

$$\begin{cases} p^*(\xi_0) \cdot x_i^*(\xi_0) + q^* \cdot \tilde{z}_i \leq p^*(\xi_0) \cdot e_i(\xi_0) \\ p^*(\xi) \cdot x_i^*(\xi) \leq p^*(\xi) \cdot e_i(\xi) + V(p^*, \xi) \cdot \tilde{z}_i, \quad \forall \xi \in \mathbb{D}_1 \end{cases}$$

and $B_i^{\mathcal{F}}(p^*, q^*) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;

Proposition continued

(b) [Market clearing condition on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i .$$

Then, $\sum_{i \in \mathcal{I}} \tilde{z}_i = 0$ *and* $((x_i^*, \tilde{z}_i), p^*, q^*)$ *is a financial equilibrium of* $\mathcal{E}^{\mathcal{F}}$.

How to discard redundant assets at no cost

Let V be a financial structure and p be a spot price vector. We know that $(V_j(p))_{j\in\mathcal{J}}$ is a spanning family of the range of $V(p)$ and we can find a maximal sub-family $\tilde{\mathcal{J}} \subset \mathcal{J}$ such that $(V_j(p))_{j\in\tilde{\mathcal{J}}}$ is still spanning the range of $V(p)$ and is linearly independent.

It means that for $j \in \mathcal{J} \setminus \tilde{\mathcal{J}}$, the asset j is redundant in the sense that $V_j(p) = \sum_{k\in\tilde{\mathcal{J}}} \mu_k^j V_k(p)$ for some $\mu^j \in \mathbb{R}^{\tilde{\mathcal{J}}}$.

$\tilde{V}$ is the substructure obtained by keeping only the assets in $\tilde{\mathcal{J}}$.

An equivalent substructure $\tilde{V}$ without redundant asset

Proposition

Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i\in\mathcal{I}}, V)$ be an unconstrained financial economy satisfying Assumption NSS. Then there exists a substructure $\tilde{V}$ composed by a subset $\tilde{\mathcal{J}}$ of the assets of V such that

a) $\tilde{V}$ has no redundant asset ;

b) If $((x_i^*, z_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ is a financial equilibrium for the structure V, then there exists $(\zeta_i^*) \in (\mathbb{R}^{\tilde{\mathcal{J}}})^{\mathcal{I}}$ such that $((x_i^*, \zeta_i^*), p^*, \tilde{q}^*)$ is a financial equilibrium for the structure $\tilde{V}$, where the price $\tilde{q}^*$ is the standard projection of q^* on $\mathbb{R}^{\tilde{\mathcal{J}}}$.

Proposition continued

c) If $((x_i^*, \tilde{z}_i^*), p^*, \tilde{q}^*) \in (\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ *is a financial equilibrium for the structure* $\tilde{V}$*, then there exists* $(z_i^*) \in (\mathbb{R}^{\tilde{\mathcal{J}}})^{\mathcal{I}}$ *such that* $((x_i^*, z_i^*), p^*, \tilde{q}^*)$ *is a financial equilibrium for the structure* V*, where the price* q^* *is computed for the asset* $j \in \mathcal{J} \setminus \{\tilde{\mathcal{J}}\}$ *according to the present value vector associated to* $\tilde{q}^*$ *and* z_i^* *is the natural embedding of* $\tilde{z}_i^*$ *in* $\mathbb{R}^{\mathcal{J}}$ *by adding* 0 *for the additional components.*

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A remark on the price of a redundant asset

Remark

Let V be a financial structure and p be a spot price vector. Let $j_0 \in \mathcal{J}$ be a redundant asset. Then there exists $\mu \in \mathbb{R}^{\mathcal{J} \setminus \{j_0\}}$ such that $V_{j_0}(p) = \sum_{j \in \mathcal{J}, j \neq j_0} \mu_j V_j(p)$. Now, let q be a no-arbitrage asset price. Then, there exists there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q = V(p)^t \lambda$. Hence,

$$q_{j_0} = V_j(p)^t \lambda = \sum_{j \in \mathcal{J}, j \neq j_0} \mu_j V_j(p)^t \lambda = \sum_{j \in \mathcal{J}, j \neq j_0} \mu_j q_j$$

So, the price of the asset j_0 is a linear combination of the prices of the other assets with the coefficient given by the fact that the payoff vector of asset j_0 is a linear combination of the payoff vectors of the other assets.

Pricing of a new asset

One commodity per state with spot prices normalised to 1 at each state and Assumptions S and NSS ;
V and q be an arbitrage free asset price ;
A new asset k represented by a payoff vector V_k is traded ;
The asset price remains unchanged, so V_k does not enlarge the budget sets of the agents so

V_k is a linear combination of the payoff vectors of the other assets.

Proof

$$(x_i(\xi) - e_i(\xi))_{\xi \in \mathbb{D}} \leq W(q)z$$

with the initial structure and

$$(x_i(\xi) - e_i(\xi))_{\xi \in \mathbb{D}} \leq W(q)z + \zeta \begin{pmatrix} -q_k \\ V_k \end{pmatrix}$$

with the additional asset.
So, under the survival assumption, the second budget set is strictly larger except if $(-q_k, V_k)$ is in the range of $W(q)$.

So, there exists $\mu \in \mathbb{R}^{\mathcal{J}}$ *such that* $V_k = \sum_{j \in \mathcal{J}} \mu_j V_j(p)$ *and* $q_k = \sum_{j \in \mathcal{J}} \mu_j q_j$.

Computation of the arbitrage price from the present value vector

If we know the present value vector λ associated to the asset price q.

$$q_k = \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V_k(\xi)$$

Remark

Note that even if we have several present value vectors associated to the asset price q, the pricing by arbitrage of the new asset is well defined.

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Definition of the over hedging pricing

Definition

Let V be a financial structure, p be a spot price and q an arbitrage free asset price associated to the present value vector λ and k an asset represented by its payoff vector $v \in \mathbb{R}^{\mathbb{D}_1}$. Then, the over hedging price of k is the value of the following minimisation problem.

$$\begin{cases} \text{Minimise } \sum_{J \in \mathcal{J}} q_j z_j \\ V(p)z \geq v \\ z \ \in \mathbb{R}^{\mathcal{J}} \end{cases}$$

A necessary and sufficient condition for the finiteness of the over hedging price

We remark that the value of the previous problem may be $+\infty$ if there is no portfolio z such that $V(p)z \geq c$.

Nevertheless, if the bond is among the existing portfolio, or, more generally, if there exists a portfolio $\underline{z}$ such that $V(p)\underline{z} \gg 0$, we are sure that the value is finite for every $v \in \mathbb{R}^{\mathbb{D}_1}$.

Actually, this is also a necessary condition.

Properties of the over hedging price

Proposition

Let V be a financial structure and (p, q) be a spot - asset price vector such that V is arbitrage free at (p, q). Let us assume that there exists a portfolio $\underline{z}$ such that $V(p)\underline{z} \gg 0$. Then the over hedging price function q^+ satisfies the following properties :

a) q^+ is a positively homogeneous convex function on $\mathbb{R}^{\mathbb{D}_1}$, so it is Lipschitz continuous.

b) If λ is a present value vector associated to q, then $q^+(v) \geq \lambda \cdot v$.

c) the restriction of q^+ to the range of $V(p)$ is the linear mapping $\lambda \cdot v$.

d) $q^+(v) = \max\{\lambda \cdot v \mid \lambda \in \mathbb{R}^{\mathbb{D}_1}_+, V(p)^t \lambda = q\}$.

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Optimality with short sale constraints

The portfolio sets are no more $\mathbb{R}^{\mathcal{J}}$ but $\{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+$, with $\underline{z}_i \leq 0$.

Proposition

Let us assume that Assumption NSS is satisfied by the financial economy $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, Z_i = \{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+)_{i\in\mathcal{I}}, V)$. For a commodity-asset price pair (p, q), if there exists a consumer i and $x_i \in X_i$, which is optimal in the budget set $B_i^{\mathcal{F}}(p, q)$, then it does not exists $\zeta \in \mathbb{R}^{\mathcal{J}}_+$ such that $W(p, q)\zeta \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$.

No arbitrage with short sale constraints

Proposition

The financial structure V with the portfolio sets $(Z_i = \{\underline{z}_i\} + \mathbb{R}^{\mathcal{J}}_+)_{i \in \mathcal{I}}$ is arbitrage free at (p, q) if and only if there exists $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ such that $q \geq \sum_{\xi \in \mathbb{D}_1} \lambda_\xi V(p, \xi) = V(p)^t \lambda$.