Master MMMEF, 2023-2024
Homeword on:
General Equilibrium Theory:
Economic analysis of financial markets
November 2023

November 22, 2023

We consider a standard two-period economy with two commodities and two states at the second period. The uncertainty is then represented by a tree $\mathbb{D} = \{\xi_0, \xi_1, \xi_2\}$. The first commodity is a standard perishable commodity whereas the second one is a durable commodity. This means that if a consumer has a quantity $x_{0,2} \geq 0$ of the second commodity at the first period, her endowments in this commodity at the second period is increased by $\mu_1 x_{0,2}$ in state ξ_1 and $\mu_2 x_{0,2}$ in state 2, where μ_1 and μ_2 belong to $]0, 1]$.

In this economy, we have I agents, $i = 1, \ldots, I$, who have preferences represented by a utility function u^i from $\mathbb{R}^6_+$ to $\mathbb{R}$. We assume that the utility functions are continuous, concave, strictly monotone and continuously differentiable on $\mathbb{R}^6_{++}$. Each agents has also an endowments $e^i \in \mathbb{R}^6_{++}$.

A contingent commodity equilibrium in this economy is then a price $p^* \in \mathbb{R}^6_{++}$ and allocations (x^{i*}) satisfying the following condition:
a) for all i, x^{i*} maximises the utility function u^i on the budget set $B^{CC}(p^*, e^i)$ defined by the unique budget constraint:

$$p^* \cdot (x^i - e^i) \leq (\mu_1 p^*_{1,2} + \mu_2 p^*_{2,2}) x^i_{0,2}$$

b) $\sum_{i=1}^I x^{i*} = \sum_{i=1}^I e^i + (0, 0, 0, \mu_1 \sum_{i=1}^I e^i_{0,2}, 0, \mu_2 \sum_{i=1}^I \mu_2 e^i_{0,2})$

1) Explain the rationale for writing the budget constraint and the market clearing condition as in the above definition.

2) Show that for $\alpha > 1$, close to 1, the allocation αe^i belongs to the contingent commodity budget set at the price p^*.

3) Show that an equilibrium allocation $(x^{i*})_{i \in I}$ is Pareto optimal.

4) Show that the equilibrium price satisfies the following inequality: $p^*_{0,2} > \mu_1 p^*_{1,2} + \mu_2 p^*_{2,2}$.

5) Using a standard result in an exchange economy with differentiable concave utility functions, show that if $(\bar{x}^i)_{i\in I} \in (\mathbb{R}^6_{++})^I$ is Pareto optimal, then it exist $q \in \mathbb{R}^6_{++}$ and $\gamma \in \mathbb{R}^I_{++}$ such that, for all i, $\nabla u^i(\bar{x}^i) = \gamma^i q$.

We are now considering a pure spot market equilibrium where the market clearing condition remains the same as for a contingent commodity equilibrium, and the consumers maximise their utility on the following budget set for the spot prices $\pi = (\pi_0, \pi_1, \pi_2) \in \mathbb{R}^6_{++}$:

$$B^{SM}(\pi, e^i) = \left\{ x^i \in \mathbb{R}^6_+ \;\middle|\; \begin{array}{l} \pi_0 \cdot (x^i_0 - e^i_0) \leq 0 \\ \pi_1 \cdot (x^i_1 - e^i_1) \leq \pi_{1,2}\mu_1 x^i_{0,2} \\ \pi_2 \cdot (x^i_2 - e^i_2) \leq \pi_{2,2}\mu_2 x^i_{0,2} \end{array} \right\}$$

6) Show that if x^i belongs to the spot market budget set for the price π, then, it belongs to the contingent commodity budget set for the same price π.

7) Show that for all $\alpha > 1$, the allocation αe^i does not belong to the spot market budget set at the price π. Conclude that the spot market budget set for π is strictly included in the contingent commodity budget set for π.

The following questions aim at proving that if the pure spot market equilibrium $(\bar{\pi}, (\bar{x}^i)^I_{i=1})$, where $\bar{x}^i \gg 0$ for all i, is Pareto optimal, then it is a contingent commodity equilibrium allocation.

8) Let $q = (q_0, q_1, q_2)$ and γ associated to $(\bar{x}^i)$ as given by Question 5. Using the first order necessary conditions for the utility maximisation problem in the pure spot market equilibrium, show that there exists $\lambda \in \mathbb{R}^3_{++}$ such that $q_1 = \lambda_1 \bar{\pi}_1$, $q_2 = \lambda_2 \bar{\pi}_2$ and $q_0 = (\lambda_0 \bar{\pi}_{0,1}, \lambda_0 \bar{\pi}_{0,2} - \lambda_1 \mu_1 \bar{\pi}_{1,2} - \lambda_2 \mu_2 \bar{\pi}_{2,2})$.

9) Let $\tilde{q}$ be the price with the same coordinates than q except for the coordinate $(0, 2)$ which is equal to $\tilde{q}_{0,2} = q_{0,2} + \mu_1 q_{1,2} + \mu_2 q_{2,2}$. Show that $\bar{x}^i$ belongs to the contingent commodity budget set for the price $\tilde{q}$ and that the budget constraint is binding.

10) Let x^i be an element of the contingent commodity budget set for the price $\tilde{q}$. For all i, show that $q \cdot x^i \leq q \cdot \bar{x}^i$ and, then, $u^i(x^i) \leq u^i(\bar{x}^i)$. Conclude that $(\tilde{q}, (\bar{x}^i)^I_{i=1})$ is a contingent commodity equilibrium.