Master MMMEF, 2022-2023
Lectures notes on:
General Equilibrium Theory:
Economic analysis of financial markets[1]

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Contents

Chapter 6

Existence of financial equilibria

The general equilibrium theory is a powerful tool to check whether or not a model is consistent. Indeed, it is quite easy to build a model but it is useful only if we are working on an object which exists in a quite large framework. Since the 50', the benchmark model is the Arrow-Debreu existence result and the associated assumptions. The purpose of this section is to present existence results for financial equilibria under assumptions which are at the same level of generality as the one for a competitive equilibrium.

Actually, in the first subsection, we show how to deduce a financial equilibrium from a competitive equilibrium of an auxiliary exchange economy in the one commodity case. We also show which assumptions are needed on the initial economy to get the necessary ones on the exchange economy. So we remark that we are really at the same level of generality as in the Arrow-Debreu existence result.

The second subsection is just a presentation of the most advanced existence results without proofs since they are far beyond the scope of this course.

6.1 Existence for the one commodity case

In this section, we provide an existence result for a two-period economy in the particular case where there is only one commodity per state, $\ell = 1$, which is a pure wealth model as it is commonly assumed in the literature in finance. The consumers take care only of their wealths in the different states.

In this framework, we use a correspondence due to Hart [12] between a financial equilibrium and a Walras equilibrium of an auxiliary economy where the commodities are the consumption at the initial node ξ_0 and the assets.

We posit the following additional assumptions to complete Assumption C and S. First of all, $\ell = 1$, so $\mathbb{L} = \mathbb{D}$.

Assumption C1. For all $i \in \mathcal{I}$,

a) $X_i = \mathbb{R}^{\mathbb{D}}_+$;

b) u_i is strictly increasing on X_i.

Note that u_i strictly increasing is just the translation of Assumption NSS when there is only one commodity per state. The financial structure is composed by a finite set of $\mathcal{J}$ real assets and the payoff asset j in state ξ is $p(\xi)V_j(\xi)$, where $V_j(\xi)$ is an amount of the unique commodity. We denote by V the $\mathbb{D}_1 \times \mathcal{J}$ matrix whose entries are $V_j(\xi)$. We assume that

Assumption F1.

a) For all $j \in \mathcal{J}$, $V_j \in \mathbb{R}_+^{\mathbb{D}_1} \setminus \{0\}$;

b) For all $\xi \in \mathbb{D}_1$, there exists $j \in \mathcal{J}$ such that $V_j(\xi) > 0$.

c) for all $i \in \mathcal{I}$, $Z_i = \mathbb{R}^{\mathcal{J}}$.

As already noticed in the previous section, assuming that the payoffs are non negative is not so restrictive since, if it is not satisfied, there exists an equivalent financial structure satisfying it under the mild sufficient condition that at least one portfolio has positive returns in all states.

Let us consider a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$. From the strict monotonicity of the utility functions, we deduces that $p^*(\xi) > 0$ for all $\xi \in \mathbb{D}$ and the budget constraints are binding. So, for all $i \in \mathcal{I}$, for all $\xi \in \mathbb{D}_1$, $p^*(\xi)x_i^*(\xi) = p^*(\xi)e_i(\xi) + \sum_{j\in\mathcal{J}} z_{ij}^* \ V_j(\xi)p^*(\xi)$. Hence, $x_i^*(\xi) = e_i(\xi) + \sum_{j\in\mathcal{J}} z_{ij}^* \ V_j(\xi)$. So, the consumption at date 1 is completely determined by the portfolio chosen on the financial market at date 0. Hence, we can reduce the choice of the consumer to her consumption at date 0 and her portfolio. Furthermore, the equilibrium portfolios must satisfy the constraints $e_i(\xi) + \sum_{j\in\mathcal{J}} z_{ij}^* \ V_j(\xi) \geq 0$ for all $\xi \in \mathbb{D}_1$. That is why we consider the following exchange economy $\tilde{\mathcal{E}}$ with the same set of consumers $\mathcal{I}$ than $\mathcal{E}$:

The commodity space is $\mathbb{R} \times \mathbb{R}^{\mathcal{J}}$. For each $i \in \mathcal{I}$, the consumption set is $\Xi_i = \{(x_i(\xi_0), z_i) \in \mathbb{R} \times \mathbb{R}^{\mathcal{J}} \mid x_i(\xi_0) \geq 0, e_i^1 + Vz_i \geq 0\}$ where $e_i^1 \in \mathbb{D}_1$ is the restriction of e_i to the states in $\mathbb{D}_1$. The utility function is: $\tilde{u}_i(x_i(\xi_0), z_i) = u_i(x_i(\xi_0), e_i^1 + Vz_i)$ and the endowments are $\tilde{e}_i = (e_i(\xi_0), 0)$.

The following proposition shows the link between a Walras equilibrium of $\tilde{\mathcal{E}}$ and a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$.

Proposition 23 *Let $((x_i^*, z_i^*), p^*, q^*)$ be a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$. Then, $((x_i^*(\xi_0), z_i^*), (p^*(\xi_0), q^*))$ is a Walras equilibrium of $\tilde{\mathcal{E}}$.*

Conversely, let $((\tilde{x}_i(\xi_0), \tilde{z}_i), (\tilde{p}(\xi_0), \tilde{q}))$ be a Walras equilibrium of $\tilde{\mathcal{E}}$, then $((\tilde{x}_i, \tilde{z}_i), \tilde{p}, \tilde{q})$ is a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$ with for all $\xi \in \mathbb{D}_1$

a) $\tilde{p}(\xi) = 1$;

b) $\tilde{x}_i(\xi) = e_i(\xi) + \sum_{j\in\mathcal{J}} \tilde{z}_{ij} \ v_j(\xi)$.

Proof. Let $((x_i^*, z_i^*), p^*, q^*)$ be a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$. Then $\sum_{i\in\mathcal{I}} x_i^*(\xi_0) = \sum_{i\in\mathcal{I}} e_i(\xi_0)$ and $\sum_{i\in\mathcal{I}} z_i^* = 0$ so the market clearing conditions are satisfied for the economy $\tilde{\mathcal{E}}$. Since x_i^* is affordable for z_i^*, the first budget constraint $p^*(\xi_0)x_i^*(\xi_0) + q^* \cdot z_i^* \leq p^*(\xi_0)e_i(\xi_0)$ holds true, so $(x_i^*(\xi_0), z_i^*)$ belongs to the Walras budget set for the price $(p^*(\xi_0), q^*)$ in the economy $\tilde{\mathcal{E}}$. If there exists $(x_i(\xi_0), z_i)$ in the Walras budget set such that $\tilde{u}_i(x_i(\xi_0), z_i) > \tilde{u}_i(x_i^*(\xi_0), z_i^*)$, then, x_i, defined by $x_i(\xi) = e_i(\xi) + \sum_{j\in\mathcal{J}} z_{ij}\ V_j(\xi)$ for all $\xi \in \mathbb{D}_1$, satisfies $u_i(x_i) > u_i(x_i^*)$ and x_i is affordable by z_i for the prices (p^*, q^*) in $\mathcal{E}$. So, we get a contradiction with the fact that x_i^* is optimal as an equilibrium allocation. So, $(x_i^*(\xi_0), z_i^*)$ is an optimal consumption in the Walras budget set. Consequently, $((x_i^*(\xi_0), z_i^*), (p^*(\xi_0), q^*))$ is a Walras equilibrium of $\tilde{\mathcal{E}}$.

Conversely, let $((\tilde{x}_i(\xi_0), \tilde{z}_i), (\tilde{p}(\xi_0), \tilde{q}))$ be a Walras equilibrium of $\tilde{\mathcal{E}}$. Since $(\tilde{x}_i(\xi_0), \tilde{z}_i)$ belongs to Ξ, $\tilde{x}_i$ belongs to $\mathbb{R}_+^{\mathbb{D}}$. One easily check that $\tilde{x}_i$ is affordable for the portfolio $\tilde{z}_i$ for the price $(\tilde{p}, \tilde{q})$. From the market clearing conditions, $\sum_{i\in\mathcal{I}} x_i^*(\xi_0) = \sum_{i\in\mathcal{I}} e_i(\xi_0)$ and $\sum_{i\in\mathcal{I}} \tilde{z}_i = 0$, so one deduces that $\sum_{i\in\mathcal{I}} \tilde{x}_i(\xi) = \sum_{i\in\mathcal{I}} e_i(\xi)$ for all $\xi \in \mathbb{D}_1$. Finally, if there exists x_i affordable for a portfolio z_i in the financial budget set and $u_i(x_i) > u_i(\tilde{x}_i)$, one has for all $\xi \in \mathbb{D}_1$, $x_i(\xi) \leq e_i(\xi) + \sum_{j\in\mathcal{J}} z_i(\xi)V_j(\xi)$ since $\tilde{p}(\xi) = 1$. So, since u_i is strictly increasing, the consumption x'_i define by $x'_i(\xi_0) = x_i(\xi_0)$ and $x'_i(\xi) = e_i(\xi) + \sum_{j\in\mathcal{J}} z_{ij}\ V_j(\xi)$ is financially affordable for z_i and $u_i(x'_i) > u_i(\tilde{x}_i)$. So, from the definition of $\tilde{u}_i$, one deduces that $\tilde{u}_i(x'_i(\xi_0), z_i) > \tilde{u}_i(\tilde{x}_i(\xi_0), \tilde{z}_i)$ and $(x'_i(\xi_0), z_i)$ belongs to the Walras budget set of $\tilde{\mathcal{E}}$. So, we get a contradiction with the optimality of $(\tilde{x}_i(\xi_0), \tilde{z}_i)$ as an equilibrium allocation. Consequently, $((\tilde{x}_i, \tilde{z}_i), \tilde{p}, \tilde{q})$ is a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$. $\square$

This proposition tells us that the existence of a financial equilibrium in $\mathcal{E}_{\mathcal{F}}$ is equivalent to the existence of a Walras equilibrium in the economy $\tilde{\mathcal{E}}$. We now check that the economy $\tilde{\mathcal{E}}$ satisfies the necessary conditions à la Arrow-Debreu for the existence of a Walras equilibrium but the boundedness of feasible allocations that we discuss specifically.

Proposition 24 *If the economy $\mathcal{E}_{\mathcal{F}}$ satisfies Assumption C, C1, S and F1, then the economy $\tilde{\mathcal{E}}$ satisfies: for all $i \in \mathcal{I}$*

a) Ξ_i is nonempty, convex, closed;

b) $\tilde{u}_i$ is continuous, strictly increasing and quasi-concave on Ξ_i.

c) $\tilde{e}_i \in \mathrm{int}\Xi_i$.

Proof. a) Ξ_i is nonempty, convex, closed since $(0, e_i^1)$ belongs to Ξ_i and Ξ_i is defined by a finite set of affine inequality constraints.

b) $\tilde{u}_i$ is continuous since u_i is so. It is strictly increasing since u_i is so and if $z_i \geq z'_i$, $z_i \neq z'_i$, then there exists $j \in \mathcal{J}$ such that $z_{ij} > z'_{ij}$. Since V_j is non negative and not equal to 0, there exists $\xi \in \mathbb{D}^1$ such that $V_j(\xi) > 0$, so $e_i(\xi) + \sum_{j\in\mathcal{J}} z_{ij}\ V_j(\xi) > e_i(\xi) + \sum_{j\in\mathcal{J}} z'_{ij}\ V_j(\xi)$. Furthermore, since V has only non negative entries, $e_i^1 + Vz_i \geq e_i^1 + Vz_i'$ and $e_i^1 + Vz_i \geq e_i^1 + Vz_i' \neq \tilde{u}_i(x_i(\xi_0), z_i)$. So, $\tilde{u}_i(x_i(\xi_0), z_i) = u_i(x_i(\xi_0), e_i^1 + Vz_i) > u_i(x_i(\xi_0), e_i^1 + V\tilde{z}_i) = \tilde{u}_i(x_i(\xi_0), \tilde{z}_i)$.

Let $(x_i(\xi_0), z_i)$, $(\tilde{x}_i(\xi_0), \tilde{z}_i)$ and $(\hat{x}_i(\xi_0), \hat{z}_i)$ three elements of Ξ_i such that $\tilde{u}_i(x_i(\xi_0), z_i) \leq \tilde{u}_i(\tilde{x}_i(\xi_0), \tilde{z}_i)$ and $\tilde{u}_i(x_i(\xi_0), z_i) \leq \tilde{u}_i(\hat{x}_i(\xi_0), \hat{z}_i)$.

Let $x_i = (x_i(\xi_0), e_i^1 + Vz_i)$, $\tilde{x}_i = (\tilde{x}_i(\xi_0), e_i^1 + V\tilde{z}_i)$ and $\hat{x}_i = (\hat{x}_i(\xi_0), e_i^1 + V\hat{z}_i)$. Then, from the definition of $\tilde{u}_i$, $u_i(x_i) \leq u_i(\tilde{x}_i)$ and $u_i(x_i) \leq u_i(\hat{x}_i)$. Since u_i is quasi-concave, for all $t \in [0,1]$, $u_i(x_i) \leq u_i(t\tilde{x}_i + (1-t)\hat{x}_i) = u_i(t\tilde{x}_i(\xi_0) + (1-t)\hat{x}_i(\xi_0), t(e_i^1 + V\tilde{z}_i) + (1-t)(e_i^1 + V\hat{z}_i)) = u_i(t\tilde{x}_i(\xi_0) + (1-t)\hat{x}_i(\xi_0), e_i^1 + V(t\tilde{z}_i + (1-t)\hat{z}_i)) = \tilde{u}_i(t(x_i(\xi_0), \tilde{z}_i) + (1-t)(x_i(\xi_0), \hat{z}_i))$. So $\tilde{u}_i$ is quasiconcave.

c) We remark that $\tilde{e}_i = (e_i(\xi_0), 0)$ belongs to the interior of Ξ since $e_i(\xi_0) > 0$ and $e_i^1 + V0 = e_i^1 \gg 0$ so, no constraint are binding in the definition of Ξ. □

We now study the feasible set of the economy $\tilde{\mathcal{E}}$ which is

$$\mathcal{A} = \{(\tilde{x}_i) \in \prod_{i \in \mathcal{I}} \Xi_i \mid \sum_{i \in \mathcal{I}} \tilde{x}_i = \sum_{i \in \mathcal{I}} \tilde{e}_i\}$$

So $(\tilde{x}_i = (x_i(\xi_0), z_i))$ belongs to $\mathcal{A}$ if $\sum_{i \in \mathcal{I}} x_i(\xi_0) = \sum_{i \in \mathcal{I}} e_i(\xi_0)$ and $\sum_{i \in \mathcal{I}} z_i = 0$. The key issue for the existence of an equilibrium is to prove that this set is bounded. Then the Arrow-Debreu Theorem implies the existence of an equilibrium for the exchange economy $\tilde{\mathcal{E}}$ and so, the existence of a financial equilibrium for the financial economy $\mathcal{E}_{\mathcal{F}}$.

For the boundedness of $\mathcal{A}$, we have no problem for the first component since $x_i(\xi_0) \geq 0$ for all i. We focus on the portfolio component. Under which condition is the set

$$\mathcal{A}_Z = \{(z_i) \in \prod_{i \in \mathcal{I}} \mathbb{R}^{\mathcal{J}} \mid \sum_{i \in \mathcal{I}} z_i = 0, \forall i, e_i^1 + Vz_i \geq 0\}$$

bounded?

Indeed, contrary to the usual framework, the set Ξ_i are not necessarily bounded below. This is obvious when the matrix V is not one to one, which means that we have redundant assets. Indeed, in this case, for any non zero useless portfolio $\zeta \in \mathrm{Ker}V$, then $(0, \zeta) \in \Xi_i$ for all consumers. So, Ξ is not bounded from below. But it may be also true with a one to one payoff matrix V. Let us take an example. We consider a date-event tree with three states of nature at date 1. A financial structure has two assets and the payoff matrix is:

$$V = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Then,

$$\Xi_i = \left\{ (x(\xi_0), z_1, z_2)\mathbb{R} \times \mathbb{R}^2 \;\middle|\; x(\xi_0) \geq 0, \left\{ \begin{array}{c} e_i(\xi_1) + z_1 + z_2 \geq 0 \\ e_i(\xi_2) + z_1 + \frac{1}{2}z_2 \geq 0 \\ e_i(\xi_3) + \frac{1}{2}z_1 + z_2 \geq 0 \end{array} \right. \right\}$$

We remark that for all $t \geq 0$, $(0, 2t, -t)$ and $(0, -t, 2t)$ belongs to Ξ_i so it is not bounded from below.

The next proposition shows that even if the consumption sets are not bounded from below, the asset attainable set $\mathcal{A}_Z$ is bounded when V is one to one. We know that this assumption is not really restrictive for the existence of an equilibrium in $\mathcal{E}_{\mathcal{F}}$. Indeed, if V is not one to one, we can consider the reduced financial structure obtained by suppressing the redundant assets. This reduced financial structure is free of useless portfolio or, equivalently, the reduced matrix is one to one. We have shown that we can build a financial equilibrium for the initial financial structure starting from a financial equilibrium for the reduced financial structure. Furthermore, we also check that the reduced financial structure satisfies Assumption F1. This is obvious for Assertions F1 (a) and (c) and is an exercise for Assertion F1 (b).

Proposition 25 *If $\sharp I \geq 2$, $\mathcal{A}_Z$ is bounded if and only if V is one to one.*

Proof. Let us assume that V is one to one. If $\mathcal{A}_Z$ is not bounded, there exists a sequence (z^ν) of $\mathcal{A}_Z$ such that $(m^\nu = \max\{\|z_i^\nu\| \mid i \in \mathcal{I}\})$ tends to $+\infty$. Let us consider the sequence $(\zeta^\nu = \frac{1}{m^\nu} z^\nu)$. From the definition of m^μ, one deduces that this sequence is bounded and $\mu^\nu = \max\{\|\zeta_i^\nu\| \mid i \in \mathcal{I}\} = 1$ for all ν. So, without any loss of generality, we can assume that this sequence converges to $\bar{\zeta}$ and $\max\{\|\bar{\zeta}_i\| \mid i \in \mathcal{I}\}) = 1$, so $\bar{\zeta} \neq 0$.

Since (z^ν) of $\mathcal{A}_Z$, for all $i \in \mathcal{I}$, $e_i^1 + V z_i^\nu \geq 0$, so $\frac{1}{m^\nu}(e_i^1 + V z_i^\nu) = \frac{1}{m^\nu} e_i^1 + V \zeta_i^\nu) \geq 0$. At the limit, we get $V\bar{\zeta}_i \geq 0$ since (m^ν) tends to $+\infty$. Furthermore, since $\sum_{i\in\mathcal{I}} z_i^\nu = 0$, we also get $\sum_{i\in\mathcal{I}} \bar{\zeta}_i = 0$. So, $0 \leq \sum_{i\in\mathcal{I}} V\bar{\zeta}_i = V(\sum_{i\in\mathcal{I}} \bar{\zeta}_i) = V0 = 0$. Hence, one deduces that $V\bar{\zeta}_i = 0$ for all i, and, since V is one to one, $\bar{\zeta}_i = 0$ for all i, which is in contradiction with $\bar{\zeta} \neq 0$. So $\mathcal{A}_Z$ is bounded.

Conversely, if $\sharp\mathcal{I} \geq 2$ and V is not one to one. Let $\zeta \in \mathrm{Ker} V \setminus \{0\}$. Let i and ι two elements of $\mathcal{I}$. We check that for all $t \in \mathbb{R}$, z^t defined by $z_i^t = t\zeta$, $z_\iota^t = -t\zeta$, $z_{i'}^t = 0$ for i' different from i and ι belongs to $\mathcal{A}_Z$ so $\mathcal{A}_Z$ is not bounded. □

So, summarising the previous discussion, we get the following existence result of a financial equilibrium.

Proposition 26 *If the unconstrained financial economy $\mathcal{E}_{\mathcal{F}}$ satisfies Assumption C, C1, S and $\mathrm{Im} V \cap \mathbb{R}^{\mathbb{D}_1}_{++} \neq \emptyset$, then a financial equilibrium exists.*

Proof. First, we have shown that there exists an equivalent financial structure V' such that for all $j \in \mathcal{J}$, $V'_j \in \mathbb{R}^{\mathbb{D}_1}_+$. Second, by eliminating the redundant assets, there exists an equivalent financial structure $\bar{V}'$ such that $\bar{V}'$ is one to one and for all $j \in \bar{\mathcal{J}}$, $\bar{V}'_j \in \mathbb{R}^{\mathbb{D}_1}_+ \setminus \{0\}$. Since the range of the financial structure is the same than the one of V, $\mathrm{Im}\bar{V}' \cap \mathbb{R}^{\mathbb{D}_1}{++} \neq \emptyset$, so Assumption F1 (b) is satisfied. So, the financial economy with the financial structure $\bar{V}'$ satisfies all necessary conditions so that the associated exchange economy $\tilde{\mathcal{E}}$ satisfies the assumptions for the existence of a Walras equilibrium: Assumptions C and S, u_i locally non satiated for all i and the attainable set $\mathcal{A}$ is bounded. From a Walras equilibrium of $\tilde{E}$, one deduces the existence of a financial equilibrium with the financial structure $\bar{V}'$ and, by the equivalence of the financial structures, an equilibrium for $\mathcal{E}_{\mathcal{F}}$. $\square$

6.2 Beyond the one commodity case

In this section, we consider a financial economy $\mathcal{E}_{\mathcal{F}}$ which satisfies the basic Assumptions C, S, NSS and F. We add an assumption on the portfolio sets:

Assumption Z: for all $i \in \mathcal{I}$, Z_i is closed convex and contains 0 in its interior.

We also define the consumption feasible set as follows:

$$\mathcal{A}_X = \{(x_i) \in \prod_{i \in \mathcal{I}} X_i \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i\}$$

This set is bounded since the individual consumption sets are bounded from below.

For a competitive equilibrium, we need a weaker non satiation assumption. We first provide an example of an economy without financial equilibrium since Assumption NSS is not satisfied whereas each utility function is locally non-satiated.

There are two states of nature at date 1, ξ_1 and ξ_2, only one commodity at each state and no financial asset, that is a pure spot market framework. They are two consumers $\mathcal{I} = \{i_1, i_2\}$, the consumption sets are $\mathbb{R}^3_+$ and the utility functions are:

$$u_1(x) = x(\xi_0) - x(\xi_1) + x(\xi_2) \quad u_2(x) = x(\xi_0) + x(\xi_1) + x(\xi_2)$$

The initial endowments are $e_1 = e_2 = (1, 1, 1)$. Assumption NSS is the only one which is not satisfied since at the allocation $(1, 0, 2)$, it is impossible to increase the welfare of the first agent by moving only her consumption at state ξ_1. There is no equilibrium since on the spot market at ξ_1, the demand of the second consumer is infinite when the price is non positive and is equal to 1 when the price is positive. But, for a positive price, the demand of the first consumer is equal to 0. So the sum of the demand is strictly smaller than the endowments at this node, which is equal to 2. Nevertheless, note that a contingent commodity equilibrium exists, which is $x_1^* = \left(\frac{3}{2}, 0, \frac{3}{2}\right)$, $x_2^* = \left(\frac{1}{2}, 2, \frac{1}{2}\right)$ and $p^* = (1, 1, 1)$.

6.2.1 Bounded portfolio sets or nominal assets

We first state the Radner [18] existence results which assume that the set of attainable portfolio $\mathcal{A}_Z$ defined by

$$\mathcal{A}_Z = \{(z_i) \in \prod_{i \in \mathcal{I}} Z_i \mid \sum_{i \in \mathcal{I}} z_i = 0\}$$

is bounded. Note that the original result of Radner was considering bounded below portfolio sets for which $\mathcal{A}_Z$ is obviously bounded but the proof works under this more general condition.

Theorem 2 *The financial economy has a financial equilibrium under Assumptions C, S, NSS, F and Z and if $\mathcal{A}_Z$ is bounded.*

We can remark that the existence of a pure spot market equilibrium is a consequence of this theorem.

We now consider the case of a nominal asset structure, that is with a payoff matrix V independent of the spot price p. In this case, we can fix the present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ and the asset price $q = V^t\lambda$.

Theorem 3 *Let $\mathcal{E}_\mathcal{F}$ be an unconstrained nominal financial economy satisfying Assumptions C, S, NSS. Then, for all $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, there exists a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$ such that $q^* = V^t\lambda$.*

The proof of this theorem is very similar to the proof for bounded attainable portfolios but it uses the Cass trick [7] presented in Subsection 6.1.

6.2.2 Numéraire assets

We now consider a numéraire asset financial structure where $\nu \in \mathbb{R}^\ell \setminus \{0\}$ is the numéraire and R the matrix of payoffs stated in units of the numéraire. In this case, the payoff matrix $V(p)$ is equal to:

$$V(p) = \begin{pmatrix} p(\xi_1)\cdot\nu & 0 & \dots & 0 \\ 0 & p(\xi_2)\cdot\nu & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p(\xi_{\sharp\mathbb{D}_1})\cdot\nu \end{pmatrix} R$$

and we check that the rank of the matrix $V(p)$ is the rank of R if $p(\xi)\cdot\nu > 0$ for all $\xi \in \mathbb{D}_1$. This properties is crucial to deduce the existence of a financial equilibrium for numéraire asset structures from the one for bounded attainable portfolios. Nevertheless, we need to assume that the numéraire commodity basket ν is strongly desirable at a feasible allocation in each state as precisely stated below in Assumption NNS.

Theorem 4 *Let $\mathcal{E}_\mathcal{F}$ be an unconstrained numéraire asset financial economy satisfying Assumptions C, S, NSS. ν denotes the numéraire basket of commodity in $\mathbb{R}^\ell$. We assume that:*
Assumption NNS: there exists $\rho > 0$ such that for every $x \in \mathcal{A}_X$, for every $\xi \in \mathbb{D}_1$, there exists $i \in \mathcal{I}$ such that for all $x' \in \mathbb{R}^\mathbb{D}$ satisfying $x'(\xi') = 0$ and $x'(\xi) \in B_\ell(\nu, \rho)$, there exists $\tau > 0$ such that $u_i(x_i) < u_i(x_i + tx')$;

Then, there exists a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$ such that $p^*(\xi)\cdot\nu > 0$ for all $\xi \in \mathbb{D}$.

6.2.3 The real asset case

The real asset financial structure beyond the numéraire case exhibits a particular difficulty since the rank of the return matrix $V(p)$ may drop at some prices leading to a sharp reduction of the transfer possibilities offer by the financial structure, so a discontinuous demand for the consumers. This explain why the notion of pseudo-equilibrium was introduced as an intermediate concept. The main difficulty is to prove that a pseudo-equilibrium exists. Then a genericity argument shows that the pseudo-equilibrium is actually a financial equilibrium almost everywhere.

Hart [13] provides the first example of a real asset financial structure without financial equilibrium. We now present an example which is an adaptation by Cornet of an example of Magill and Shafer [15].

There are two states of nature at date 1, two commodities at each state and two consumers. The financial structure is composed of two real assets. The consumption sets are $\mathbb{R}^6_+$, the utility functions are:

$$u_i(x_i) = U_i(x_i(\xi_0))\,[U_i(x_i(\xi_1))]^{\rho_1}\,[U_i(x_i(\xi_2))]^{\rho_2}$$

with $\rho_1 > 0$, $\rho_2 > 0$, $\rho_1 + \rho_2 = 1$. For $i = 1, 2$, $U_i(a, b) = a^{\alpha_1^i} b^{\alpha_2^i}$ with $\alpha_1^i > 0$, $\alpha_2^i > 0$, $\alpha_1^i + \alpha_2^i = 1$.

$e_1 = \left(\frac{1}{2}, \frac{1}{2}, 1-\epsilon, 1-\epsilon, \epsilon, \epsilon\right)$ and $e_2 = \left(\frac{1}{2}, \frac{1}{2}, \epsilon, \epsilon, 1-\epsilon, 1-\epsilon\right)$

The matrices representing the two real assets in the two states are identical equal to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so, given the spot price p, the payoff matrix is

$$V(p) = \begin{pmatrix} p_1(\xi_1) & p_2(\xi_1) \\ p_1(\xi_2) & p_2(\xi_2) \end{pmatrix}$$

In other words, a unit of the first asset delivers the value of one unit of the first commodity and a a unit of the second asset delivers the value of one unit of the second commodity.

We remark that the rank of $V(p^*)$ at equilibrium is either 1 or 2 since the prices are positive due to the strict monotonicity of the utility function. Then, if the spot prices in the two states are colinear, the rank is 1 and if not, the rank is 2. We show that in both cases, we get a contradiction.

We now prove that it does not exists a financial equilibrium if $\alpha^1 \neq \alpha^2$ and $\varepsilon \neq \frac{1}{2}$.

Let us start by assuming that the rank $V(p^*)$ is 2. Then, in that case, the market is complete and the financial equilibrium is actually a competitive equilibrium for a price π^* which is obtained from p^* by discounting the spot prices at node ξ_1 and ξ_2 according to the unique present value vector.

Since the equilibrium allocation are strictly positive, the first order necessary condition for the demand of the consumers leads to the following equalities:

$$\frac{x^*1h(\xi_1)}{x^*1h(\xi_2)} = \frac{x^*2h(\xi_1)}{x^*2h(\xi_2)} = \frac{\pi^*h(\xi_2)\rho_1}{\pi^*h(\xi_1)\rho_2}$$

for all commodities h. Furthermore, from the market clearing condition, we get for all commodities h and all states $\xi \in \mathbb{D}_1$, $x^*1h(\xi)+x^*2h(\xi) = 1$, so, one deduces from the previous equality that $\frac{\pi^*h(\xi_2)\rho_1}{\pi^*h(\xi_1)\rho_2} = 1$. Consequently, $\pi^*(\xi_1)$ is colinear to $\pi^*(\xi_2)$, which implies that $p^*(\xi_1)$ is colinear to $p^*(\xi_2)$ and the rank of the matrix $V(p^*)$ is then equal to 1. So, there is no equilibrium with the rank of $V(p^*)$ equal to 2.

If the rank $(V(p^*)$ is equal to 1, since we have a real asset structure, we can normalise the price vectors state by state and since they are colinear, we get that $p^*(\xi_1) = p^*(\xi_2)$. Using the first order necessary condition, one gets:

$$x^*_{ih}(\xi) = \frac{\alpha^i_h(p^*(\xi) \cdot e_i(\xi) + p^*_1(\xi)z^*_{i1} + p^*_2(\xi)z^*_{i2})}{p^*_h(\xi_1)}$$

for both consumers, both commodities and both states of nature. Since, at equilibrium the market clearing condition for both commodities and both states is $x^*_{1h}(\xi) + x^*_{2h}(\xi) = 1$, if we normalise the prices so that $p^*_1(\xi) + p^*_2(\xi) = 1$, we get

$$\begin{aligned} 1 &= \frac{\alpha^1_1(1-\varepsilon+p^*_1(\xi_1)z^*_{11}+p^*_2(\xi_1))z^*_{12})+\alpha^2_1(\varepsilon+p^*_1(\xi_1)z^*_{21}+p^*_2(\xi_1)+z^*_{22})}{p^*_1(\xi_1)} \\ &= \frac{\alpha^1_2(1-\varepsilon+p^*_1(\xi_1)z^*_{11}+p^*_2(\xi_1))z^*_{12})+\alpha^2_2(\varepsilon+p^*_1(\xi_1)z^*_{21}+p^*_2(\xi_1)+z^*_{22})}{p^*_1(\xi_1)} \\ &= \frac{\alpha^1_1(\varepsilon+p^*_1(\xi_1)z^*_{11}+p^*_2(\xi_1))z^*_{12})+\alpha^2_1(1-\varepsilon+p^*_1(\xi_1)z^*_{21}+p^*_2(\xi_1)+z^*_{22})}{p^*_1(\xi_1)} \\ &= \frac{\alpha^1_2(\varepsilon+p^*_1(\xi_1)z^*_{11}+p^*_2(\xi_1))z^*_{12})+\alpha^2_2(1-\varepsilon+p^*_1(\xi_1)z^*_{21}+p^*_2(\xi_1)+z^*_{22})}{p^*_1(\xi_1)} \end{aligned}$$

This implies that $(1 - 2\varepsilon)\alpha^1 + (2\varepsilon - 1)\alpha^2 = 0$, which is not possible since α^1 and α^2 are not colinear and $\varepsilon \neq \frac{1}{2}$. So, it does not exists a financial equilibrium with the rank of $V(p^*)$ equal to 1.

As already said, we now introduce the intermediary concept of pseudo-equilibria to get a generic existence result for financial equilibrium. Let $\mathcal{G}^r$ be the set of all linear subspaces of dimension r of $\mathbb{R}^{\mathbb{D}_1}$. This set is called the r-Grassmann manifold of $\mathbb{R}^{\mathbb{D}_1}$. In the definition of a pseudo-equilibrium, instead of considering the possible transfers of wealth among the two periods and the states of nature through a financial structure, we consider a transfer space $E \in \mathcal{G}^r$ and the marketable payoff are the vectors in E.

Définition 11 A r-pseudo-equilibrium of the economy $\mathcal{E}_{\mathcal{F}}$ is an element

(x^*, p^*, q^*, E^*) in $\prod_{i\in\mathcal{I}} X_i \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}} \times \mathcal{G}^r$ such that:

(i) for every i, x_i^* is optimal for the utility function u_i in the budget set

$$B_i^G(p^*, E^*) = \{x_i \in X_i \mid \exists t_i \in E^*, p^* \,\square\, (x_i - e_i) \leq t_i\}$$

(ii $\sum_{i\in\mathcal{I}} x_i^* = \sum_{i\in\mathcal{I}} e_i$

(iii) $\mathrm{Im}W(p^*, q^*) \subset E^*$.

We now present the link between a pseudo-equilibrium and a financial equilibrium.

Proposition 27 *Let $(x^*, z^* p^*, q^*)$ be a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$, then, $(x^*, p^*, q^*; \mathrm{Im}W(p^*, q^*))$ is a r-pseudo-equilibrium where r is the rank of $W(p^*, q^*)$.*

Conversely, if (x^*, p^*, q^*, E^*) is a r-pseudo-equilibrium of $\mathcal{E}_{\mathcal{F}}$ and

$$E^* = \mathrm{Im}W(p^*, q^*)$$

then there exists $z^* \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that (x^*, z^*, p^*, q^*) is a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$.

The proof of this proposition is left as an exercise. The following existence result shows that a r-pseudo-equilibrium exists if $r \leq \sharp\mathbb{D}_1$.

Theorem 5 *Let $\mathcal{E}_{\mathcal{F}}$ be an unconstrained financial economy satisfying Assumptions C, S, NSS and F. Then for all $r \leq \sharp\mathbb{D}_1$, for all $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, there exists a r-pseudo-equilibrium (x^*, p^*, q^*, E^*) such that $E^* \subset \lambda^\perp$.*

The proof of this existence result is the most technical one among all results presented in this course. Indeed, it involves a fixed point like theorem on the Grassmann manifold, which has a structure less tractable than the convex sets that are involved in the standard fixed-point theorem.

From this result, one deduces the following generic existence result for real asset financial structure. To do it, we need to strengthen the assumptions on the consumers by considering differentiable preferences as in the standard results of the general equilibrium theory from a differentiable viewpoint (See Balasko [4], Mas-Colell [16], Carosi et al. [19]).

Assumption SC: for every $i \in \mathcal{I}$,

a) $X_i = \mathbb{R}^{\mathbb{L}}_{++}$;

b) u_i is $\mathcal{C}^2$ on X_i and, for all $x_i \in X_i$, $\nabla u_i(x_i) \in \mathbb{R}^{\mathbb{L}}_{++}$ and for all $v \in \nabla u_i(x_i)^\perp \setminus \{0\}$, $v \cdot Hu_i(x_i)(v) < 0$.

c) For all $x_i \in X_i$, the set $\{x_i' \in X_i \mid u_i(x_i') \geq u_i(x_i)\}$ is closed in $\mathbb{R}^{\mathbb{L}}$.

The real asset financial structure is represented by an element R in $(\mathcal{M}(\sharp\mathbb{D} \times \ell)^{\mathcal{J}}$.

Theorem 6 *Let $\mathcal{E}$ be an exchange economy satisfying Assumption SC. Then, there exists an open subset Ω of $(\mathbb{R}^{\mathbb{L}}_{++})^{\mathcal{I}} \times (\mathcal{M}(\sharp\mathbb{D} \times \ell))^{\mathcal{J}}$ of full Lebesgue measure such that for all $(e, R) \in \Omega$, the financial economy $\mathcal{E}_{\mathcal{F}}$ has a financial equilibrium.*

The proof of this theorem shows that generically on the pair of endowments and financial structure the matrix $W(p^*, q^*)$ associated for the pseudo-equilibrium price pair (p^*, q^*) has a maximal rank and then the existing pseudo-equilibrium is then a financial equilibrium.

Chapter 7

Production in a Finance Economy

A two-period one-good production economy is composed of the following ingredients:

- a date-event tree $\mathbb{D}$ with two dates and a unique node ξ_0 at date 0.

- a finite set $\mathcal{I}$ of consumers with a production set $X_i \subset \mathbb{R}^{\mathbb{D}}$, preferences represented by a utility function u_i from X_i to $\mathbb{R}$, and a vector e_i of initial endowments.

- a finite set $\mathcal{K}$ of technology sets $Y_k \subset \mathbb{R}^{\mathbb{D}}$ describing the income streams $y_k = (y_{k\xi})_{\xi\in\mathbb{D}}$ generated by investment projects available to the firm k. The usual sign convention is $y_{k\xi} < 0$ indicates an input or investment, while $y_{k\xi} > 0$ is an output or revenue is state ξ. Typically $y_{k\xi_0} < 0$ and $y_{k\xi} > 0$ for $\xi \in \mathbb{D}_1$.

Then, the description of the economy must be completed by describing the ownership structure of the firms by the consumers. We will discuss this point below since we will consider several ownership structures.

The basic assumption on the firms is the following:

Assumption P. For each $k \in \mathcal{K}$, the technology set Y_k is closed, convex, $-\mathbb{R}^{\mathbb{D}}_+ \subset Y_k$, $Y_k \cap \mathbb{R}^{\mathbb{D}}_+ = \{0\}$. $(\zeta + \sum_{k\in\mathcal{K}} Y_k) \cap \mathbb{R}^{\mathbb{D}}_+$ is bounded and closed for all $\zeta \in \mathbb{R}^{\mathbb{D}}_+$.

For convenience in the following results, we will assume that the technology sets can be represented by a $\mathcal{C}^1$ transformation function, that is:

Assumption P'. For each $k \in \mathcal{K}$, there exists a non-decreasing, quasi-convex, and continuously differentiable transformation function T_k from $\mathbb{R}^{\mathbb{D}}$ to $\mathbb{R}$ such that $Y_k = \{y \in \mathbb{R}^{\mathbb{D}} \mid T_k(y) \leq 0\}$ and $T_k(0) = 0$.

Définition 12 An allocation $(x, y) \in \prod_{i\in\mathcal{I}} X_i \times \prod_{k\in\mathcal{K}} Y_k$ is feasible if $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i + \sum_{k\in\mathcal{K}} y_k$.

A feasible allocation $(x, y) \in \prod_{i\in\mathcal{I}} X_i \times \prod_{k\in\mathcal{K}} Y_k$ is Pareto optimal if it does not exist a feasible allocation (x', y') such that $u_i(x'_i) \geq u_i(x_i)$ for all $i \in \mathcal{I}$ and $(u_i(x'_i))_{i\in\mathcal{I}} \neq (u_i(x_i))_{i\in\mathcal{I}}$.

7.1 Contingent Market Equilibrium

We assume that the ownership of the firms are distributed among the consumers and we denote θ_{ik} the share of Consumer i in the firm k. So, $\theta_{ik} \geq 0$ and, for all $k \in \mathcal{K}$, $\sum_{i \in \mathcal{I}} \theta_{ik} = 1$.

If we have a full set of contingent markets at date 0 as presented in Chapter 3, we denote by $p \in \mathbb{R}^{\mathbb{D}}$ the contingent price vector. Then the income of Consumer i is $p \cdot e_i + \sum_{k \in \mathcal{K}} \theta_{ik} p \cdot y_k$ if the productions are $(y_k) \in \prod_{k \in \mathcal{K}} Y_k$. So the budget set of this consumer is

$$B_i^W(p, p \cdot e_i + \sum_{k \in \mathcal{K}} \theta_{ik} p \cdot y_k) = \{x_i \in X_i \mid p \cdot x_i \leq p \cdot e_i + \sum_{k \in \mathcal{K}} \theta_{ik} p \cdot y_k\}$$

Définition 13 A contingent market equilibrium is a price-allocation pair $(p^*, x^*, y^*) \in \mathbb{R}^{\mathbb{D}} \times \prod_{i \in \mathcal{I}} X_i \times \prod_{k \in \mathcal{K}} Y_k$, which is feasible and such that:

(i) for all $i \in \mathcal{I}$, x_i^* is a maximum of u_i on the budget set $B_i^W(p^*, w_i^*)$ with $w_i^* = p^* \cdot e_i + \sum_{k \in \mathcal{K}} \theta_{ik} p \cdot y_k^*$.

(ii) for all $k \in \mathcal{K}$, y_k^* is a maximum of $p^* \cdot y$ over Y_k.

From the standard results on the existence and optimality of Walras equilibrium with production, we have the following result:

Theorem 7 *(i) Under Assumptions C, S, NSS*[1] *and P, there exists a contingent commodity equilibrium.*

(ii) If $(p^*, x^*, y^*) \in \mathbb{R}^{\mathbb{D}} \times \prod_{i \in \mathcal{I}} X_i \times \prod_{k \in \mathcal{K}} Y_k$ is a contingent commodity equilibrium, then (x^*, y^*) is Pareto optimal.

7.2 Entrepreneurial equilibrium

We assume now that each firm k has a unique owner $i(k)$ and that each consumer is the owner of at most one firm[2]. We denote by $k(i)$ the firm owned by Consumer i if she is the owner of one firm and by $\mathcal{I}_Y$ the set of consumers who are owner of a firm.

We assume now that we have spot markets at all node with a normalized spot price equal to 1 and a financial structure with $\mathcal{J}$ nominal assets and the $\sharp\mathbb{D}_1 \times \mathcal{J}$ payoff matrix V. We denote by $q \in \mathbb{R}^{\mathcal{J}}$ the asset price vector and by $W(q)$ the $\sharp\mathbb{D} \times \mathcal{J}$ full payoff matrix where the first row is the opposite of q and the remaining rows are the rows of V. We assume that there is no constraint on the asset markets.

[1]Note that in a one commodity economy, Assumption NSS merely means that the utility functions are strictly monotone

[2]An exercise below allows us to remove this last assumption.

The budget set of an agent i whose is the owner of the firm $k(i)$ is then given by:

$$B_i^{\mathcal{E}}(q) = \left\{x_i \in X_i \mid \exists (z, y_{k(i)}) \in \mathbb{R}^{\mathcal{J}} \times Y_{k(i)}\ x_i \leq e_i + W(q)z + y_{k(i)}\right\}$$

If a consumer is not the owner of a firm, $i \notin \mathcal{I}_Y$, the budget set is obtained by putting the production $y_{k(i)}$ equals to 0.

If we assume that each owner decides about the suitable production plan for her, we get the following definition of an entrepreneurial equilibrium.

Définition 14 An entrepreneurial equilibrium is a pair $(q^*, x^*, z^*, y^*) \in \mathbb{R}^{\mathcal{J}} \times \prod_{i \in \mathcal{I}} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \prod_{k \in \mathcal{K}} Y_k$ such that (x^*, y^*) is feasible and

(i) for each $i \in \mathcal{I}_Y$, $x_i^* \leq e_i + W(q^*)z_i^* + y_{k(i)}^*$ and is the maximum of u_i on the budget set $B_i^{\mathcal{E}}(q^*)$; for each $i \notin \mathcal{I}_Y$, $x_i^* \leq e_i + W(q^*)z_i^*$ and is the maximum of u_i on the budget set $B_i^{\mathcal{E}}(q^*)$;

(ii) $\sum_{i \in \mathcal{I}} z_i^* = 0$.

For the following result, we assume that
Assumption D u_i is continuously differentiable on the interior of X_i with a non-vanishing gradient vector.

Exercise 19 *Let f be a strictly monotone quasi-concave $\mathcal{C}^1$ function on an open convex set U. Prove that all partial derivatives of f are positive on U.*

Proposition 28 *Under Assumptions C, D, S, NSS and P, let (q^*, x^*, z^*, y^*) be an entrepreneurial equilibrium with $x_i^* \in \mathrm{int} X_i$ for all $i \in \mathcal{I}$. Then, for all $k \in \mathcal{K}$,*

$$\pi_{i(k)}^* \cdot y_k^* \geq \pi_{i(k)}^* \cdot y_k \ \forall y_k \in Y_k$$

where $\pi_{i(k)}^* = \nabla u_{i(k)}(x_{i(k)}^*)$. We also have $W^t(q^*)\pi_i^* = 0$ for all $i \in \mathcal{I}$.

This proposition means that the optimal production for the owner of the firm k is the one which maximizes the profit for the personal price $\pi_i^* = \nabla u_{i(k)}(x_{i(k)}^*)$. Note also that the personal price π_i^* is a present value vector compatible with the no-arbitrage asset price q^*.

Proof. Let us assume that there exists $k \in \mathcal{K}$ and $y_k \in Y_k$ such that $\pi_{i(k)}^* \cdot y_k > \pi_{i(k)}^* \cdot y_k^*$. For $t \in [0,1]$, let $x_{i(k)}^t = e_{i(k)} + W(q^*)z_{i(k)}^* + ((1-t)y_k^* + ty_k) = x_{i(k)}^* + t(y_k - y_k^*)$. For t small enough, $x_{i(k)}^t$ belongs to $X_{i(k)}$ and to $B_i^{\mathcal{E}}(q^*)$. Furthermore, $\pi_{i(k)}^* \cdot (x_{i(k)}^t - x_{i(k)}^*) = t\pi_{i(k)}^* \cdot (y_k - y_k^*) > 0$. So, since $\pi_{i(k)}^* = \nabla u_{i(k)}(x_{i(k)}^*)$, one deduces that $u_{i(k)}(x_{i(k)}^t) > u_{i(k)}(x_{i(k)}^*)$ for t small enough, which contradicts that $(x_{i(k)}^*, z_{i(k)}^*, y_k^*)$ is a maximum of u_i in the budget set $B_i^{\mathcal{E}}(q^*)$.

For the second assertion, the reasoning is the same as the one for an exchange economy, that is, if $W^t(q^*)\pi_{i(k)}^* \neq 0$, let $\zeta = W^t(q^*)\pi_{i(k)}^*$. Then for $t \in [0,1]$, let $x_{i(k)}^t = e_{i(k)} + W(q^*)(z_{i(k)}^* + t\zeta) + y_k^* = x_{i(k)}^* + tW(q^*)\zeta$. For t small enough,

$x^t_{i(k)}$ belongs to $X_{i(k)}$ and to $B^{\mathcal{E}}_i(q^*)$. Furthermore, $\pi^*_{i(k)} \cdot (x^t_{i(k)} - x^*_{i(k)}) = t\pi^*_{i(k)} \cdot W(q^*)\zeta = W^t(q^*)\pi^*_{i(k)} \cdot \zeta > 0$. So, since $\pi^*_{i(k)} = \nabla u_{i(k)}(x^*_{i(k)})$, one deduces that $u_{i(k)}(x^t_{i(k)}) > u_{i(k)}(x^*_{i(k)})$ for t small enough, which contradicts that $(x^*_{i(k)}, z^*_{i(k)}, y^*_k)$ is a maximum of u_i in the budget set $B^{\mathcal{E}}_i(q^*)$.Note that the above argument works if Consumer i is not in $\mathcal{I}_Y$ by putting $y^*_k = 0$. □

From the above result, we deduces two consequences. First, if the market is incomplete, then the personal prices are not always colinear, so the consumers may not use the same criterion to choose their optimal production. Hence, the global production may not be efficient in the sense that $\sum_{k\in\mathcal{K}} y^*_k$ does not belong to the boundary of the total production set $Y = \sum_{k\in\mathcal{K}} Y_k$. Indeed, if the sets Y_k are smooth (the transformation function is $\mathcal{C}^1$), let us assume that the personal prices of two owners differ, that is $\pi^*_{i(k)}$ is not colinear to $\pi^*_{i(k^*)}$. Then if the total production $\sum_{k\in\mathcal{K}} y^*_k$ is efficient, there exists a price $\pi \in \mathbb{R}^{\mathbb{D}}_+ \setminus \{0\}$ such that the total production maximizes the profit with respect to π on Y, which implies that the individual productions y^*_k maximize the profit over the production set Y_k. Then, $\pi^*_{i(k)}$ and $\pi^*_{i(k^*)}$ would be colinear to π, which leads to a contradiction.

Nevertheless, note that if the production set Y_k is included in the marketable set, that is the range of $W(q^*)$ then the choice of the optimal production is independent of the owner. More precisely, assume that $Y_k = \tilde{Y}_k - \mathbb{R}^{\mathbb{D}}_+$ with $\tilde{Y}_k \subset \mathrm{Im}W(q^*)$. Then, since for all i, $W^t(q^*)\pi^*_i = 0$, for all $w \in \mathrm{Im}W(q^*)$, for all pairs $(i, i') \in \mathcal{I} \times \mathcal{I}$, $(\pi^*_i - \pi^*_{i'}) \cdot w = 0$ and since $\pi^*_i \in \mathbb{R}^{\mathbb{D}}_{+}$, one can conclude that the solutions of maximizing $\pi^*_I \cdot y$ on Y_k do not depend on i.

Exercise 20 *Provide a proof of the above remark.*

So, the disagreement occurs only if the firm offers new transfert possibilities to its owner with respect to the financial market.

The second consequence is the fact that the owner-consumer can decompose her choice into two part: choosing the optimal production according to her personal price and choosing the optimal consumption and portfolio according to the asset price. More formally:

Proposition 29 *Under Assumptions C, D, S, NSS and P, let q be an asset price and let Consumer i be the owner of firm k. Then $(\bar{x}_i, \bar{z}_i, \bar{y}_k) \in \mathrm{int}X_i \times \mathbb{R}^{\mathcal{J}} \times Y_k$ maximizes her utility function over the budget set $B^{\mathcal{E}}_i(q)$ if and only if:*

a) $\bar{x}_i$ satisfies $\bar{x}_i \leq e_i + W(q)\bar{z}_i + \bar{y}_k$ and maximizes u_i over the set

$$\left\{x_i \in X_i \mid \exists z \in \mathbb{R}^{\mathcal{J}}\ x_i \leq e_i + W(q)z + \bar{y}_k\right\}$$

b) $\bar{y}_k$ maximizes $\bar{\pi}_i \cdot y_k$ over Y_k where $\bar{\pi}_i = \nabla u_i(\bar{x}_i)$.

Exercise 21 *Show that the previous results can be extended to the case of a consumer who owns more than one firm. If $\mathcal{K}(i) \subset \mathcal{K}$ is the set of firms owned by Consumer i, show that at an entrepreneurial equilibrium, $\sum_{k\in\mathcal{K}(i)} y^*_k$ is efficient for the production set $\sum_{k\in\mathcal{K}(i)} Y_k$ and $\sum_{k\in\mathcal{K}(i)} y^*_k$ maximises the profit on $\sum_{k\in\mathcal{K}(i)} Y_k$ for the personal price π^*_i.*

7.3 Partnership

In this section, we deal with the question of the joint management decision among stockholders or partners of a firm. Indeed, in the previous section, we have seen that consumers may disagree about the criterion to decide the best possible production since they disagree about the actualisation factor or about the price for which the firm maximises its profit.

As previously, we have a nominal asset structure represented by the $\mathbb{D}_1 \times \mathcal{J}$ payoff matrix V. The consumers have also the opportunity to invest at the first period to buy a share $\theta_{ik} \geq 0$ of the firm k. Then, the stream of income associated to this investment is $\theta_{ik} y_k$ if the production $y_k \in Y_k$ is chosen by the partners or shareholders at equilibrium.

Then, for given productions $(y_k) \in \prod_{k \in \mathcal{K}} Y_k$, the budget set of Consumer i is then:

$$B_i^{\mathcal{P}}(q) = \left\{ x_i \in X_i \mid \exists (z, \theta) \in \mathbb{R}^{\mathcal{J}} \times \mathbb{R}_+^{\mathcal{K}} \; x_i \leq e_i + W(q)z + \sum_{k \in \mathcal{K}} \theta_{ik} y_k \right\}$$

It is then reasonable to assume that the consumers maximize their utility functions over this budget set, but, as shareholders, it is also reasonable to assume that the choice of the production $y_k \in Y_k$ is Pareto optimal for the shareholders in the following sense.

Let $(\bar{x}, \bar{z}, \bar{\theta}, \bar{y}, \bar{q})$ in $\prod_{i \in \mathcal{I}} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I} \times \mathcal{K}} \times \prod_{k \in \mathcal{K}} Y_k \times \mathbb{R}^{\mathcal{J}}$ such that $(\bar{x}, \bar{y})$ is feasible, for all i, $\bar{x}_i \in B_i^{\mathcal{P}}(\bar{q})$, $\bar{x}_i \leq e_i + W(\bar{q})\bar{z}_i + \sum_{k \in \mathcal{K}} \bar{\theta}_{ik} \bar{y}_k$ and $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$, $\sum_{k \in \mathcal{K}} \bar{\theta}_{ik} = 1$ for all i . Then $\bar{y}_k$ is not Pareto optimal if it exists $y_k \in Y_k$ such that $u_i(\bar{x}_i + \bar{\theta}_{ik}(y_k - \bar{y}_k)) \geq u_i(\bar{x}_i)$ for all i with at least one strict inequality.

Note that the large inequality is satisfied if Consumer i is not a shareholder of firm k since $\bar{\theta}_{ik} = 0$ and then the strict improvement is only for shareholders. In other words, this criterion involves only the shareholders.

The next proposition shows that a production $\bar{y}_k$ is Pareto optimal for the shareholders if and only if it maximizes the profit on Y_k for some non-negative combination of the supporting prices of the shareholders $\bar{\pi}_i = \nabla u_i(\bar{x}_i)$.

Proposition 30 *We posit Assumptions C, D, S, NSS and P and we strengthen Assumption C by assuming that u_i is strictly quasi-concave on the interior of X_i.*

Let $(\bar{x}, \bar{z}, \bar{\theta}, \bar{y}, \bar{q})$ in $\prod_{i \in \mathcal{I}} \mathrm{int} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I} \times \mathcal{K}} \times \prod_{k \in \mathcal{K}} Y_k \times \mathbb{R}^{\mathcal{J}}$ such that $(\bar{x}, \bar{y})$ is feasible, for all i, $\bar{x}_i \in B_i^{\mathcal{P}}(\bar{q})$, $\bar{x}_i \leq e_i + W(\bar{q})\bar{z}_i + \sum_{k \in \mathcal{K}} \bar{\theta}_{ik} \bar{y}_k$ and $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$, $\sum_{k \in \mathcal{K}} \bar{\theta}_{ik} = 1$ for all i. Let $k \in \mathcal{K}$ and $\mathcal{I}(k) = \{i \in \mathcal{I} \mid \bar{\theta}_{ik} > 0\}$. $\bar{y}_k$ is Pareto optimal among the shareholders $\mathcal{I}(k)$ if and only if it exists some coefficients $\alpha \in \mathbb{R}_+^{\mathcal{I}(k)} \setminus \{0\}$ such that $\bar{y}_k$ maximises the profit on Y_k for the price $\sum_{i \in \mathcal{I}(k)} \alpha_i \bar{\theta}_{ik} \bar{\pi}_i$ where $\bar{\pi}_i = \nabla u_i(\bar{x}_i)$.

Proof. Let us assume that $\bar{y}_k$ is Pareto optimal among the shareholders $\mathcal{I}(k)$. Let $P_i(\bar{x}_i) = \{\xi_i \in X_i \mid u_i(\xi_i) > u_i(\bar{x}_i)$. Then, from the very definition, for all $i \in \mathcal{I}(k)$, $\bar{\theta}_{ik}(Y_k - \bar{y}_k) \cap \left(\cap_{i \in \mathcal{I}(k)} (P_i(\bar{x}_i) - \bar{x}_i)\right) = \emptyset$. Since $\bar{\theta}_{ik} \in]0,1]$, $\bar{\theta}_{ik}(Y_k -$

$\bar{y}_k) = Y_k - \bar{y}_k$. Since the sets $Y_k - \bar{y}_k$ and $\left(\cap_{i\in\mathcal{I}(k)}(P_i(\bar{x}_i) - \bar{x}_i)\right)$ are convex, nonempty and disjoint, there exists $\pi \in \mathbb{R}^{\mathbb{D}} \setminus \{0\}$ such that $\sup \pi \cdot (Y_k - \bar{y}_k) \leq \inf \pi \cdot \left(\cap_{i\in\mathcal{I}(k)}(P_i(\bar{x}_i) - \bar{x}_i)\right)$. Since $-\mathbb{R}^{\mathbb{D}}_+ \subset Y_k$, one deduces that $\pi \geq 0$. Since $0 \in Y_k - \bar{y}_k$ and 0 belongs to the closure of $\left(\cap_{i\in\mathcal{I}(k)}(P_i(\bar{x}_i) - \bar{x}_i)\right)$, one deduces that the above sup and inf are both equal to 0. So, $\pi \cdot \bar{y}_k \geq \pi \cdot y_k$ for $y_k \in Y_k$. By a continuity argument, one also deduces that the infimum over the intersection of the closures of the cone generated by $(P_i(\bar{x}_i) - \bar{x}_i)$ is also equal to 0. But, since u_i is differentiable and quasi-concave, the closure of the cone generated by $(P_i(\bar{x}_i) - \bar{x}_i)$ is the set $\{\xi \in \mathbb{R}^{\mathbb{D}} \mid \bar{\pi}_i \cdot \xi \geq 0\}$. From the Farkas Lemma, or by noticing that 0 is a solution of the following minimisation problem and by applying the Karush-Kuhn-Tucker Theorem,

$$\begin{cases} \text{Minimize } \pi \cdot \xi \\ \bar{\pi}_i \cdot \xi \geq 0 \text{ for all } i \in \mathcal{I}(k) \end{cases}$$

one deduces that there exists $\beta \in \mathbb{R}^{\mathcal{I}(k)}_+ \setminus \{0\}$ such that $\pi = \sum_{i\in\mathcal{I}(k)} \beta_i \bar{\pi}_i$. So, it suffices to take $\alpha_i = \beta_i / \bar{\theta}_{ik}$.

Conversely, if there exists $\alpha \in \mathbb{R}^{\mathcal{I}(k)}_+ \setminus \{0\}$ such that $\bar{y}_k$ maximises the profit on Y_k for the price $\sum_{i\in\mathcal{I}(k)} \alpha_i \bar{\theta}_{ik} \bar{\pi}_i$. Let us assume that there $y_k \in Y_k$ such that $u_i(\bar{x}_i + \bar{\theta}_{ik}(y_k - \bar{y}_k)) \geq u_i(\bar{x}_i)$ for all $i \in \mathcal{I}(k)$ with at least one strict inequality. Then $y_k - \bar{y}_k$ is a non zero vector and, from the strict quasi-concavity of u_i for all i, $u_i(\bar{x}_i + \frac{1}{2}\bar{\theta}_{ik}(y_k - \bar{y}_k)) \geq u_i(\bar{x}_i)$ for all $i \in \mathcal{I}(k)$. Then from the differentiability of u_i and the quasi-concavity, we get that $\bar{\pi}_i \cdot \frac{1}{2}\bar{\theta}_{ik}(y_k - \bar{y}_k) > 0$ for all $i \in \mathcal{I}(k)$, so $(\sum_{i\in\mathcal{I}(k)} \alpha_i \bar{\theta}_{ik} \bar{\pi}_i) \cdot (y_k - \bar{y}_k) > 0$, which contradicts the fact that $\bar{y}_k$ is profit maximising. $\square$.

Even if this Pareto optimal criterion is valuable, we remark that it is to weak to provide a normative criterion for the joint decision of the shareholders since all coefficients $\alpha \in \mathbb{R}^{\mathcal{I}(k)}_+ \setminus \{0\}$ are acceptable but the production maximizing the profit for the associated price are very different, so there is a large indeterminacy of the final output.

That is why we introduce the notion of constrained Pareto optimality taken the point of view of a planner. The planner can choose the portfolios and the productions but she is constrained by the existing financial market for the transfert of wealth among the two periods and among the state of the world at the second period.

Let us define first a constrained feasible allocation.

Définition 15 Let us consider $(x, z, \theta, y) \in \prod_{i\in\mathcal{I}} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I}\times\mathcal{K}} \times \prod_{k\in\mathcal{K}} Y_k$. Then this plan is constrained feasible if:

(i) $\sum_{i\in\mathcal{I}} x_{i0} = \sum_{i\in\mathcal{I}} e_{i0} + \sum_{k\in\mathcal{K}} y_{k0}$:

(ii) for all $(i, \xi) \in \mathcal{I} \times \mathbb{D}$, $x_{i\xi} = e_{i\xi} + V_\xi \cdot z_i + \sum_{k\in\mathcal{K}} \theta_{ik} y_{k\xi}$;

(iii) $\sum_{i\in\mathcal{I}} z_i = 0$;

(iv) for all $k \in \mathcal{K}$, $\sum_{i\in\mathcal{I}} \theta_{ik} = 1$

Note that the set of constrained feasible allocation is not convex due to the products $\theta_{ik} y_{k\xi}$ for each node, firm and state. We now derive the notion of Constrained Pareto Optimality.

Définition 16 A plan $(x, z, \theta, y) \in \prod_{i\in\mathcal{I}} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I}\times\mathcal{K}} \times \prod_{k\in\mathcal{K}} Y_k$ is Constrained Pareto Optimal if it constrained feasible and it does not exist a constrained feasible plan $(x', z', \theta', y') \in \prod_{i\in\mathcal{I}} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I}\times\mathcal{K}} \times \prod_{k\in\mathcal{K}} Y_k$ such that $u_i(x'_i) \geq u_i(x_i)$ for all $i \in \mathcal{I}$ and $(u_i(x'_i))_{i\in\mathcal{I}} \neq (u_i(x_i))_{i\in\mathcal{I}}$.

Now, we characterise the first order condition for a constrained Pareto optimal allocation.

Proposition 31 *We posit Assumptions C, D, S, NSS, P, and P'.*
Let $(\bar{x}, \bar{z}, \bar{\theta}, \bar{y}) \in \prod_{i\in\mathcal{I}} \mathrm{int} X_i \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I}\times\mathcal{K}} \times \prod_{k\in\mathcal{K}} Y_k$ be a Constrained Pareto Optimal plan. Then for all $k \in \mathcal{K}$, $\bar{y}_k$ maximises the profit over Y_k for the price $\pi_k = \sum_{i\in\mathcal{I}} \bar{\theta}_{ik} \bar{\pi}_i$ where $\bar{\pi}_i = \nabla u_i(\bar{x}_i)$ for all $i \in \mathcal{I}$.

This means that if the shareholders wish to achieve at least a constrained Pareto optimal allocation, then they must choose as a criterion for the choice of the production the maximisation with respect to the price which is a convex combination of their personal prices with the coefficient given by the shares in the firm.

Proof. For all $i \in \mathcal{I}$, let $(x_0, z, \theta, y) \in \mathbb{R}^{\mathcal{I}} \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times [0,1]^{\mathcal{I}\times\mathcal{K}} \times \prod_{k\in\mathcal{K}} Y_k$ and let us define the function $\tilde{u}_i$ by

$$\tilde{u}_i(x_0, z, \theta, y) = u_i\left(x_{i0}, \left(e_{i\xi} + V_\xi \cdot z_i + \sum_{k\in\mathcal{K}} \theta_{ik} y_{k\xi}\right)_{\xi\in\mathbb{D}_1}\right)$$

Since $\bar{x}_i$ belongs to the interior of X_i, this function is well defined and continuously differentiable in a neighbourhood of $(\bar{x}_0, \bar{z}, \bar{\theta}, \bar{y})$. Let C be the affine subspace of $\mathbb{R}^{\mathcal{I}} \times (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}\times\mathcal{K}} \times (\mathbb{R}^{\mathbb{D}})^{\mathcal{K}}$ defined by the following equation:

$$\begin{cases} \sum_{i\in\mathcal{I}} x_{i0} - \sum_{k\in\mathcal{K}} y_{k0} = \sum_{i\in\mathcal{I}} e_{i0} \\ \sum_{i\in\mathcal{I}} z_i = 0 \\ \text{for all } k \in \mathcal{K},\ \sum_{i\in\mathcal{I}} \theta_{ik} = 1 \end{cases}$$

If $(\bar{x}, \bar{z}, \bar{\theta}, \bar{y})$ is Constrained Pareto Optimal then $(0, \bar{x}_0, \bar{z}, \bar{\theta}, \bar{y})$ is a solution of the following maximisation problem:

$$\begin{cases} \text{Maximise } t \\ t + \tilde{u}_i(\bar{x}_0, \bar{z}, \bar{\theta}, \bar{y}) - \tilde{u}_i(x_0, z, \theta, y) \leq 0, \quad \text{for all } i \in \mathcal{I} \\ (x_0, z, \theta, y) \in C \\ T_k(y_k) \leq 0, \quad \text{for all } k \in \mathcal{K} \\ -\theta_{ik} \leq 0,\ \theta_{ik} \leq 1 \quad \text{for all } (i,k) \in \mathcal{I} \times \mathcal{K} \end{cases}$$

Indeed, if not, we could find a constrained feasible allocation for which the utility level of all consumers be strictly greater than the one at $\bar{x}_i$.

We also remark that the Mangasarian-Fromovitz constraint qualification condition is satisfied. Indeed, let Δ_{y_k} be the vector $-\mathbf{1}_{\mathbb{D}}$, let $\Delta_{x_{i0}} = \sharp\mathcal{K}/\sharp\mathcal{I}$, $\Delta_{z_i} = 0_{\mathcal{J}}$ and $\Delta_{\theta_i} = 0_{\mathcal{K}}$. Let $w = ((\Delta_{x_{i0}})_{i\in\mathcal{I}}, 0_{\mathcal{I}\times\mathcal{J}}, 0_{\mathcal{I}\times\mathcal{K}}, (\Delta_{y_k})_{k\in\mathcal{K}})$. Let us choose w_0 strictly greater than the maximum over i of $|w\cdot\nabla\tilde{u}_i(\bar{x}_0,\bar{z},\bar{\theta},\bar{y})|$. Then the vector $(-w_0, w)$ satisfies the MF condition.

So, there exists KKT multipliers associated to the solution $(0,\bar{x}_0,\bar{z},\bar{\theta},\bar{y})$. Let $\alpha \in \mathbb{R}^{\mathcal{I}}_+$ be the multipliers associated to the constraints $t + \tilde{u}_i(\bar{x}_0,\bar{z},\bar{\theta},\bar{y}) - \tilde{u}_i(x_0,z,\theta,y) \leq 0$. Let λ be the multiplier associated to the constraints $\sum_{i\in\mathcal{I}} x_{i0} - \sum_{k\in\mathcal{K}} y_{k0} = \sum_{i\in\mathcal{I}} e_{i0}$. Let $\mu \in \mathbb{R}^{\mathcal{K}}_+$ be the multipliers associated to the constraints $T_k(y_k) \leq 0$.

From the KKT theorem, one deduces that:

$$\begin{cases} \alpha_i \frac{\partial u_i}{\partial x_{i0}} = \lambda \quad \text{for all } i\in\mathcal{I} \\ 0 = -\lambda + \mu_k \frac{\partial T_k}{\partial y_{k0}} \quad \text{for all } k\in\mathcal{K} \\ \sum_{i\in\mathcal{I}} \alpha_i \bar{\theta}_{ik} \frac{\partial u_i}{\partial x_{i\xi}} = \mu_k \frac{\partial T_k}{\partial y_{k\xi}} \quad \text{for all } (k,\xi)\in\mathcal{K}\times\mathbb{D}_1 \end{cases}$$

From which, one deduces that $\nabla T_k(\bar{y}_k)$ is positively colinear to the vector

$$\sum_{i\in\mathcal{I}} \bar{\theta}_{ik} \begin{pmatrix} 1 \\ \left(\frac{\frac{\partial u_i}{\partial x_{i\xi}}(\bar{x}_i)}{\frac{\partial u_i}{\partial x_{i0}}(\bar{x}_i)}\right)_{\xi\in\mathbb{D}_1} \end{pmatrix}$$

So, $\nabla T_k(\bar{y}_k)$ is positively colinear to the vector $\sum_{i\in\mathcal{I}} \bar{\theta}_{ik}\nabla u_i(\bar{x})$. This means that $\bar{y}_k$ maximisez the profit with respect to the price $\sum_{i\in\mathcal{I}} \bar{\theta}_{ik}\nabla u_i(\bar{x})$ over Y_k. □

Example: Consider a one-good two period production economy with two states of nature at date 1 adn two agents $i = 1, 2$. The agents' utility functions, which do not depend on consumption at date 0

$$u^1(x^1_1, x^1_2) = 5\sqrt{x^1_1} + x^1_2, \quad u^2(x^2_1, x^2_2) = x^2_1 + 5\sqrt{x^2_2}$$

Both agents have 1 unit of the good at date 0 and no resources at date 1: $e^1 = e^2 = (1, 0, 0)$. Two technologies are available with the following production sets:

$$\begin{array}{l} Y^1 = \{(y^1_0, y^1_1, y^1_2) \in \mathbb{R}^3 \mid y^1_0 \leq 0, y^1_1 \leq 2y^1_0, y^1_2 \leq y^1_0, y^1_1 + 2y^1_2 \leq 2y^1_0\} \\ Y^2 = \{(y^2_0, y^2_1, y^2_2) \in \mathbb{R}^3 \mid y^2_0 \leq 0, y^2_1 \leq y^2_0, y^2_2 \leq 2y^2_0, 2y^2_1 + y^2_2 \leq 2y^2_0\} \end{array}$$

There is no financial market. Each agent chooses a share portfolio $(\theta^i_1, \theta^i_2) \in [0,1]^2$.

We can show that $\bar{y}_1 = (-1, 0, 1)$ and $\bar{y}_2 = (-1, 1, 0)$ with the portfolios $\bar{\theta}^1 = (0, 1)$ and $\bar{\theta}^2 = (1, 0)$ is a partnership equilibrium which is not productive efficient and constrained Pareto suboptimal.

We can show that $\hat{y}_1 = (-1, 2, 0)$ and $\hat{y}_2 = (-1, 0, 2)$ with the portfolios $\hat{\theta}^1 = (1, 0)$ and $\hat{\theta}^2 = (0, 1)$ is a partnership equilibrium which is Pareto optimal.

We can prove that $(\hat{x}^1 = (0, 2, 0), \hat{x}^2 = (0, 0, 2), \hat{y}_1, \hat{y}_2, \hat{\pi} = (1, 1/2, 1/2))$ is a contingent market equilibrium.

Exercise 22 *We consider an economy with $\mathcal{K}$ technologies represented by the productions sets which are half line in the space $\mathbb{R}^{\mathbb{D}}$. For all $k \in \mathcal{K}$, there exists $\eta^k \in \mathbb{R}_+^{\mathbb{D}_1}$ such that:*

$$Y^k = \{(y_0^k, (y_\xi^k)_{\xi \in \mathbb{D}_1}) \in \mathbb{R}^{\mathbb{D}} \mid (y_\xi^k)_{\xi \in \mathbb{D}_1} = -\eta^k y_0^k, y_0^y \leq 0\}$$

There is only one asset, the riskless bond with a return equal to 1 for each $\xi \in \mathbb{D}_1$ and a price q. Each agent chooses the quantity b^i of bond and the shares $(\theta_k^i)_{k \in \mathcal{K}}$. We put the standard assumption on the utility functions and we only consider interior consumption in $\mathbb{R}_{++}^{\mathbb{D}}$.

1) Prove that at equilibrium all partners agree on the optimal plan for each venture. Show that at an equilibrium, the economy functions as if each agent chose the scale at which to operate each technology, the global scale being simply the sum of the levels chosen by each individual.

2) Show that the set of constrained feasible allocations for a planner is convex.

3) Prove that a partnership equilibrium is constrained Pareto optimal.