Economic analysis of financial market
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Complete financial structures, equivalent financial structures, and existence of financial equilibrium

Outline

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Definition

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The unconstrained financial structure V is complete at the price p, if for every present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, the two following sets are equal :

$$\mathcal{B}^W(\pi) = \{x \in \mathbb{R}^{\mathbb{L}} \mid \pi \cdot x \leq 0\}$$

and

$$\mathcal{B}^{\mathcal{F}}(p,q) = \left\{ x \in \mathbb{R}^{\mathbb{L}} \middle| \exists z \in \mathbb{R}^{\mathcal{J}} \begin{array}{ll} p(\xi_0) \cdot x(\xi_0) + q \cdot z \leq 0 & \\ p(\xi) \cdot x(\xi) \leq V(p,\xi) \cdot z, & \forall \xi \in \mathbb{D}_1 \end{array} \right\}$$

where $q = V(p)^t \lambda$ and $\pi = (p(\xi_0), (\lambda_\xi p(\xi))_{\xi \in \mathbb{D}_1})$.

Examples

Remark

The financial structure associated to the full set of Arrow securities and the financial structure with all contingent commodity contracts are complete.

Remark

If V is complete, the two following budget sets

$$B_i^W(\pi, \pi \cdot e_i) = \{x_i \in X_i \mid \exists \chi_i \in \mathcal{B}^W(\pi), x_i = e_i + \chi_i\}$$

and

$$B_i^{\mathcal{F}}(p, q) = \{x_i \in X_i \mid \exists \chi_i \in \mathcal{B}^{\mathcal{F}}(p, q), x_i = e_i + \chi_i\}$$

are equal.

Consequence on the financial equilibrium

Proposition

Let $\mathcal{E}_{\mathcal{F}} = ((X_i, u_i, e_i, \mathbb{R}^{\mathcal{J}})_{i \in \mathcal{I}}, V)$ be an unconstrained financial economy, which satisfies Assumption NSS. We assume that the financial structure is complete at a spot price $\bar{p}$. Let $(\bar{x}_i) \in \prod_{i \in \mathcal{I}} X_i$ and $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$. Then, the two conditions are equivalent :

a) There exists $(\bar{z}_i) \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that $((\bar{x}_i, \bar{z}_i), \bar{p}, \bar{q})$ is a financial equilibrium with $\bar{q} = V(\bar{p})^t \lambda$.

b) $((\bar{x}_i), \bar{\pi})$ is a contigent commodity equilibrium with $\bar{\pi} = (\bar{p}(\xi_0), (\lambda_\xi \bar{p}(\xi))_{\xi \in \mathbb{D}_1})$.

Characterisation of complete financial structures

Proposition

Let us consider an unconstrained financial structure V and a spot price vector $p \in \mathbb{R}^{\mathbb{L}} \setminus \{0\}$. Then :

a) If the rank of $V(p)$ is equal to $\sharp\mathbb{D}_1$, that is $V(p)$ is onto, then V is complete at p.

b) If $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$ and V is complete at p, then the rank of $V(p)$ is equal to $\sharp\mathbb{D}_1$.

Example : complete financial structure without V onto

Let us take a simple date-event tree $\mathbb{D}$ with two nodes at period 1, ξ_1 and ξ_2. There is a unique nominal asset j with the payoff vector $V_j = (1, 0)$, so the payoff matrix V is not onto.
Nevertheless, if we consider the spot price $p = (1, 1, 0)$ and the present value vector $\lambda = (1, 1)$ and the associated asset price $q = 1$, we check that :

$$\mathcal{B}^W(\pi) = \{x \in \mathbb{R}^3 \mid x(\xi_0) + x(\xi_1) \leq 0\}$$

and

$$\mathcal{B}^{\mathcal{F}}(p, q) = \{x \in \mathbb{R}^3 \mid \exists z \in \mathbb{R}, x(\xi_0) + z \leq 0, x(\xi_1) \leq z\}$$

are equal.

Over hedging pricing with a complete financial structure

If V is onto at p, so the market is complete, then the cost function or over hedging price q^+ is linear on $\mathbb{R}^{\mathbb{D}_1}$ and just equal to $\lambda \cdot v$ for the unique present value vector λ associated to q.

Regular financial structure

In the absence of redundant asset, $V(p)$ is onto if and only if $V(p)$ is regular. So, in many articles, the authors simply assume that the return matrix is regular, which implies that the markets are complete and that there exists a unique portfolio associated to any wealth transfer v on $\mathbb{D}_1$.

Completion using options

Let us assume that we have a unique asset, which discriminates the state of the world tomorrow, in the sense that if $(\xi_1, \xi_2, \ldots, \xi_k)$ *are the* k *states at period* 1, $v(\xi_1) < v(\xi_2) < \ldots < v(\xi_k)$. *Then, an option at the strike price* σ *is a financial asset which promises to deliver* $v(\xi)$ *for the states such that* $v(\xi) \geq \sigma$ *and* 0 *otherwise. We introduce* $k-1$ *options at the strike price* $v(\xi_\kappa)$ *for* $\kappa = 2, \ldots, k$. *Then the financial structure build with the initial asset and the* $k-1$ *options is complete and without redundant assets.*

Nominal and numéraire asset structures

Remark

For the nominal asset structures or for the numéraire asset structures, the rank of the payoff matrix does not depend on the spot price as long as the value of the numéraire is not vanishing. So, the market is complete for all spot prices.

For real asset structure, the rank of the matrix depends on the spot price and the rank can drop, which means that the market is no more complete for some spot prices whereas it is complete for almost all spot prices.

Outline

1. Complete financial structures
2. Equivalent financial structures
3. Existence result for the one commodity case

Definition

Definition

Let p be a spot price vector and V_1 and V_2 be two financial structures. V_1 is equivalent to V_2 at the price p if for all $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, $\mathcal{B}^{\mathcal{F}}_{V_1}(p, V_1(p)^t\lambda) = \mathcal{B}^{\mathcal{F}}_{V_2}(p, V_2(p)^t\lambda)$.

Equilibrium and equivalence

Proposition

Let V_1 and V_2 two financial structures. If $\mathcal{E}$ satisfies Assumption NSS, then, if $((\bar{x}_i, \bar{z}_i), \bar{p}, \bar{q})$ is a financial equilibrium for the financial structure V_1 and V_1 and V_2 are equivalent at $\bar{p}$, then there exists $\bar{\zeta} \in (\mathbb{R}^{\mathcal{J}_2})^{\mathcal{I}}$ and an asset price vector $\bar{\chi} \in \mathbb{R}^{\mathcal{J}_2}$ such that $((\bar{x}_i, \bar{\zeta}_i), \bar{p}, \bar{\chi})$ is a financial equilibrium for the financial structure V_2.

A new interpretation of a complete financial structure

Remark

According to the previous sub-section and the definition of equivalent financial structures, we remark that a financial structure is complete if and only if it is equivalent to the financial structure with all contingent commodities or to the structure with all Arrow securities.

Characterisation of equivalent financial structures

Proposition

Let V_1 and V_2 two financial structures. For a spot price p such that $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$, V_1 and V_2 are equivalent at p, if and only if the range of $V_1(p)$ is equal to the range of $V_2(p)$.

Remark

In the unconstrained case, each structure is equivalent to a structure without redundant asset.

Outline

Assumptions

$\ell = 1$, so $\mathbb{L} = \mathbb{D}$.

Assumption C1. For all $i \in \mathcal{I}$,

a) $X_i = \mathbb{R}_+^{\mathbb{D}}$;

b) u_i is strictly increasing on X_i.

Assumption F1.

a) For all $j \in \mathcal{J}$, $V_j \in \mathbb{R}_+^{\mathbb{D}_1} \setminus \{0\}$;

b) For all $\xi \in \mathbb{D}_1$, there exists $j \in \mathcal{J}$ such that $V_j(\xi) > 0$.

c) for all $i \in \mathcal{I}$, $Z_i = \mathbb{R}^{\mathcal{J}}$.

Analysis of the equilibrium equations

Let us consider a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$. From the strict monotonicity of the utility functions, we deduces that $p^*(\xi) > 0$ for all $\xi \in \mathbb{D}$ and the budget constraints are binding. So, for all $i \in \mathcal{I}$, for all $\xi \in \mathbb{D}_1$,

$$p^*(\xi)x_i^*(\xi) = p^*(\xi)e_i(\xi) + \sum_{j \in \mathcal{J}} z_{ij}^* \; V_j(\xi)p^*(\xi)$$

Hence, $x_i^*(\xi) = e_i(\xi) + \sum_{j \in \mathcal{J}} z_{ij}^* \; V_j(\xi)$. So, the consumption at date 1 is completely determined by the portfolio chosen on the financial market at date 0. Hence, we can reduce the choice of the consumer to her consumption at date 0 and her portfolio. Furthermore, the equilibrium portfolios must satisfy the constraints $e_i(\xi) + \sum_{j \in \mathcal{J}} z_{ij}^* \; V_j(\xi) \geq 0$ for all $\xi \in \mathbb{D}_1$.

An auxiliary exchange economy $\tilde{\mathcal{E}}$

Commodity space $\mathbb{R} \times \mathbb{R}^{\mathcal{J}}$.

$\forall i \in \mathcal{I}$, *the consumption set*
$\Xi_i = \{(x_i(\xi_0), z_i) \in \mathbb{R} \times \mathbb{R}^{\mathcal{J}} \mid x_i(\xi_0) \geq 0, e_i^1 + Vz_i \geq 0\}$ *where* $e_i^1 \in \mathbb{D}_1$ *is the restriction of* e_i *to the states in* $\mathbb{D}_1$.

The utility function : $\tilde{u}_i(x_i(\xi_0), z_i) = u_i(x_i(\xi_0), e_i^1 + Vz_i)$

The endowments : $\tilde{e}_i = (e_i(\xi_0), 0)$.

Financial equilibrium and Walras equilibrium of the auxiliary economy

Proposition

Let $((x_i^*, z_i^*), p^*, q^*)$ *be a financial equilibrium of* $\mathcal{E}_{\mathcal{F}}$. *Then,* $((x_i^*(\xi_0), z_i^*), (p^*(\xi_0), q^*))$ *is a Walras equilibrium of* $\tilde{\mathcal{E}}$.
Conversely, let $((\tilde{x}_i(\xi_0), \tilde{z}_i), (\tilde{p}(\xi_0), \tilde{q}))$ *be a Walras equilibrium of* $\tilde{\mathcal{E}}$*, then*
$((\tilde{x}_i, \tilde{z}_i), \tilde{p}, \tilde{q})$ *is a financial equilibrium of* $\mathcal{E}_{\mathcal{F}}$ *with for all* $\xi \in \mathbb{D}_1$

a) $\tilde{p}(\xi) = 1$ *;*

b) $\tilde{x}_i(\xi) = e_i(\xi) + \sum_{j \in \mathcal{J}} \tilde{z}_{ij} \; v_j(\xi)$.

Assumptions on $\tilde{\mathcal{E}}$

Proposition

If the economy $\mathcal{E}_{\mathcal{F}}$ satisfies Assumption C, C1, S and F1, then the economy $\tilde{\mathcal{E}}$ satisfies : for all $i \in \mathcal{I}$

a) Ξ_i is nonempty, convex, closed ;

b) $\tilde{u}_i$ is continuous, strictly increasing and quasi-concave on Ξ_i.

c) $\tilde{e}_i \in \mathrm{int}\Xi_i$.

On the attainable set of $\tilde{\mathcal{E}}$

$$\mathcal{A} = \{(\tilde{x}_i) \in \prod_{i \in \mathcal{I}} \Xi_i \mid \sum_{i \in \mathcal{I}} \tilde{x}_i = \sum_{i \in \mathcal{I}} \tilde{e}_i\}$$

$(\tilde{x}_i = (x_i(\xi_0), z_i))$ *belongs to* $\mathcal{A}$ *if* $\sum_{i \in \mathcal{I}} x_i(\xi_0) = \sum_{i \in \mathcal{I}} e_i(\xi_0)$ *and* $\sum_{i \in \mathcal{I}} z_i = 0$.

For the boundedness of $\mathcal{A}$, *no problem for the first component. Under which condition is the set*

$$\mathcal{A}_Z = \{(z_i) \in \prod_{i \in \mathcal{I}} \mathbb{R}^{\mathcal{J}} \mid \sum_{i \in \mathcal{I}} z_i = 0, \forall i, e_i^1 + V z_i \geq 0\}$$

bounded ?

An example

$$V = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Then,

$$\Xi_i = \left\{ (x(\xi_0), z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}^2 \;\middle|\; \begin{array}{l} e_i(\xi_1) + z_1 + z_2 \geq 0 \\ e_i(\xi_2) + z_1 + \frac{1}{2} z_2 \geq 0 \\ e_i(\xi_3) + \frac{1}{2} z_1 + z_2 \geq 0 \end{array} \right\}$$

We remark that for all $t \geq 0$, $(0, 2t, -t)$ and $(0, -t, 2t)$ belongs to Ξ_i so it is not bounded from below.

A necessary and sufficient condition for the boundedness of $\mathcal{A}$

Proposition

If $\sharp I \geq 2$, $\mathcal{A}_Z$ is bounded if and only if V is one to one.

The existence result

Proposition

If the unconstrained financial economy $\mathcal{E}_{\mathcal{F}}$ satisfies Assumption C, C1, S and $\operatorname{Im} V \cap \mathbb{R}^{\mathbb{D}_1}_{++} \neq \emptyset$, *then a financial equilibrium exists.*