Ex 12

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (\overbrace{x^2-y^2+z^2-1}^{f_1}, \overbrace{xyz-1}^{f_2})$$

$$f(x_0,y_0,z_0) = 0$$

$$f : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, \underset{(y,z)}{\underset{\parallel}{u}}) \longmapsto (f_1(x,u), f_2(x,u))$$

f est polynomiale, donc $\mathcal{C}^1$ sur $\mathbb{R}^3$ et

$$\mathrm{Jac}_f(x_0,y_0,z_0) = \begin{pmatrix} 2x_0 & -2y_0 & 2z_0 \\ y_0 z_0 & x_0 z_0 & x_0 y_0 \end{pmatrix}$$

$D_x f(x_0,y_0,z_0)$ $\quad$ $D_u f(x_0,y_0,z_0)$

$\det D_u f(x_0,y_0,z_0) = -2x_0(y_0^2+z_0^2)$

Supposons, par l'absurde que $\det(D_u f(x_0, y_0, z_0)) = 0$

alors

- soit $x_0 = 0$ donc $x_0 y_0 z_0 = 0$

  donc $f_2(x_0, y_0, z_0) = -1 \neq 0 \rightsquigarrow$ contredit $f(x_0, y_0, z_0) = 0$

- soit $y_0^2 + z_0^2 = 0$, ie $\cancel{y_0} = z_0 = 0$ donc

  $f_2(x_0, y_0, z_0) = x_0 y_0 z_0 - 1 = -1 \neq 0$

  $\rightsquigarrow$ contredit $f(x_0, y_0, z_0) = 0$

Donc $\det D_u f(x_0, y_0, z_0) \neq 0$

Par le TFI, $\exists I$ voisinage de x_0, V voisinage de (y_0, z_0)

et $\varphi : I \longrightarrow V$ $\mathcal{C}^1$ tq $\forall (x, y, z) \in I \times V$ $D_u f(x, y, z)$

inversible et $\Big(f(x, y, z) = 0$ ssi $\varphi(x) = (y, z) \Big)$

En particulier pour $(y,z) = \varphi(x)$ on a $f(x, \varphi(x)) = 0 \quad \forall x \in I$

De plus, $D\varphi(x) = - \underline{D_u f(x, \varphi(x))}^{-1} \cdot D_x f(x, \varphi(x)) \quad \forall x \in I$

On note $\varphi(x) = (\varphi_1(x), \varphi_2(x))$

alors $D_u f(x, \varphi(x)) = \begin{pmatrix} -2\varphi_1(x) & 2\varphi_2(x) \\ x\varphi_2(x) & x\varphi_1(x) \end{pmatrix}$

Donc $D_u f(x, \varphi(x))^{-1} = \dfrac{-1}{2x(\varphi_1(x)^2 + \varphi_2(x)^2)} \begin{pmatrix} x\varphi_1(x) & -2\varphi_2(x) \\ -x\varphi_2(x) & -2\varphi_1(x) \end{pmatrix}$

et $D_x f(x, \varphi(x)) = \begin{pmatrix} 2x \\ \varphi_1(x)\varphi_2(x) \end{pmatrix}$

$$D\varphi(x) = \frac{1}{2x\left(\varphi_1(x)^2+\varphi_2(x)^2\right)} \begin{pmatrix} x\varphi_1(x) & -2\varphi_2(x) \\ -x\varphi_2(x) & -2\varphi_1(x) \end{pmatrix} \begin{pmatrix} 2x \\ \varphi_1(x)\varphi_2(x) \end{pmatrix}$$

$$= \frac{1}{x\left(\varphi_1(x)^2+\varphi_2(x)^2\right)} \begin{pmatrix} x^2\varphi_1(x) - \varphi_1(x)\varphi_2(x)^2 \\ -x^2\varphi_2(x) - \varphi_1^2(x)\varphi_2(x) \end{pmatrix} = \underbrace{\frac{1}{x\|\varphi(x)\|^2} \begin{pmatrix} \varphi_1(x)\left(x^2-\varphi_2(x)^2\right) \\ \varphi_2(x)\left(-x^2+\varphi_1(x)^2\right) \end{pmatrix}}_{=:\varphi'(x) \in \mathbb{R}^2}$$

$$D\varphi(x) : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad h \longmapsto \underbrace{\varphi'(x)}_{\in \mathbb{R}^2} \cdot h$$

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)) \qquad \varphi'(x) = (\varphi_1'(x), \varphi_2'(x))$$

$$(x_1 \; \cdots \; x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle$$