Calibration in Quantitative Finance

Noufel Frikha

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Who am I ?

- Noufel FRIKHA, email : *noufel.frikha@univ-paris1.fr*
- Full Professor (professeur d'université), université Paris 1 Panthéon-Sorbonne
- Associate Professor (maître de conférences) at Université Paris Cité (formerly known as Université Paris Diderot)

When and where ?

- 6 lectures
- Tuesdays starting 9th January from 9am to 12am.

Prerequisites ?

- Solid knowledge in probability (convergence of r.v., conditional expectation, martingales, ...)
- Solid knowledge in stochastic calculus (Brownian motion, Itô's calculus, Girsanov's theorem, SDEs, ...) and basic concepts of quantitative finance.

Assessment ?

- Exam (5th March from 2 :30pm to 4 :30pm)
- Homeworks ($2 \sim 3$)

References and outline at the end of the introduction

Calibration in Quantitative Finance

Question : Calibration of what ?

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Answer : Models

The modeling puzzle

Question : What is a model ?

Answer : A model is an umbrella term that refers to a simplified description of a complex reality embodying different concepts such as economic models, scale models, scientific models.

For scientists, a model is more precisely a mathematical representation of the world used to

- Understand
- Describe
- Predict

Yet, it still has to verify certain properties to be recognized as a successful model before being adopted in practice.

$\rightsquigarrow$ Two such key properties are **consistency** and **tractability**.

Consistency : A model is said to be consistent if it captures, up to some extent, the stylized facts of the observations. For example, the stylized features of electricity/gaz prices :

- Seasonality
- Stationarity and mean reversion
- Multiscale autocorrelation structure
- Heavy tails & Spikes : fast upward movements followed by quick return to initial level.

The more complex a model is, e.g. with a large number of parameters, the more detailed the description of the reality, the more simplistic, the more it drifts away from real life.

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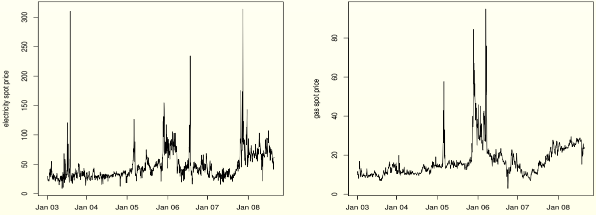

FIGURE – Electricity/Gas spot prices on the Powernext market and at the NBP for the period 14th January 2003 till 20th August 2008.

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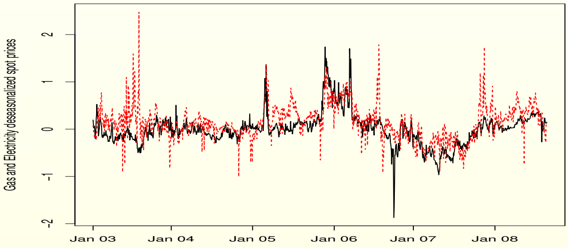

FIGURE – The log-deseasonalized gas (normal line) and electricity spot

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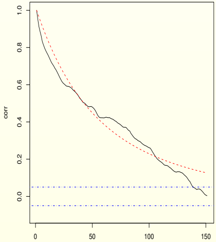
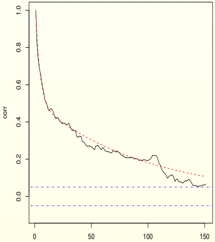
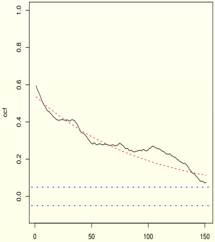


(a) ACF of Y^g



(b) ACF of Y^e



(c) CCF of (Y^g, Y^e)

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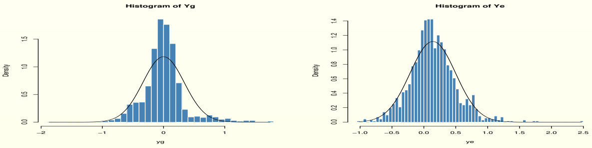

FIGURE – Histograms of Y^g and Y^e with normal density curves.

The two residuals time series Y^g and Y^e are far from being normally distributed. Excess-kurtosis of Y^g and Y^e are equal to 4.5 and 2.3 meaning that the two distributions are peaked and have heavy tails. The skewness of Y^g and Y^e are respectively equal to 0.77 and 0.57 meaning that the two distributions are not symmetric.

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In a utopian world, ideal models would provide a perfect balance between **consistency** and **tractability** :

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Finding such models is without any doubt a myth.

The modeling puzzle

How much **consistency** is a scientist willing to sacrifice for **tractability** ?

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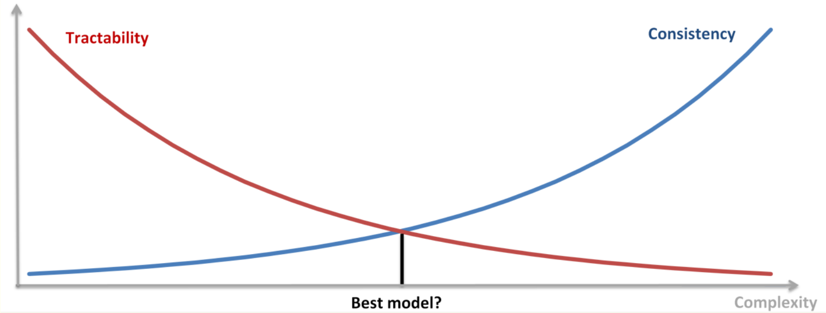
To answer this question, a picture is worth a thousand words.



“Everything should be made as simple as possible, but not simpler” - Albert Einstein

Building a good model

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Focus on one particular class of assets : **equities**

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Focus on one particular class of assets : **equities**

- Flexibility : replicate stylized facts
- Tractability : fast computations of options prices/Greeks

Daily price evolution of S&P500 index

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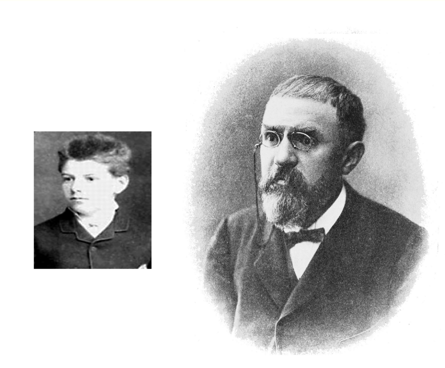
The price is always positive and random. Several justifications of this last assumption. See e.g. De Meyer, B., & Saley, H. M. (2003). *On the strategic origin of Brownian motion in finance*.

Bachelier model

First stochastic model for the stock price has been introduced in 1903 by Louis Bachelier in his PhD thesis under the supervision of the celebrated mathematician Henri Poincaré.



- Bachelier is considered as the forefather of mathematical finance and a pioneer in the study of stochastic processes.
- PhD not well received because it attempted to apply mathematics to an unfamiliar area for mathematicians
- Although Bachelier's work on random walks predated Einstein's celebrated study of Brownian motion by five years, the pioneering nature of his work was recognized only after several decades, first by Andrey Kolmogorov

Bachelier model

In the Bachelier model, the stock price is stochastic and follows a Brownian motion

$$S_t = S_0 + \sigma W_t, \quad 0 \leqslant t \leqslant T,$$

where $S_0 > 0$ is the initial price of the asset, T is the investment horizon, and $\sigma > 0$.

- ◦ The price process is a martingale (no arbitrage/fair game in Bachelier terms)
- ◦ S can become negative but T is assumed to be short, so that the probability of the event $\{S_T < 0\}$ is very small.
- ◦ For the same reason, short rate $r = 0$.

Bachelier model

$$S_t = S_0 + \sigma W_t, \quad 0 \leqslant t \leqslant T,$$

Bachelier was interested in computing European call option prices with payoff $(S_T - K)^+$

$$C_t = \mathbb{E}\left[(S_T - K)^+ | \mathcal{F}_t\right]$$

Using properties of the Gaussian distribution, one can show that

$$C_t = (S_t - K)\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\varphi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right)$$

where

$$\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy.$$

Exercise : Prove the above formula.

Black-Scholes model

No interest in the community in the work of Bachelier, although the Russian mathematician Kolmogorov has cited Bachelier... Growing interest starting 1960s, Fisher Black and Myron Scholes model

Scholes received the 1997 Nobel Memorial Prize in Economic Sciences, Black wasn't awarded because he died in 1995.

Back to calibration

Once we have built a good/flexible parametric model and computed call prices, we calibrate the model to the market : i.e. we find the parameters of the model such that we can replicate at best the (liquid) European call prices for a range of maturities $(T_i)_{1\leqslant i\leqslant N}$ and strikes $(K_j)_{1\leqslant j\leqslant M}$.

Example : using optimization problem

$$\arg\min_{\theta\in\Theta}\sum_{i=1}^{N}\sum_{j=1}^{M} w_{i,j}(\text{Call}^{\text{model}}(T_i,K_j,\theta)-\text{Call}^{\text{Market}}(T_i,K_j))^2+\text{regularization}$$

Once the model is calibrated, we postulate that the model provides good prices for *illiquid/exotic options* ...

We will say that the model is tractable if pricing/calibration can be done in a reasonable computational time.

As a summary

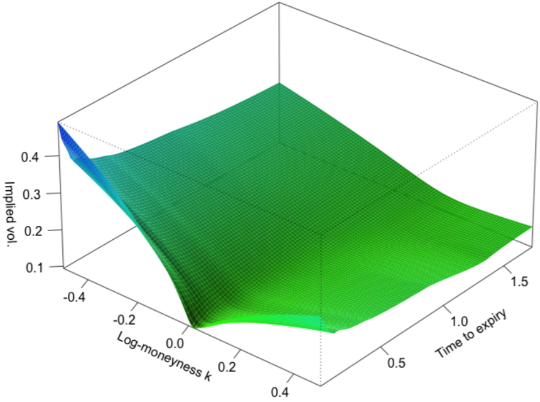
This course will cover the **mathematics** regarding stock/volatility models and their calibration.

Outline (preliminary)

1. Black-Scholes model and implied volatility
2. Local volatility models
3. Stochastic volatility models
4. Rough volatility models (if time permits)

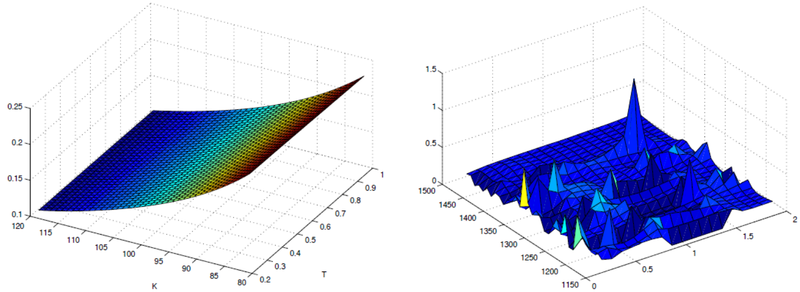
References

- Gatheral, J. (2006) The volatility surface, a practicioner's guide.
- Tankov, P. (2015). Surface de volatilité, `www.proba.jussieu.fr/pageperso/tankov/`



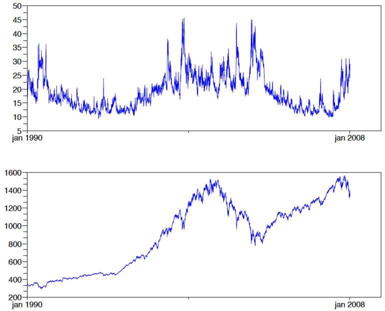
The SPX volatility surface as of August 14, 2013.

(taken from Gatheral (2006))



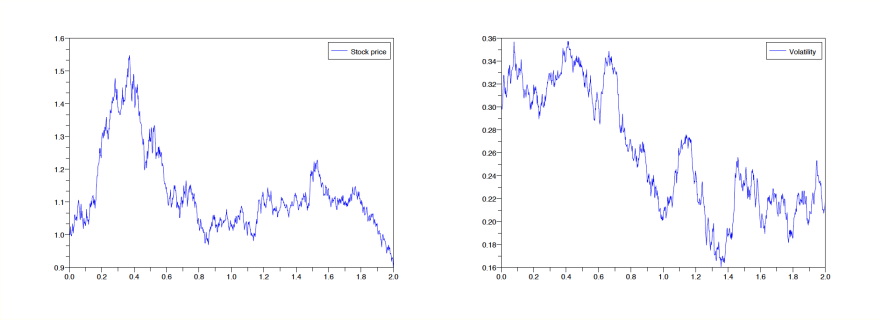
Implicit diffusion $(K,T) \to \sigma(K,T)$: generated data from loc vol model (left), real data on S&P 500. (taken from Tankov (2005))

Appendix
VIX and S&P500 index



Real data on VIX index (top) and S&P 500 index. (taken from Tankov (2005))

The VIX Index is a calculation designed to produce a measure of constant, 30-day expected volatility of the U.S. stock market, derived from real-time, mid-quote prices of S&P 500 Index call and put options. On a global basis, it is one of the most recognized measures of volatility – closely followed by a variety of market participants as a daily market indicator.



One trajectory of stock prices on the left and volatility process on the right. (taken from Tankov (2005))