Université Paris 1 - Panthéon Sorbonne
MMMEF, 2023-2024

Homework 1
Calibration in quantitative finance

Due date: January 31th before 10 am.

Instructions:

- One answer sheet per group preferably typed in Latex.
- Please clearly indicate the **composition of your group** in your email.
- You must rigorously justify your answers. The number of points for each question is indicated for information purposes only.

1 Properties of the running local time

In this exercise, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions and supporting a one-dimensional $\mathbb{F}$ Brownian motion W. We consider a one-dimensional Itô's process Z with dynamics

$$Z_t = Z_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s, \quad 0 \leqslant t, \tag{1.1}$$

where $(\mu_s)_{s\geqslant 0}, (\sigma_s)_{s\geqslant 0}$ are $\mathbb{F}$-adapted and continuous processes. For a fix $x \in \mathbb{R}$, we define the running local time $(L_t^x(Z))_{t\geqslant 0}$ of the process Z at x by

$$L_t^x(Z) := 2\left\{(Z_t - x)_+ - (Z_0 - x)_+ - \int_0^t \mathbf{1}_{\{Z_s > x\}}\, dZ_s\right\}.$$

1. (1pt) Justify that the process $(L_t^x(Z))_{t\geqslant 0}$ defined above has continuous sample paths and satisfies $L_0^x(Z) = 0$.

   We will assume that $(t, x) \mapsto L_t^x(Z)$ has a continuous modification which is jointly continuous in x and t. A proof of this result is outside the scope of this homework. From now on, we will work with this modification.

2. (1pt) The aim of this question is to prove that for all continuous function f with compact support, it holds

$$\int_{\mathbb{R}} f(x) L_t^x(Z)\, dx = \int_0^t f(Z_s)\, \sigma_s^2\, ds. \tag{1.2}$$

   We define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(y) = \int_{-\infty}^y \int_{-\infty}^z f(u)\, du\, dz.$$

   (a) Prove that

$$F(y) = \int_{\mathbb{R}} (y - z)_+ f(z)\, dz.$$

   (b) Prove that F is twice continuously differentiable. Provide explicit expressions of F' and F'' in terms of f.

3. (1pt) Using the previous question, prove the identity (1.2) and extend it to any measurable function with compact support.

4. (1pt) Prove that for any $x \in \mathbb{R}$, it holds

$$L_t^x(Z) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{x-\varepsilon < Z_s < x+\varepsilon\}} \sigma_s^2\, ds.$$

   Hint: Use the identity (1.2) *together with the continuity of* $x \mapsto L_t^x(Z)$.

5. (1pt) Deduce from the previous question that $t \mapsto L_t^x(Z)$ is non-decreasing *a.s.*

6. (1pt) Assume that $\sigma_t = \sigma(t, Z_t)$ for some deterministic function σ defined on $\mathbb{R}_+ \times \mathbb{R}$ and that for any $t > 0$ the random variable Z_t admits a density $z \mapsto p(t, z)$. Show that for any continuous function f with compact support and for any $x \in \mathbb{R}$

$$\mathbb{E}\Big[\int_0^t f(s)\, dL_s^x\Big] = \int_0^t f(s)\, \sigma^2(s, x)\, p(s, x)\, ds.$$

7. (1pt) Extend the previous identity to any non-negative continuous function f.

2 Application

In this part, we assume that the following local volatility model with dynamics

$$S_t = S_0 + \int_0^t rS_s\,ds + \int_0^t S_s\sigma(s, S_s)\,dW_s$$

admits a unique strong solution.

1. (1pt) Can you provide a simple assumption under which the random variable S_t admit a density function $z \mapsto p(t, z)$ for all $t > 0$?

2. (2pts) For a fixed strike K, interest rate r and maturity $T > 0$, using the results established in the first section, prove that

$$e^{-rT}(S_T - K)_+ = (S_0 - K)_+ - r\int_0^T e^{-rs}(S_s - K)_+\,ds + \int_0^T e^{-rs}\mathbf{1}_{\{S_s > K\}}\,dS_s + \frac{1}{2}\int_0^T e^{-rs}\,dL_s^K.$$

3. (2pts) Deduce from the previous identity that

$$e^{-rT}\mathbb{E}[(S_T - K)_+] = (S_0 - K)_+ + rK\int_0^T e^{-rs}\mathbb{P}(S_s > K)\,ds + \int_0^T e^{-rs}p(s, K)\,K^2\sigma^2(s, K)\,ds.$$

4. (1pt) What can be deduced from the previous identity?