Université Paris 1 - Panthéon Sorbonne
MMMEF

Homework 2
Calibration in quantitative finance

Due date: February 14th before 10 am.

Instructions:

- One answer sheet per group preferably typed in Latex.
- Please clearly indicate the **composition of your group**.
- You must rigorously justify your answers.

1 On existence and uniqueness of SDEs

Consider the Stochastic Differential Equation (SDE for short) with dynamics

$$Z_t = Z_0 + \int_0^t b(Z_s)\, ds + \int_0^t \sigma(Z_s)\, dB_s, \quad 0 \leqslant t, \tag{1.1}$$

for some coefficients b, $\sigma : \mathbb{R} \to \mathbb{R}$, $Z_0 \in \mathbb{R}$ and where B is a standard one-dimensional Brownian motion. We recall the following notions of weak and strong solutions to (1.1).

- We say that (1.1) admits a weak solution denoted by the couple (Z, B), if there exists a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ supporting an $\mathbb{F}$-adapted Brownian motion B and a $\mathbb{F}$-adapted process Z such that (1.1) is satisfied $\mathbb{P}$-a.s.
- We say that (1.1) admits a strong solution, if for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a Brownian motion B there exists a $\mathbb{F}$-adapted process Z such that (1.1) holds $\mathbb{P}$-a.s. where $\mathbb{F}$ is the filtration generated by B.
- We say that strong uniqueness holds for (1.1) if pathwise uniqueness is satisfied for (1.1), that is, $Z^1_t = Z^2_t$ for all $t \geqslant 0$ almost surely, for any two solutions (Z^1, B) and (Z^2, B) defined on the same probability space equipped with the same Brownian motion B such that $Z^1_0 = Z^2_0$.

Let us recall the following theoretical result:

weak existence $+$ strong uniqueness $\Rightarrow$ strong existence and uniqueness.

For a proof, one may refer to the textbook *Brownian motion and stochastic calculus* by Ioannis Karatzas and Steven E. Shreve.

1. (a) (1pt) What is the main difference between the notions of strong and weak solutions of SDE?
   (b) (1pt) Is any strong solution a weak one?
2. (1pt) Give an example of conditions on the coefficients b and σ that ensure the strong existence and uniqueness of a solution. Theorem 3.1 in the appendix provides sufficient conditions for uniqueness. You could use Theorem 3.1 without justification throughout the exercise.
3. In this question, we set $\sigma(x) = \text{sign}(x)$ where $\text{sign}(x) = 1$ if $x \geqslant 0$ and -1 otherwise. We consider the following SDE

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\, dB_s \tag{1.2}$$

   where B is a one-dimensional Brownian motion and $Z_0 \in \mathbb{R}$.

   (a) (2pts) Prove that the SDE (1.2) admits a weak solution.
   (b) (1pt) Prove that pathwise uniqueness is not satisfied for the SDE (1.2).
4. (1pt) For which values of $\alpha \in [0, 1]$, the SDE with coefficients $b = 0$ and $\sigma(x) = \sigma x^\alpha$, $\sigma > 0$, satisfies pathwise uniqueness. *Hint: use Theorem 3.1.*
5. (2pts) **Bonus question:** Provide a sketch of proof for Theorem 3.1 of the appendix.

2 On the CIR process

In the following section, we study the SDE

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dB_t \tag{2.1}$$

where $V_0 = v \in \mathbb{R}_+$, $\kappa, \sigma > 0$. Throughout the exercise, it is assumed that there exists an integer n such that

$$\theta = n\frac{\sigma^2}{4\kappa}.$$

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a n-dimensional Brownian motion $W = (W^1, \cdots, W^n)$, we consider the following SDEs with dynamics

$$dU_t^i = -aU_t^i\,dt + \ \eta\,dW_t^i,\ U_0^i \in \mathbb{R},$$

where $a, \eta > 0$.

1. (a) (1pt) Justify that the above SDEs admits a unique strong solution.
   (b) (1pt) Solve the SDE satisfied by U^i, that is, derive their closed form expression. What is the name usually associated to the process U^i?
   (c) (1pt) Show that the law of U_t^i is the same as the one of $e^{-at}(U_0^i + W^i_{\frac{\eta^2}{2a}(e^{2at}-1)})$ for $i = 1, \cdots, n$.
2. (2pts) We now consider a process $(Z_t)_{t\geqslant 0}$ defined by

$$Z_t := \|U_t\|^2 := \sum_{i=1}^n (U_t^i)^2.$$

   Show that

$$Z_t = Z_0 + \int_0^t \kappa(\theta - Z_s)\,ds + \sigma\int_0^t \sum_{i=1}^n U_s^i\,dW_s^i$$

   for well-chosen parameters κ, θ, σ and Z_0.
3. (2pts) Show that the process B defined by

$$B_t = \int_0^t \sum_{i=1}^n \frac{U_s^i}{\sqrt{Z_s}}\,dW_s^i,$$

   is a one-dimensional $(\mathbb{F}, \mathbb{P})$-Brownian motion where $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ is a filtration to be specified.
4. (1pt) Deduce that the process Z constructed above is the unique non-negative strong solution to (2.1).
5. (2pts) Compute for all $u \leqslant 0$, all $0 \leqslant t \leqslant T$ and all $i = 1, \cdots, n$, the quantities

$$\mathbb{E}[\exp(u(W_T^i)^2)|\mathcal{F}_t].$$

6. (2pts) Deduce that the conditional Laplace transform of V writes for any $u \geqslant 0$ and $0 \leqslant t \leqslant T$

$$\begin{aligned}\mathbb{E}[\exp(uV_T)|\mathcal{F}_t] = &\Big(1 - \frac{\sigma^2}{2\kappa}(1 - \exp(-\kappa(T-t)))u\Big)^{-\frac{2\kappa\theta}{\sigma^2}} \\ &\times \exp\Big(\frac{u\exp(-\kappa(T-t))}{1 - \frac{\sigma^2}{2\kappa}(1-\exp(-\kappa(T-t)))u} V_t\Big).\end{aligned} \tag{2.2}$$

7. (1pt) Comment the results proved in this exercise.

8. (2pts) Can you sketch another way for obtaining (2.2) for more general coefficients θ, κ, σ?

3 Appendix

Theorem 3.1. *Assume that $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz function and that $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies*

$$|\sigma(x) - \sigma(y)|^2 \leqslant \phi(|x-y|),$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies for any $\varepsilon > 0$

$$\int_0^\varepsilon \frac{1}{\phi(z)}\, dz = \infty.$$

Then, pathwise uniqueness holds for the following SDE with dynamics

$$X_0 = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s.$$