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Probabilistic Methods in Finance - Tutorial

Exercise 1. Currency risk: We consider a US company due to receive M euros at a known future time T (because it exports in Europe). It costs are in dollars and it will have to change euros against dollars.

1. Explain what is the risk for the company. The value of 1 euro at a time s will be denoted by S_s.

2. The company decides to hedge its currency risk by entering a forward contract today, time $t < T$. Explain the derivative that it will use. What is its total payoff at T?

3. Explain how the company could hedge its risk using options. What would be its total payoff at T?

Exercise 2.

1. We consider an investment paying a monthly interest, with compounded interests.
The interest rate for 1 year is $r_d = 10\%$.
Compute the corresponding monthly rate (annualised and not annualised).

2. Prove that $\forall r_d > 0$, $\forall m \in \mathbb{N}$ with $m \geq 2$, $\left(1 + \dfrac{r_d}{m}\right)^m > 1 + r_d$.

3. Prove that if r_m is given by $\left(1 + \dfrac{r_m}{m}\right)^m = 1 + r_d$, when $m \to +\infty$, r_m converges toward $\ln(1 + r_d)$.

Exercise 3. Model of **Population dynamics**: we assume that the growth rate of a population at t (change in number of individuals in a population over time) is given by: $\frac{dN(t)}{dt} = rN(t)$ with $N(t)$ population size at time t, r *per capita* rate of increase or continuous compound rate of growth.

1. Compute the function N. This model is used for example for bacteria growth (bacteria reproduce by binary fission, i.e. splitting in half, the time between divisions is about an hour for many bacterial species).

2. A similar model is used for decay of radioactive substances (in that case r is the continuous compound rate of decay). How long will it take a certain amount of radium to decay to half the original amount (half-life of the substance)?

Exercise 4.

1. Prove that under the NAO assumption, if two portfolios have same value at a future time T, then they must have same value at any earlier time, t.

We consider European call and put with a same strike price K and same maturity T on a stock paying no dividend. Let C_t be the call price at a time $t < T$ and P_t the put price.

2. At a time $t < T$, we consider a portfolio made of 1 call and K 0-coupon bond with maturity T. What is the content of this portfolio at time T? Same question for a portfolio made of 1 put and 1 stock.

3. Combine the 2 options with the stock in order to have a portfolio that is risk-free between t and T.

4. From **2.** and then **3.**, deduce a relationship between their prices at t (for example with r the continuous risk-free rate between t and T).

5. Is this relationship still observed if short sales are not allowed (but selling options still allowed)?

Exercise 5. Different formulations of the no-arbitrage principle.
We make the usual assumptions on the market, on an interval $[0, T]$.

def (A01): an arbitrage opportunity is a portfolio satisfying:
$V_0 = 0$, $V_T \geq 0$ P-as ie $P(V_T \geq 0) = 1$, and $P(V_T > 0) > 0$.

def (A02): an arbitrage opportunity is a portfolio satisfying: $V_0 \leq 0$, $V_T \geq 0$ P-as, and $P(V_T > 0) > 0$.

def (A03): an arbitrage opportunity is a portfolio satisfying:
$V_0 \leq 0$, $V_T \geq 0$ P-as, and $V_0 \neq 0$ or $P(V_T \neq 0) > 0$.

1. Prove that an A01 is an A02 and an A02 is an A03.

2. The no-arbitrage principle states that there is no arbitrage opportunity. Prove that it is equivalent to assume it with any of the three above definitions.

3. Prove that assuming that there exists a portfolio satisfying $V_0 > 0$ and $V_T > 0$ P-as, instead of a risk-free asset, gives the same result.

Exercise 6.

1. What is the forward price for maturity T of the 0-coupon maturing at time $T' > T$? (proof needed)

2. A forward interest rate is an interest rate which is specified now for a loan that will occur at a specified future date.
Compute the forward interest rate contracted at date t, for a loan done at date T and to be reimbursed at date T' (with $t < T < T'$).
You will consider 2 cases: continuous and discrete interest rates.

Exercise 7. Currency forward and carry trade

1. Compute the forward exchange rate $F(t, T)$ in exercise 1 (with proof).

2. An investor wants to take advantage of a high interest rate in a foreign currency, by investing an amount M of his currency for 1 year, but he decides to enter a forward contract to completely hedge the implied currency risk.

a. Explain the positions he will take, in particular the details of the forward contract.

The interest rate for 1 year in the domestic currency is denoted by r and the same rate in the foreign currency by r_f.

b. What is his final rate of return? Give an interpretation of this result.

Exercise 8. Under the usual assumptions on the markets, we consider a commodity having a continuous storage cost with an annualised rate α.

1. Give a concrete example where this modeling would be chosen.

2. Determine the future price of this commodity at a time $t < T$. The continuous interest rate will be denoted by r.

Exercise 9. Option prices properties.
To prove the below properties, you will build an arbitrage opportunity if they do not hold.

1. For European or American options, the call option price is a decreasing function of the strike price.

2. Usual notations. C_0 (respt C_0^A) price at 0 of the European (respt American) call with strike K and maturity T. Prove that: $S_0 - \frac{K}{(1+r)^T} \leq C_0 \leq C_0^A \leq S_0$

$$\frac{K}{(1+r)^T} - S_0 \leq P_0 \leq \frac{K}{(1+r)^T}.$$

3. When the underlying asset pays no dividend, we have $C_t^A = C_t$.

   Is there an equivalent result for the put option?

4. Analogue of the call-put parity relationship for the American options. Prove that:
   $S_0 - K \leq C_0^A - P_0^A \leq S_0 - \frac{K}{(1+r)^T}$.

Exercise 10. We consider the financial market in discrete-time, with two dates $t = 0$ and $t = 1$, same than in the course (the risk-free asset is worth 1 at $t = 0$ and $1 + r$ at $t = 1$ for any state of the world, the risky asset is an equity paying no dividend; this equity is worth S at $t = 0$ and S^u (with probability p) or S^d at $t = 1$, with $S^d < S(1 + r) < S^u$.
We consider a European call option on the equity, with strike price K such that $S^d < K < S^u$. and maturity $t = 1$.

1. We define the volatility of an asset as the standard deviation of its return between $t = 0$ and $t = 1$.

   a. Compute σ, the volatility of the risky asset.

   b. Prove that $\sigma_C \geq \sigma$, where σ_C is the volatility of the call option.

2. **a.** By building a risk-free portfolio constituted of one short call option and some risky asset, compute the call option price at $t = 0$.

   b. We assume $r = 10\%$, $S = 100$, $S^d = 90$, $S^u = 130$ and $K = 108$. Build an Arbitrage Opportunity if the option quote at $t = 0$ is 9.

   Is there any Arbitrage Opportunity when the assets are not supposed to be divisible: the numbers of assets involved in the transactions can only be integers.

3. The risk premium of an asset (or Sharpe ratio) is computed as its expected excess return (above the risk-free rate), divided by its volatility. It depends of the market's risk aversion, i.e. the average risk aversion of all market participants.

   a. Let μ be the expected return of the equity, and $\Pi = \frac{\mu - r}{\sigma}$ its risk premium.

   Prove that $p - p^* = \Pi\sqrt{p(1-p)}$, where p^* is the usual notation: $p^* = P^*(\{\omega_1\})$, P^* being the risk-neutral probability.

   b. Prove that any option (not necessarily a call or a put) written on the equity has a risk premium equal to $\pm\Pi$.

Exercise 11. We consider a financial asset on 3 dates. For $t = 0, 1$ or 2, the asset price at time t is denoted by S_t. The dynamics of (S_t) is described by the following tree:

$$\begin{tikzcd}
 & & s_2^4 \\
 & s_1^2 \arrow[ur] \arrow[dr] & \\
 & & s_2^3 \\
s_0 \arrow[uur] \arrow[ddr] & & \\
 & & s_2^2 \\
 & s_1^1 \arrow[ur] \arrow[dr] & \\
 & & s_2^1
\end{tikzcd}$$

The randomness is described by a probability space $(\Omega, \mathcal{F}, P)$, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ where $\omega_i = S_2^{-1}(s_2^i)$, i.e. we consider no other randomness than the one associated to the asset price moves.
S_1 and S_2 are two random variables on $(\Omega, \mathcal{F}, P)$.

1. The natural filtration of $(S_t)_{0 \le t \le 2}$ is denoted by $(\mathcal{F}_t)_{0 \le t \le 2}$. Describe $\mathcal{F}_1$ and $\mathcal{F}_2$.

2. We consider a random variable Y on $(\Omega, \mathcal{F}, P)$.
Give a necessary and sufficient condition for Y to be $\mathcal{F}_1$-measurable.

3. Compute $\mathbb{E}(X|\mathcal{F}_1)$ for X $\mathcal{F}_2$-measurable. Generalisation to a complete set of n events.

Exercise 12. Let X and Y be independent Bernoulli random variables with parameters p and q in $]0,1[$ (reminder: $X = 1$ with probability p and 0 with probability $1-p$).
Let $Z = \mathbb{1}_{\{X+Y \neq 0\}}$ and $\mathcal{B} = \sigma(Z)$.

1. Compute $E(X|\mathcal{B})$ and $E(Y|\mathcal{B})$

   (you will consider the following partition of Ω: $\Omega = \{\omega | Z(\omega) = 0\} \cup \{\omega | Z(\omega) = 1\}$).

2. Are these two random variables independent?

Exercise 13. $(\Omega, \mathcal{F}, P)$ probability space

1. $(X_n)_{n \in \mathbb{N}^*}$ sequence of independent r.v. in $L^2(\Omega, \mathcal{F}, P)$ such that $\forall n \in \mathbb{N}^*$, $\mathbb{E}(X_n) = 0$ (fair play).
For $n \in \mathbb{N}^*$, let $\mathcal{F}_n = \sigma\{X_1, ..., X_n\}$ (σ-field generated by $\{X_1, ..., X_n\}$),
and $M_n = X_1 + ... + X_n$ (gain after n trials).
Prove that $(M_n)_{n \in \mathbb{N}^*}$ is an $(\mathcal{F}_n)$-martingale.

2. Consider a binomial experiment with a probability of success $p \in]0,1[$. Let N_n denote the number of successes after n independent trials. Find a constant α such that $(N_n - n\alpha)_{n \ge 1}$ is a martingale.

Exercise 14. $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n \in \mathbb{N}})$ is a filtered probability space.
Let $(M_n)_{n \in \mathbb{N}}$ an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale s.t. $\forall n \in \mathbb{N}$, $E(M_n^2) < +\infty$.

1. Prove that $\forall n \le m$, $E(M_m^2 - M_n^2 | \mathcal{F}_n) = E((M_m - M_n)^2 | \mathcal{F}_n)$.

2. Deduce that: $\forall n \in \mathbb{N}$, $E(M_{n+1}^2 | \mathcal{F}_n) \ge M_n^2$. Such a process $(M_n^2)_{n \in \mathbb{N}}$ is called a sub-martingale.

3. Generalisation: prove that if $\varphi : \mathbb{R} \to \mathbb{R}$ is convex with $\forall n \in \mathbb{N}$, $\varphi(M_n) \in L^1$, then $(\varphi(M_n))_{n \in \mathbb{N}}$ is a sub-martingale.

Exercise 15. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$.
Let $(M_n)_{n \in \mathbb{N}}$ be an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale and $(\Delta_n)_{n \in \mathbb{N}}$ an adapted stochastic process that is bounded, ie there exists $c \in \mathbb{R}$ such that: $\forall n \in \mathbb{N}$, $|\Delta_n| \le c$ P-a.s..
We consider a process $(X_n)_{n \in \mathbb{N}}$ defined by $X_n = \sum_{k=1}^{n} \Delta_{k-1}(M_k - M_{k-1})$.
Prove that $(X_n)_{n \in \mathbb{N}}$ is an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale (called the martingale transform of (M_n) by (Δ_n)).

Exercise 16. $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}})$ filtered probability space.
Def: An $(\mathcal{F}_n)$-stopping time is a random variable $\tau : (\Omega, \mathcal{F}, P) \to \mathbb{N} \cup \{\infty\}$ st $\forall n \in \mathbb{N}$, $\{\tau = n\} \in \mathcal{F}_n$.

We consider a bounded stopping time $\tau : (\Omega, \mathcal{F}, P) \to \{1, ..., T\}$ with $T \in \mathbb{N}^*$.

1. Prove that for any $n \in \{1, ..., T\}$, $\{\tau > n\} \in \mathcal{F}_n$.

2. Let $(M_n)_{n\in\mathbb{N}^*}$ an $(\mathcal{F}_n)$-martingale.
M_τ denotes the random variable defined by $(M_\tau)(\omega) = M_{\tau(\omega)}(\omega)$ (martingale stopped at time τ).
Writing M_T as $\sum_{k=1}^T M_T \mathbb{1}_{\{\tau=k\}}$, prove that $\mathbb{E}(M_T) = \mathbb{E}(M_\tau)$.

3. The game of exercise 12.1. is played T times.

   Prove that if the player decides to play until the time where his total gain is above a given $G > 0$, then its expected gain is 0 (you will consider $\tau = \inf\{n \in \{1, ..., T\} | M_n \geq G\}$, where $\inf \emptyset = T$).

Exercise 17. We consider a classical binomial model on a time interval $[0, T]$ divided in N periods. Let $\Delta t = \frac{T}{N}$.
The risky asset is an equity. At each period its price is multiplied by d or u.
The risk-free asset is worth $e^{rn\Delta t}$ after n periods, with $d < e^{r\Delta t} < u$.

At time 0, a bank sells a European call on the equity, with strike price K and maturity date T.
We assume $Su^3d^{N-3} < K < Su^{N-3}d^3$.

The bank hedges its risk by selling/buying some equity at each date in order to keep a risk-free portfolio from time 0 until time T. Describe and interpret the number of equities in the bank portfolio at dates $T - \Delta t$ and $T - 2\Delta t$, for the highest and lowest values of the equity.

Exercise 18. We consider a classical binomial model (N periods).

1. Deduce the put-call parity relationship from the option pricing formula in this model.

2. Deduce a relationship at time 0 between the delta of a European put and the delta of the call with same maturity date and same strike price.

Exercise 19. Change of probability $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}})$ is a filtered probability space.
Let Z a random variable s.t. $Z > 0$ Pa.s. and $E(Z) = 1$. We denote by P^* the probability with density Z with respect to P. In particular $P^*(A) = E(Z\mathbb{1}_A)$, for all $A \in \mathcal{F}$.

1. For any $n \in \mathbb{N}^*$, let $Z_n = E(Z|\mathcal{F}_n)$. Prove that $(Z_n)_{n\in\mathbb{N}}$ is an $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale under the probability P.

2. Prove that for $k \in \mathbb{N}$ and $Y \in L^1(\Omega, \mathcal{F}, P^*)$, $\mathcal{F}_k$-measurable, $\mathbb{E}^*(Y) = E(Z_k Y)$.

3. Prove that for $n \in \mathbb{N}^*$ and $X \in L^1(\Omega, \mathcal{F}, P^*)$, $\mathcal{F}_n$-measurable, $\mathbb{E}^*(X|\mathcal{F}_{n-1}) = \dfrac{1}{Z_{n-1}}\mathbb{E}(XZ_n|\mathcal{F}_{n-1})$.

4. Let (M_n) a $(\mathcal{F}_n)$-martingale under P. To check if (M_n) can be a $(\mathcal{F}_n)$-martingale under P^*, compute $\mathbb{E}^*(M_n - M_{n-1}|\mathcal{F}_{n-1})$ for $n \geq 1$.

   Deduce that $\left(M_n - \displaystyle\sum_{k=1}^n \frac{1}{Z_{k-1}}\mathbb{E}((M_k - M_{k-1})Z_k|\mathcal{F}_{k-1})\right)_{n\in\mathbb{N}^*}$ is a $(\mathcal{F}_n)$-martingale under P^*.

Exercise 20. We consider a family of random variables on $(\Omega, \mathcal{F}, P)$, $(B_t)_{t\geq 0}$, with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (ie $\mathbb{R}$ equipped with the Borel σ-field), such that $B_0 = 0$ and:

$$\forall 0 \leq s \leq t, \begin{cases} B_t - B_s \sim \mathcal{N}(0, t-s) \\ B_t - B_s \text{ is independent of } \mathcal{F}_s = \sigma(B_u, u \leq s) \text{ (the past)} \end{cases}$$

1. Prove that $\forall t$, $B_t \sim \mathcal{N}(0,t)$ and $\forall s,t \geq 0$, $E(B_s B_t) = min(s,t)$.

2. Let $\lambda \in \mathbb{R}$. Prove that $(B_t)_{t\geq 0}$, $(B_t^2 - t)_{t\geq 0}$, and $(e^{\lambda B_t - \frac{\lambda^2}{2}t})_{t\geq 0}$ are $(\mathcal{F}_t)$-martingales.

3. Let $t_0 = 0 < t_1 < ... < t_n = T$ a subdivision of $[0,T]$ and $\delta = \max_{0\leq k \leq n-1} |t_{k+1} - t_k|$ (subdivision norm).

a. Let $X_k = (B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)$ for $0 \leq k \leq n-1$.

Prove that $\dfrac{\mathbb{E}\left[(X_k)^2\right]}{(t_{k+1} - t_k)^2}$ is the same for any k.

b. Deduce that $\displaystyle\sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2 \xrightarrow{L^2} T$ when $\delta \to 0$.

Exercise 21. Let $(B_t)_{t\geq 0}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.
$(\mathcal{F}_t)_{t\geq 0}$ its natural filtration (ie, for $t \geq 0$, $\mathcal{F}_t = \sigma(B_s, s \leq t)$).

For a given $\lambda \in \mathbb{R}$, let $L_t = e^{-\lambda B_t - \frac{\lambda^2}{2}t}$ for $t \geq 0$.

1. Compute $\mathbb{E}(L_t)$ for $t \geq 0$.
For $T > 0$, let P^* be the probability with density L_T with respect to P.

2. Prove that, for $t \leq T$ and $Z \in L^1(\Omega, \mathcal{F}, P)$ $\mathcal{F}_t$-mesurable, we have $\mathbb{E}^*(Z) = \mathbb{E}(Z L_t)$.

3. Let $W_t = B_t + \lambda t$ for $t \geq 0$.

a. Prove that, for $s \leq t$, the characteristic (or generating) function of $W_t - W_s$ under P^* is the characteristic function of the law $\mathcal{N}(0, t-s)$.

b. Prove that $(W_t)_{t\in[0,T]}$ is a Brownian motion under P^*.
You will use the following lemma:

> If a r.v. X satisfies: $\forall u \in \mathbb{R}$, $\mathbb{E}(e^{iuX}|\mathcal{B}) = \mathbb{E}(e^{iuX})$ Pa.s., then X is independent of $\mathcal{B}$.

4. For $T, \sigma > 0$ and $\mu \in \mathbb{R}$, let $S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T}$. Compute $P^*(S_T \geq K)$ for $\lambda = \frac{\mu - r}{\sigma}$.
The cumulative probability distribution function for a standardized normal variable will be denoted by N:

$$\text{for } d \in \mathbb{R}, \quad N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx.$$

For **exercises 22., 23., & 24.**, let $(B_t)_{t\geq 0}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration.

The Ito integral (on $[0,T]$) has been defined for a simple process $(H_t)_{0\leq t\leq T}$ as follows:

let $(H_t)_{0\leq t \leq T}$ such that, with $t_0 = 0 < t_1 < ... < t_n = T$, $\quad H_t(\omega) = \displaystyle\sum_{k=0}^{n-1} H^k(\omega) \mathbb{1}_{[t_k, t_{k+1}[}(t)$

where for $0 \leq k \leq n-1$, $H^k \in L^2(\Omega, \mathcal{F}, P)$ and is $\mathcal{F}_{t_k}$-measurable.

Then $\int_0^T H_t dB_t$ is defined as $\sum_{k=0}^{n-1} H^k (B_{t_{k+1}} - B_{t_k})$.

Exercise 22.

1. Let $t_0 = 0 < t_1 < ... < t_n = T$ and for $t \le T$, $f(t) = \sum_{k=0}^{n-1} a_k \, 1\!\!1_{[t_k, t_{k+1}[}(t)$ with $a_0, ..., a_{n-1} \in \mathbb{R}$.

   What is the law of $\int_0^T f(t) dB_t$?

2. For $n \in \mathbb{N}^*$, we set $t_k = \frac{kT}{n}$ for $0 \le k \le n$, and $B_t^n = \sum_{k=0}^{n-1} B_{t_k} \, 1\!\!1_{[t_k, t_{k+1}[}(t)$.

   By approximating $(B_t)_{0 \le t \le T}$ by $(B_t^n)_{0 \le t \le T}$ in L^2, compute $\int_0^T B_t dB_t$

   (you will check that for any n, $(B_t^n)_{0 \le t \le T}$ is a simple process).

Exercise 23. Let $(H_t)_{0 \le t \le T}$ be a simple process.

1. Prove that $\left(\int_0^t H_s dB_s\right)_{0 \le t \le T}$ and $\left(\left(\int_0^t H_s dB_s\right)^2 - \int_0^t H_s^2 ds\right)_{0 \le t \le T}$ are continuous $(\mathcal{F}_t)$-martingales.

2. Compute $\mathbb{E}\left[\left(\int_0^T H_s dB_s\right)^2\right]$.

   Interpret the result as an isometry property for the Ito integral on the set of simple processes.

Exercise 24. Use the Ito formula to write each of the following stochastic processes $(X_t)_{t \ge 0}$ as Ito integrals (plus constants):

1. $X_t = e^{\frac{t}{2}} cos B_t$, **2.** $X_t = e^{\frac{t}{2}} sin B_t$, **3.** $X_t = (B_t + t) e^{-B_t - \frac{t}{2}}$.

Exercise 25. We consider the Black-Scholes model with the risky asset being a stock.

The stock price is assumed to follow the process $dS_t = S_t(\mu dt + \sigma dB_t)$ where $(B_t)_{t \ge 0}$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and μ, σ are some constants, with $\sigma > 0$.

We assume that the stock pays a continuous dividend yield at a constant annualised rate of δ.

A. Let $F(t, T)$ the future price (or delivery price) of the stock at date t for maturity T.

1. Establish the value of $F(t, T)$ for $t \in [0, T[$.

2. Prove that $(F(t, T))_{t \in [0, T]}$ is a geometric Brownian motion.

B. We consider a European option on the stock with maturity T.

1. Recall why the option price at time t can be written $F(t, S_t)$ (not the same F!).

2. Establish the partial differential equation satisfied by F.

3. How can F be identified among all the solutions to this equation?

4. Considering a forward contract, give an example of function satisfying this equation.

Exercise 26. Using Exercise 25. results, for given T, K, δ, r, write a system of 2 equations satisfied by the function

$$F:\ (t,x) \mapsto xe^{-\delta(T-t)}N(d_1(t, xe^{-\delta(T-t)})) - Ke^{-r(T-t)}N(d_2(t, xe^{-\delta(T-t)}))$$

d_1, d_2 and $N(d)$ correspond to the usual notations, that will be recalled.

Exercise 27. We consider the Black-Scholes model, with the risky asset being a stock paying no dividends. The stock price is then assumed to follow the process $dS_t = S_t(\mu dt + \sigma dB_t)$ where $(B_t)_{t\geq 0}$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and μ, σ are some constants, with $\sigma \neq 0$. The risk-free rate is denoted by r.

1. Using the call-put parity, compute the price at time 0 of a European put option with maturity T and strike price K.

2. Recall the steps allowing to prove that the price at 0 of the European option paying $h(S_T)$ at T is $e^{-rT}\mathbb{E}^*(h(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}))$, where $(W_t)_{t\in[0,T]}$ is a Brownian motion under a probability P^*.

3. Compute the value at time 0 of the European option paying $h(S_T) = S_T^2$ at T.

4. We consider a European option on the stock, paying $h(S_T)$ at its maturity T, with $h : [0, +\infty[\to \mathbb{R}$ of class C^∞ (except potentially at one point). We denote by F_t its price and Δ_t its delta at time $t \in [0,T]$. We know that F_t can be written $F(t, S_t)$ for $t \in [0,T]$.

- **a.** Write $dF(t, S_t)$, variation of the option price between t and $t+dt$, in terms of dt, dS_t and the partial derivatives of F.
- **b.** Recall the value of Δ_t and explain your choice.
- **c.** m_t and s_t are defined by writing the expression found in **a.** as $dF_t = F_t(m_t dt + s_t dB_t)$. Then $|s_t|$ corresponds to the volatility of the option at time t. Compute s_t in terms of σ, S_t, Δ_t and C_t.
- **d.** The option is supposed to be a call. Compare $S_t\Delta_t$ and $F(t, S_t)$ and deduce that the call is more volatile than its underlying asset (same result as in exercise 15).
- **e.** Do we have the same result for a put? (think to the case where S_t is small).

5. We consider two European options on this stock, with same maturity T.
The notations of 2. become, for the option i with $i = 1$ or 2 : $h^i(S_T)$, F_t^i, Δ_t^i, m_t^i et s_t^i.

- **a.** Prove that a portfolio constituted of Δ_t^2 options 1 and $-\Delta_t^1$ options 2 is locally risk-free (the quantities of each option are assumed unchanged between t and $t+dt$).
- **b.** Deduce that for $t \in [0,T]$, we have:

$$\Delta_t^2 m_t^1 F_t^1 - \Delta_t^1 m_t^2 F_t^2 = r[\Delta_t^2 F_t^1 - \Delta_t^1 F_t^2]$$

- **c.** Deduce that $\dfrac{m_t^i - r}{s_t^i}$ is independent of i. Compare to exercise **10**, question **4**.