Université Paris 1 Panthéon Sorbonne
MAEF, MMEF, QEM, IMAEF, 2022-2023

Mid Term Exam: Portfolio Theory

22th March 2023

- Documents and cell phone are prohibited.
- The duration of the exam is **1h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions. We freely use the notations introduced in the course.

Exercice 1 (Questions on the lectures)

1. An investor is said to satisfy the mean-variance preference assumption if for his utility function U, it holds
$$\mathbb{E}[U(X^T R)] = f(\mathbb{E}[X^T R], \sigma(X^T R))$$
where X is the vector of his position, R is the random vector of assets return and $\sigma(X^T R)$ stands for the standard deviation of $X^T R$.

   What are the properties that the function f must satisfy?

2. Provide an explicit example of utility function U for which the above properties are satisfied.

3. If Σ is the covariance matrix of the random vector R, prove that
$$\mathrm{Var}(X^T R) = X^T \Sigma X.$$

Exercice 2

We assume that the market contains only two risky assets with $\mathbb{E}[R_1] = 2$, $\mathbb{E}[R_2] = 4$, standard deviations $\sigma_1 = 1$, $\sigma_2 = 3$ and $\mathrm{Cov}(R_1, R_2) = -3$. The vector of portfolio weight is given by $(x, 1-x)^T$.

1. What is the correlation coefficient between R_1 and R_2?
2. Determine the covariance matrix Σ?
3. Which property of Σ ensures that the two risky assets can be combined into a risk-free portfolio?
4. Determine the set of feasible portfolios and the Global Minimum Variance portfolio.
5. In the mean/standard deviation plane, represent the set of feasible portfolios. Indicate the coordinate of asset 1, asset 2 and of the Global Minimum Variance portfolio, specify the set of efficient portfolios, the portfolios that involve short-selling of asset 1 and the portfolios that involve short selling of asset 2.

Exercice 3

We here follow the notations of the course in the presence of N risky assets without any risk-free asset. The vector of the weights is given by $X \in \mathbb{R}^N$ and we denote by R the N-dimensional random vector of assets return, Σ is its covariance matrix which is assumed to be invertible throughout the exercise.

1. Which constraint is associated to X when the portfolio is fully invested?
2. Write the optimization problem under constraints for the fully invested portfolio having the smallest possible variance for an expected return given by μ.
3. Write the associated Lagrangian to the aforementioned optimization problem.
4. State the sufficient conditions for a global minimum and explain why they are sufficient.

5. Exhibit the three cases depending on the value of $\mathbb{E}[R]$, one in which there is no solution to the above optimization problem, two in which there exists a unique solution: one in which it coincides with the Global Minimum Variance (GMV for short) portfolio and the last one. Give the explicit value of X on the two last cases.

6. We now consider only the value of X found in the last case considered in the previous question.

   (a) Under which condition on the parameter μ is the portfolio with weight X called efficient?

   (b) Compute the covariance of X with the portfolio with weight $Y = \frac{1}{c}\Sigma^{-1}\mathbf{1}$, where $\frac{1}{c}$ stands for the variance of the GMV portfolio and and $\mathbf{1} = (1, \cdots, 1)$ is the N-dimensional vector with all its components being equal to 1.