Probabilistic Methods in Finance
QEM, MMEF, MAEF, IMMAEF 2022-2023
Mid-term, 21 March 2023

Please mention your degree (MAEF,...) on your sheet, and do not copy the questions.
English or French can be used.

I. We consider the usual binomial model with 2 basis assets on 1 period:
- the risk-free asset is worth 1 at time 0 and $1 + r$ at time 1,
- the risky asset is a stock paying no dividend; it is worth S at time 0 and S^d or S^u at time 1, with $S^d < S^u$.

We make the usual assumptions on the market, including the no arbitrage opportunity assumption.

1. What is the consequence of the no arbitrage opportunity assumption on the parameters of the model (full proof required, considering any case).

2. We consider a general option with maturity $T = 1$ on the stock (not necessarily a call or a put). It is worth F^u when the stock is worth S^u and F^d when the stock is worth S^d.

a. What quantity Δ of stock does the option seller have to add to its position to get a risk-free portfolio? (prove the result without replicating the option)
What is the rate of return of such a portfolio?

b. Let $K \in]S^d, S^u[$. Compute Δ and give its sign:
- if the option is a call with strike price K,
- if the option is a put with strike price K.

c. Build a portfolio with positions in the call, the put and the stock that is risk-free between 0 and 1 and deduce a link between the deltas computed in **2.b**.

II. General model

A N-periods model with $(d + 1)$ financial assets is built on a finite probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$, making the vector of prices at time n, $\mathcal{S}_n = (S_n^0, S_n^1, ...S_n^d)$, being $\mathcal{F}_n$-measurable. Asset 0 is risk-free, the other assets are risky.

1. Recall in this context:

a. how a self-financing strategy is defined (financially and mathematically),

b. what an Equivalent Martingale Measure (EMM) is,

c. what the existence of an EMM implies,

d. what the existence and unicity of an EMM implies.

2. We assume that there exists a unique Equivalent Martingale Measure P^*. We consider a European option on the risky assets, given by its payoff $F_N \geq 0$ at its maturity $T = N\Delta t$.

a. Prove that the discounted option price process is a martingale under P^* (full proof expected).

b. Deduce the price of the option at time 0.

Correction

I.

1. NAO assumption implies $S^d < S(1+r) < S^u$

Proof: if $S(1+r) \leq S^d$, at time $t=0$, borrow S \$ at rate r, buy the stock.
At time $t=1$, you get $S^u - S(1+r)$ \$ or $S^d - S(1+r)$ \$. Both are non-negative values, and the first one is positive. This is an AO.

If $S(1+r) \geq S^u$, sell short 1 stock for S \$, invest this amount at rate r.
At time $t=1$, you get $S(1+r)$ and you reimburse 1 stock. In dollars, you have $S(1+r) - S^u \geq 0$ or $S(1+r) - S^d > 0$. This is an AO.

2.a The possible values at time 1 of the portfolio $\begin{cases} -1 \text{ option} \\ \Delta \text{ U.A.} \end{cases}$ are $-F^u + \Delta S^u$ and $-F^d + \Delta S^d$.
With $\Delta = \frac{F^u - F^d}{S^u - S^d}$, both are equal and the portfolio is then risk-free.
The rate of return of such a portfolio is r.

b. $\Delta_{call} = \frac{S^u - K}{S^u - S^d} > 0$ and $\Delta_{put} = -\frac{K - S^d}{S^u - S^d} < 0$.

c. -1 call $+1$ put $+$ 1 U.A. equivalent to K in cash. Then $-C_t + P_t + S_t = \frac{K}{1+r}$.
1 call $-\Delta_{call}$ UA is risk-free.
1 call + some risk-free position is equivalent to 1 put + 1 UA,
then 1 put + 1 UA $-\Delta_{call}$ UA is risk-free, from which we get: $\Delta_{put} = \Delta_{call} - 1$.

II.

1.a self-financing strategy: stochastic process $\Theta = ((\theta_n^0, \theta_n^1, ...\theta_n^d))_{0 \leq n \leq N}$ where $\theta_n^i \in \mathbb{R}$ denotes the quantities of assets i in the portfolio at time n,
· predictable, i.e., Θ_0 is $\mathcal{F}_0$-measurable, and, for $1 \leq n \leq N$, Θ_n is $\mathcal{F}_{n-1}$-measurable and
· $\forall 0 \leq n \leq N-1,\ \Theta_n \cdot \mathcal{S}_n = \Theta_{n+1} \cdot \mathcal{S}_n$
i.e. the investor readjusts his position from Θ_n to Θ_{n+1} without bringing or consuming any wealth external to the portfolio.

b. An Equivalent Martingale Measure is a probability P^* equivalent to P such that the discounted prices of the basis assets are martingales under P^*.

c. existence of an EMM $\Rightarrow$ the market is without arbitrage ("viable market").

d. existence and unicity of an EMM $\Rightarrow$ no arbitrage and complete market (any payoff $\mathcal{F}_N$-measurable is replicable).

2.a Unique P^* $\Rightarrow$ option replicable by a self-financing portfolio strategy Θ. The value of the portfolio at time n is $V_n^\Theta = \Theta_n \cdot \mathcal{S}_n$. The self-financing property implies: $\tilde{V}_n^\Theta = \tilde{V}_0^\Theta + \sum_{k=0}^{n-1} \Theta_{k+1} \cdot (\tilde{S}_{k+1} - \tilde{S}_k)$.
$\forall 1 \leq n \leq N$, · $\tilde{V}_n^\Theta$ is $\mathcal{F}_n$-measurable: for $k \leq n-1$, Θ_{k+1}, $\tilde{S}_k$ and $\tilde{S}_{k+1}$ are $\mathcal{F}_k$ or $\mathcal{F}_{k+1}$-measurable then $\mathcal{F}_n$-measurable. · $\tilde{V}_n^\Theta \in L^1$ (as Ω is finite).
· $\mathbb{E}^*(\tilde{V}_{n+1}^\Theta - \tilde{V}_n^\Theta | \mathcal{F}_n) = \mathbb{E}^*(\Theta_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n) | \mathcal{F}_n) = \sum_{i=0}^{d} \theta_{n+1}^i \mathbb{E}^*(\tilde{S}_{n+1}^i - \tilde{S}_n^i | \mathcal{F}_n)$ as θ_{n+1}^i is $\mathcal{F}_n$-measurable for each i (predictable process). Then $\mathbb{E}^*(\tilde{V}_{n+1}^\Theta - \tilde{V}_n^\Theta | \mathcal{F}_n) = 0$.

b. Price given by $V_0^\Theta = e^{-rT} \mathbb{E}^*(F_N)$.