Portfolio choice theory and asset pricing

Tutorials : Risk measures

Exercise 1

Assume that the loss distribution of a portfolio follows a Gaussian law $L \sim \mathcal{N}(\mu, \sigma^2)$. Then, prove that

$$\mathrm{Var}_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha),$$

where Φ is the cdf of $\mathcal{N}(0,1)$ and $\Phi^{-1}(\alpha)$ is the α-quantile of Φ.

Exercise 2

We consider a portfolio consisting in a long position of $\beta = 10$ shares of a stock with initial price $S_0 = 100$. The intra-day log return of the asset is given by $\Delta_1 Y_{t+1} = \log(S_{t+1}/S_t)$ for $t \geq 0$ are assumed to be iid according to a Gaussian law of mean 0 and standard deviation $\sigma = 0.1$.

a. Compute the $\mathrm{VaR}_\alpha(L_1)$ where L_1 is the portfolio loss between today and tomorrow for $\alpha = 99\%$.
   Hint : Use the fact that $\Phi^{-1}(\alpha) \approx 2.3$.
b. We keep the long position on the portfolio during 100 days. Compute $\mathrm{VaR}_\alpha(L_{100})$ where L_{100} is the portfolio loss during 100 days.

Exercise 3

Let L be a loss with law $\mathcal{N}(\mu, \sigma^2)$. As usual, we denote by Φ the cdf of $\mathcal{N}(0,1)$ and by ϕ its density. Prove that

$$\mathrm{ES}_\alpha(L) = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

Exercise 4

Let $X = (X_1, X_2) \sim \mathcal{N}(\mu, \Sigma)$ with $\mu = (\mu_1, \mu_2)$ and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \rho \in (-1, 1).$$

a. Show that
$$\mathrm{Var}_\alpha(X_1 + X_2) \leq \mathrm{Var}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2).$$
b. What's the financial interpretation of this inequality ?

Exercise 5

Prove the following relations :

a. If L follows a Laplace distribution (double exponential) with density $f(\ell) = \frac{\lambda}{2}\exp(-\lambda|\ell|)$ then

$$\mathrm{Var}_\alpha(L) = \begin{cases} -\frac{1}{\lambda}\ln(2(1-\alpha)), & \text{if} \quad \alpha \geq 1/2, \\ \frac{1}{\lambda}\ln(2\alpha), & \text{otherwise} \end{cases}$$

and

$$\mathrm{ES}_\alpha(L) = \begin{cases} \frac{1}{\lambda}(1-\ln(2(1-\alpha))), & \text{if} \quad \alpha \geq 1/2, \\ \frac{1}{\lambda}(\frac{1}{2}+\alpha(1-\ln(2\alpha))), & \text{otherwise.} \end{cases}$$

b. If L follows a Pareto distribution with index p with density function $f(\ell) = \frac{p}{\ell^{p+1}}\mathbf{1}_{\ell\geq 1}$, with $p > 1$, then

$$\mathrm{Var}_\alpha(L) = (1-\alpha)^{-\frac{1}{p}}, \quad \mathrm{ES}_\alpha(L) = \frac{p}{p-1}(1-\alpha)^{-\frac{1}{p}}.$$

Exercise 6 (Coherence and independence)

We could think that the risk of two independent risks aggregate together, namely, if L_1 and L_2 are two independent real-valued random variables and ρ is a risk measure, then

$$\rho(L_1 + L_2) = \rho(L_1) + \rho(L_2).$$

In general, this is wrong.

Exhibit two independent random variables L_1 and L_2 as well as a risk measure ρ such that

$$\rho(L_1 + L_2) < \rho(L_1) + \rho(L_2).$$

Exercise 7 (Conditioning a Gaussian vector)

With the notation introduced in the course, we let $X = (X_1, X_2)$ be a Gaussian vector with mean $\mu = (\mu_1, \mu_2)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{pmatrix}$$

a. Prove that

$$\mathbb{E}[X_2|X_1] = \mu_2 + \Sigma_{2,1}\Sigma_{1,1}^{-1}(X_1 - \mu_1)$$

b. Deduce from the previous question that $X_2 - \mathbb{E}[X_2|X_1]$ is independent of X_2.

Exercise 8 (Characteristic function of normal mixture)

a. Recall the definition of a normal mixture with non-negative weight random variable W and parameters $\mu \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times k}$.
b. Prove that the characteristic function of the normal mixture X is given by
$$\phi_X(u) = \mathbb{E}[\exp(iu^T X)] = \exp(iu^T\mu)F_W(\frac{1}{2}u^T\Sigma u),$$
where for $\theta \in \mathbb{R}$
$$F_W(\theta) = \int_0^\infty \exp(-\theta w)\,\mathbb{P}_W(dw).$$

Exercise 9 (Stochastic representation of Elliptical distributions)

Prove that $X \sim E_d(\mu, \Sigma, \psi)$ if and only if
$$X \overset{d}{=} \mu + RAS$$
where S is uniformly distributed on the unit sphere, $R \geq 0$ is a radial random vector independent of S and $A \in \mathbb{R}^{d\times k}$ such that $\Sigma = AA^T$.

Exercise 10

a. Let $X \sim \mathcal{N}_d(\mu, \Sigma)$. Prove that $X \sim E_d(\mu, \Sigma, \psi)$ for an explicit function ψ.
b. Let $X \sim E_d(\mu, \Sigma, \psi)$. Prove that for any $B \in \mathbb{R}^{d\times k}$, $b \in \mathbb{R}^k$, it holds
$$BX + b \sim E_k(B\mu + b, B\Sigma B^T, \psi).$$
c. Prove that if $X = (X_1, X_2) \sim E_d(\mu, \Sigma, \psi)$, $X_1 \in \mathbb{R}^k$ and $X_2 \in \mathbb{R}^{d-k}$ with $\mu = (\mu_1, \mu_2)$ and $\Sigma = (\begin{smallmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{smallmatrix})$ then
$$X_1 \sim E_k(\mu_1, \Sigma_{11}, \psi), \quad X_2 \sim E_{d-k}(\mu_2, \Sigma_{22}, \psi)$$
d. Prove that if $X \sim E_d(\mu, \Sigma, \psi)$ and $Y \sim E_d(\tilde{\mu}, \Sigma, \tilde{\psi})$ are independent then $X+Y \sim E_d(\mu + \tilde{\mu}, \Sigma, \psi\tilde{\psi})$.

Exercise 11 (sub-additivity of VaR for elliptical distributions)

Let $X \sim E_d(\mu, \Sigma, \psi)$. Prove that for any $u, w \in \mathbb{R}^d$ and $\alpha \in (0, 1)$,
$$\mathrm{VaR}_\alpha(u^T X + w^T X) \leq \mathrm{VaR}_\alpha(u^T X) + \mathrm{VaR}_\alpha(w^T X).$$

Exercise 12 (Correlation bounds)

Let X_1, X_2 be two log-normal distributions : $\log(X_1) \sim \mathcal{N}(0, 1)$ and $\log(X_2) \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$.

a. Compute the minimal and maximal correlation $\rho_{\min}$, $\rho_{\max}$ of the vector (X_1, X_2).
b. What are the limits of $\rho_{\min}$, $\rho_{\max}$ when $\sigma \uparrow \infty$?

Exercise 13 (The Clayton Copula)

The Clayton copula is defined by

$$C(u_1, u_2) = \max(u_1^{-\theta} + u_2^{-\theta} - 1, 0)^{-\frac{1}{\theta}}, \quad \theta \geq -1.$$

a. For which value of θ, does C correspond to the minimal copula $C_{\min}(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$?
b. Prove that when C goes to $C_{\max}(u_1, u_2) = \min(u_1, u_2)$ when $\theta \to +\infty$.
c. To which copula C converges when $\theta \to 0$?