Probabilistic Methods in Finance
QEM, MMEF, MAEF, IMMAEF 2022-2023
Mid-term, 19 March 2024

Please mention your degree (MAEF,...) on your sheet, and do not copy the questions.
English or French can be used.

I. We consider the usual binomial model with 2 basis assets on 1 period:
- the risk-free asset is worth 1 at time 0 and $1 + r$ at time 1,
- the risky asset is a stock paying no dividend; it is worth S at time 0 and S^u or S^d at time 1, with $S^d < S^u$.

We make the usual technical assumptions on the market (frictionless market), except regarding short selling.

1. In this question only, we assume $S^u = S(1 + r)$. Is there any arbitrage opportunity in each of the following situations?

a. Short selling allowed, with no cost.

b. Short selling allowed, with a cost: if a stock is borrowed at time 0, $(1 + \varepsilon)$ stock has to be reimbursed at time 1, for a given $\varepsilon > 0$.

2. We make the usual assumptions on the market, including the no arbitrage opportunity assumption. In this question, we assume $r = 0$.

We consider a general option with maturity $T = 1$ on the stock. It is worth F at time 0, and at time 1: F^u when the stock is worth S^u and F^d when the stock is worth S^d.

a. Let $\Delta = \frac{F^u - F^d}{S^u - S^d}$. Prove that $F - \Delta S = F^u - \Delta S^u$.

b. The option is quoted at $F^u - \Delta(S^u - S) + \varepsilon$ with $\varepsilon > 0$. Build an arbitrage opportunity.

II. General model

A N-periods model with $(d + 1)$ financial assets is built on a finite probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$, making the vector of prices at time n, $\mathcal{S}_n = (S_n^0, S_n^1, ...S_n^d)$, being $\mathcal{F}_n$-measurable. Asset 0 is risk-free with a constant price (the risk-free rate is again taken equal to 0), the other assets are risky.

1. Recall in this context how a self-financing strategy Θ is defined.
You will give two different writings, with their financial interpretation.
The value at time n of the corresponding portfolio will be denoted as V_n^Θ.

2. Prove that if the prices of the basis assets are martingales under a probability P^* equivalent to P, then, for a self-financing strategy Θ:

a. $(V_n^\Theta)_{0 \leq n \leq N}$ is a martingale under P^* (prove everything),

b. Θ cannot be an arbitrage opportunity.