Université Paris 1 Panthéon Sorbonne
MAEF, MMEF, QEM, IMAEF, 2022-2023

Final Exam: Portfolio Theory

10th May 2023

- Documents, calculators and cell phone are prohibited.
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions. We freely use the notations introduced in the course.

Exercice 1 (Questions on the lectures)

1. For a given loss L and confidence level $\alpha \in (0,1)$, recall the definitions of the Value-at-Risk at level α of L denoted by $\mathrm{VaR}_\alpha(L)$ and the Expected Shortfall at level α of L denoted by $\mathrm{ES}_\alpha(L)$.
2. Prove that for any $c \geqslant 0$ and $d \in \mathbb{R}$, one has
$$\mathrm{VaR}_\alpha(cL + d) = c\mathrm{VaR}_\alpha(L) + d.$$
3. We now assume that $L \overset{d}{=} \mathcal{N}(\mu, \sigma^2)$ for some parameters $\mu \in \mathbb{R}$ and $\sigma \geqslant 0$. Prove that for any $\alpha \in (0,1)$
$$\mathrm{VaR}_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha).$$
where Φ is cumulative distribution function of $\mathcal{N}(0,1)$.
4. What is the main drawback of assessing the risk of L using $\mathrm{VaR}_\alpha(L)$?
5. Does the $\mathrm{ES}_\alpha(L)$ allow to circumvent the previous drawback?
6. Provide the definition of a copula of dimension N.

Exercice 2

In this exercise, we assume that investors have mean-variance preferences. We follow the notations introduced in the course. We consider N risky assets. We denote by $R = (R_1, \cdots, R_N)$ where R_i stands for the return of the asset i for $i = 1, \cdots, N$. We assume that the expected returns $(\mathbb{E}[R_i])_{1 \leqslant i \leqslant N}$ are not all identical. The covariance matrix of R is denoted by Σ and is assumed to be invertible. The rate of return of the risk free asset is denoted by r. We also recall the following notations introduced in the course: $a = \mathbf{1}^T\Sigma^{-1}\mathbb{E}[R]$, $b = \mathbb{E}[R]^T\Sigma^{-1}\mathbb{E}[R]$, $c = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, $d = bc - a^2$, $f^2 = b - 2ar + cr^2$.

We consider the Lagrangian $\mathcal{L} = X^T\mathbb{E}[R] + (1 - \mathbf{1}^T X)r + \lambda(X^T\Sigma X - \sigma^2)$ where $\sigma^2 > 0$ is a given parameter and $\mathbf{1} = (1, \cdots, 1)$.

1. Write explicitly the optimization problem to which the above Lagrangian $\mathcal{L}$ is associated.
2. Write explicitly the two first order conditions associated to the Lagrangian.
3. Express the optimal weights X satisfying the two first order conditions as a function of the parameter λ.
4. Prove that one has
$$2\lambda = \pm\frac{\sqrt{L}}{\sigma},$$
where L is a quantity you will determine. Justify that L is positive.
5. Deduce the optimal weights X as a function of σ satisfying the optimality conditions of question 2).
6. Compute the expected return of the optimal portfolio composed of risky assets only. You will distinguish the two cases $\lambda = \frac{\sqrt{L}}{\sigma}$ and $\lambda = -\frac{\sqrt{L}}{\sigma}$.
7. Determine the fraction of wealth X_0 invested in the risk free asset. You will again distinguish the two cases $\lambda = \frac{\sqrt{L}}{\sigma}$ and $\lambda = -\frac{\sqrt{L}}{\sigma}$.

8. Deduce the return μ of the optimal portfolio composed of N risky assets and the risk free asset. Again distinguish the two cases as previously done.
9. When X is an efficient portfolio, give the expression of σ as a function of $\mu - r$.
10. Deduce from the previous question the expression of the weights $X(\mu)$ as a function of μ.

Exercice 3

For x, we let $x_+ = \max(x, 0)$. We define the map $\rho : L^1(\mathbb{P}) \to \mathbb{R}$ by

$$\rho(X) = \mathbb{E}[X] + \frac{1}{2}\mathbb{E}[(X - \mathbb{E}[X])_+].$$

1. Prove that if $X \leqslant 0$ a.s. then $(X - \mathbb{E}[X])_+ \leqslant -\mathbb{E}[X]$ a.s.
2. Deduce that if $X \leqslant 0$ a.s. then $\rho(X) \leqslant 0$.
3. By writing $Y = X + Y - X$ deduce that that $\rho(X) \leqslant \rho(Y)$ if $X \leqslant Y$ a.s.
4. Prove that for any $c \geqslant 0$ and any $d \in \mathbb{R}$, it holds

$$\rho(cX + d) = c\rho(X) + d.$$

   What is the name of the above properties in the language of risk measures? What is the financial interpretation?
5. Prove that ρ is sub-additive.
6. Is ρ a coherent risk measure?