APPENDIX: consequence of the **non-zero quadratic variation** for the Brownian motion

Integration of a function g with respect to another function f: $\int_a^b g(t)df(t)$.

For a class C^1 function f, we take: $\int_a^b g(t)df(t) = \int_a^b g(t)f'(t)dt$.

For f less smooth: if f is of bounded variation on $[a,b]$, the Stieljes integral is used.

Functions of bounded variation

The total variation of a real-valued function f, defined on an interval $[a,b] \subset \mathbb{R}$ is the quantity $V_{a,b}(f) = \sup_{\Delta \in \mathcal{P}} \sum_k |f(t_{k+1}) - f(t_k)|$ where the supremum is taken over the set of all partitions Δ of the considered interval.

For $\Delta = \{a = t_0 < t_1 < ... < t_p = b\}$ partition of $[a,b]$, the norm (or step, or mesh) of the partition is the length of the longest of its subintervals: $|\Delta| = \max\{t_i - t_{i-1} \mid i = 1, ..., p\}$.

Then $V_{a,b}(f) = \sup_{|\Delta| \to 0} \sum_k |f(t_{k+1}) - f(t_k)|$,

Def: $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ iff $V_{a,b}(f) < +\infty$.

$f : \mathbb{R} \to \mathbb{R}$ is of finite variation iff f is of bounded variation on any $[a,b] \subset \mathbb{R}$.

Ex: If $f : [a,b] \to \mathbb{R}$ is differentiable with a derivative that is Riemann-integrable, its total variation is $V_{a,b}(f) = \int_a^b |f'(t)|dt$, i.e. the vertical component of the arc-length of its graph.

For a continuous function of a single variable, being of bounded variation means that the distance along the direction of the y-axis (not considering the contribution of motion along x-axis) traveled by a point moving along the graph has a finite value.

Functions of bounded variation are precisely those with respect to which one may find Stieltjes integrals of all continuous functions g, defined as:

$$\int_a^b g(t)df(t) = \sup_{|\Delta| \to 0} \sum_k g(t_k)[f(t_{k+1}) - f(t_k)]$$

Def: A stochastic process is of bounded variation (or finite) iff its paths are P-as of bounded variation.

Quadratic variation

Def: A sequence (X_n) of random variables converges in probability towards the random variable X iff $\forall \varepsilon > 0$, $\lim_{n \to +\infty} P(|X_n - X| > \varepsilon) = 0$.

Note that convergence in L^2 implies convergence in probability.

Def : quadratic variation of a process $(X_t)_{t \geq 0}$ in $[0,T]$, denoted $[X,X]_T$

$=$ limit in probability of $\sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})^2$ when the norm of the partition $t_0 = 0 < t_1 < ... < t_n = T$ converges to 0.

Ex : the quadratic variation of the Brownian motion in $[0,T]$ is T:
(B_t) $(\mathcal{F}_t)$-Brownian motion, we have $[B,B]_T = T$ (convergence in L^2).

Th : A continuous process of bounded variation in $[0,T]$ has a quadratic variation on $[0,T]$ equal to 0.

Dém : (X_t) such a process. $\Delta = \{t_0 = 0 < t_1 < ... < t_n = T\}$

when $|\Delta| \to 0$, $\sum_{k=0}^{n-1}(X_{t_{k+1}} - X_{t_k})^2 \leq \sup_k |X_{t_{k+1}} - X_{t_k}| \cdot \sum_k |X_{t_{k+1}} - X_{t_k}|$ converges to 0 P-as,

Indeed P-as in ω: $\sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)| \leq M(\omega)$ $((X_t)$ P-as of bounded variation)

and $\sup_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)| \longrightarrow 0$ when $|\Delta| \to 0$ as P-as, $t \mapsto X_t(\omega)$ uniformly C^0 on $[0,T]$.

Then the convergence holds also in probability. $\square$

Consequence: the Brownian motion is such that:
for any $T \geq 0$, as $[B,B]_T = T$,
P-as, $t \mapsto B_t(\omega)$ is not a function of bounded variation in $[0,T]$,

hence $\int_0^T H_t(\omega)dB_t(\omega)$ **cannot be defined** as a Stieljes integral.

One has to define globally a r.v. $\int H_t dB_t$. It **cannot be defined pathwise** (i.e. ω by ω).