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Probabilistic Methods in Finance - Tutorial solutions

Exercise 1. **1.** The company doesn't know how much dollars it will get at time T, because the future rate (S_T = number of dollars for 1€) is not known today.
The company is therefore exposed to a foreign exchange risk (precisely, risk that S_T willvbe. It has two solutions to hedge this risk:

2. A forward contract locks in the exchange rate for a future transaction. The company will assume a *short position*: agrees to sell the asset (each euro) at time T for the given price.
(currency future on EUR: contract size $= 125,000$€, use the number of futures to match M – or to be the closest possible).
Total payoff at $T = MS_T + M(F(t,T) - S_T) = MF(t,T)$: the FX rate is locked at $F(t,T)$.

3. The company can hedge its risk by buying M put options on euro which mature at time T.
At time T, the company receive M euros and $(K - S_T)^+$ for each put, where S_T is the value of one euro at time T. The resulting cash-flow is

$$M[S_T + (K - S_T)^+] = \begin{cases} MK \text{ if } S_T \leq K \text{ (the puts are exercised)} \\ MS_T \text{ if } S_T \geq K \text{ (the puts are not exercised)} \end{cases} = max(K, S_T)\ \$$$

This guarantees that the value of the euro will not be less than the exercise price K, while allowing the company to benefit from any favorable exchange-rate movements.

Exercise 2. **1.** $\left(1 + \frac{r_m}{m}\right)^m = 1 + r_d$ with $m = 12$. Then $\frac{r_m}{m} = (1.1)^{\frac{1}{12}} - 1 = 0.797\%$ and $r_m = 9.569\%$.

2. $\forall x > 0$, $\forall m \in \mathbb{N}$ with $m \geq 2$, $(1 + \frac{x}{m})^m > 1 + x$, indeed $x \mapsto (1 + \frac{x}{m})^m - x$ is increasing on $\mathbb{R}^+$, strictly on $\mathbb{R}^{*+}$, and is worth 1 at 0.

Or $x \mapsto (1+x)^m$ strictly convex function on $\mathbb{R}^+$, then above its tangent at 0.

3. See lecture notes.

Exercise 3. **1.** $\forall t > 0$, $\ln \frac{N'(t)}{N(t)} = r$ then $(\ln N)'(t) = r$. We get $\ln N(t) = \ln N(0) + rt$ and $N(t) = N(0)e^{rt}$.

2. Decay: $N(t) = N(0)e^{-rt}$. Half-life: $t = \frac{\ln 2}{r}$.

Exercise 4. **1.** See lecture notes.

2. The 2 portfolios contain 1 stock if $S_T > K$ and $K\$$ else.

3. We consider a portfolio at time t made of 1 put, 1 underlying stock and -1 call.
"-1 call" means that the call has been sold (for example written at that time).

At T, if $S_T \geq K$, call exercised, put not exercised: the stock is sold against $K\$$.
if $S_T < K$, put exercised, call not exercised: the stock is sold against $K\$$.

In both cases, we end up with $K\$$ in the portfolio.

4. Then the portfolio has same value at t as a portfolio containing K 0-coupons maturing at T.
We get: $P_t + S_t - C_t = KB(t,T)$.

5. If short sales are not allowed (but one can still borrow cash and write options), the call-put parity relationship reduces to $C_t + KB(t,T) \leq P_t + S_t$.

Indeed $C_t + KB(t,T) < P_t + S_t$ does not lead to an arbitrage opportunity, so can be observed.

Exercise 7.

1. The U.A. is the euro, let S_t its quote in \$ at t.

The holder of any currency can earn interest at the risk-free interest rate prevailing in this currency. This interest can be regarded as a dividend yield (case of a security that provides an income).
We denote by r_f the value of the foreign risk-free interest rate with continuous compounding.

Consider the two following portfolios:
Portfolio A: one long forward contract on the security and $F(t,T)$ 0-coupons in \$, with maturity T
Portfolio B: 1 0-coupon in the currency.

The value of portfolio B at time t is $B_f(t,T)$ in the currency (with $B_f(t,T)$ price at t of the 0-coupon in the foreign currency), or $S_t B_f(t,T)$ \$ where S_t is the exchange rate at time t.
Both portfolios will contain one unit of the foreign currency at time T, then they must have same value at time t:

$$F(t,T) = S_t \frac{B_f(t,T)}{B_\$(t,T)} \qquad (*)$$

With continuous rates, we get: $F(t,T) = S_t e^{(r-r_f)(T-t)}$. This is called the "interest rate parity relation".

It involves the interest rate differential (domestic minus foreign $r - r_f$).

2.a. M changed at t in $\frac{M}{S_t}$ of the currency, and invested $\Rightarrow \frac{M}{S_t}(1+r_f)$ of the currency at time 1.
The rate that can be locked by a forward contract is $F(t,T) = S_t \dfrac{1+r}{1+r_f}$.

b. Then at time 1 we have: $\frac{M}{S_t}(1+r_f)F(t,T) = M(1+r)$. The return is then r.
Interpretation: the gain on the highest interest rate is compensated by a loss in the FX (the foreign currency looses some power purchase because of its high interest rate).
Otherwise stated: the interest rate parity (*) states that hedged returns from investing in different currencies should be the same.

Exercise 8. **1.** Commodity (e.g. wheat) submitted to some continuous losses due to the storage (mould).
2. We compare 2 portfolios at t:
Portf A: one long forward contract on the security plus $F(t,T)$ 0-coupons with maturity T (or an amount of cash equal to $F(t,T)e^{-r(T-t)}$).
Portfolio B: $e^{\alpha(T-t)}$ U.A.

Let $n(s)$ be the number of securities held in the portfolio B at time $s \in]t,T[$.
Between s and $s+ds$, each unit of the commodity held in portfolio B has a storage cost equal to $\alpha S_s ds$,or δds commodity. Then $dn(s) = -n(s)\alpha ds$, hence $n'(s) = -\alpha n(s)$ for any $s \in]t,T[$, and $n(s) = n(t)e^{-\alpha(s-t)}$.
We get $n(T) = 1$.

Portfolios A and B are therefore worth the same at time T, the AOA assumption implies that they have the same value at time t: we obtain, with $r = r(t,T)$ the continuous risk-free interest rate: $0 + F(t,T)e^{-r(T-t)} = S_t e^{\alpha(T-t)}$. Thus

$$F(t,T) = S_t e^{(r+\alpha)(T-t)}$$

Exercise 10.

1.a The risky asset return is worth at $t = 1$: $\frac{S^u - S}{S}$ with proba $p = P(S = S^u)$,
$\frac{S^d - S}{S}$ with proba $1 - p$.

If $X = x$ with proba p and y with proba $1 - p$, then the variance of X is

$$V(X) = px^2 + (1-p)y^2 - [px + (1-p)y]^2 = p(1-p)(x^2 - 2xy + y^2) = p(1-p)(x-y)^2.$$

Then the volatility of the risky asset is $\sigma = \sqrt{p(1-p)}\frac{S^u - S^d}{S}$.

b. Volatility of the call: $= \sqrt{p(1-p)}\frac{S^u - K}{C}$. Prove that $\frac{S^u - K}{C} \geq \frac{S^u - S^d}{S}$.
$C = \frac{p^*}{1+r}(S^u - K)$ with $p^* = \frac{S(1+r) - S^d}{S^u - S^d}$. Then $\frac{S^u - K}{C} = \frac{1+r}{p^*} = \frac{(1+r)(S^u - S^d)}{S(1+r) - S^d}$.
Prove that $\frac{(1+r)}{S(1+r) - S^d} \geq \frac{1}{S}$. OK

2.a The portfolio "-1 call + Δ UA" is risk-free when $\Delta = \frac{F^u - F^d}{S^u - S^d}$ (usual notations),
as its 2 possible values are: $-F^u + \Delta S^u$ and $-F^d + \Delta S^d$, which are equal for this choice of Δ.

Then its return can only be r, so we get, F being the option price at 0:
$(-F + \Delta S)(1 + r) = -F^d + \Delta S^d$ (also equal to $-F^u + \Delta S^u$).

We deduce $F(1 + r) = F^d + \frac{F^u - F^d}{S^u - S^d}[S(1 + r) - S^d] = F^d + p^*(F^u - F^d)$, with $p^* = \frac{S(1+r) - S^d}{S^u - S^d}$.
We obtain the usual formula $F = \frac{1}{1+r}(p^* F^u + (1 - p^*)F^d$.

b. We get $\Delta = \frac{22}{40}$ then $(1.1)F = 22 - \frac{22}{40}[130 - 110] = 11$ then $F = 10$ (or from $p^* = \frac{110-90}{130-90} = \frac{1}{2}$ and pricing formula).
The call being mispriced, we should be able to build an arbitrage opportunity (AO).
We know that the call at $t = 1$ can be replicated by a portfolio containing Δ equities and some cash.
The call is cheap given the price of this portfolio, which is 10.

So we will buy the call and sell this portfolio:

Starting with nothing (no cash, no asset), we sell short $\frac{22}{40}$ equities and buy 1 call. We get $\frac{22}{40}100 - 9 = 46\$$ that we invest for 1 year at rate 10%.

After 1 year, 2 cases:
- either the equity is worth 130, then we exercise the call: we pay 108 to receive 1 equity, we reimburse $\frac{22}{40}$ equities and are left, in value, with: $130 - 108 - \frac{22}{40}130 + 46 \times 1.1 = (20 - \frac{20}{40}130 + 46) \times 1.1 = 1.1\$$.
- either the equity is worth 90, then we do not exercise the call: we reimburse $\frac{22}{40}$ equities and are left with $46 \times 1.1 - \frac{22}{40} \times 90 = (46 - \frac{20}{40} \times 90) \times 1.1 = 1.1\$$.

Both values are positive, so we have an AO.

Note that the portfolio constituted in above AO is risk-free as well: the call is bought at $C - \varepsilon$ instead of C, obtained as $\alpha + \Delta S$.
$\Delta S - (C - \varepsilon) = -\alpha + \varepsilon$ is invested at rate r.
The unique value of above portfolio at time 1 can be computed in the lower state:
it is $-\Delta S^d + (\varepsilon - \alpha)(1 + r) = \varepsilon(1 + r) > 0$, as $\Delta S^d + \alpha(1 + r) = 0$ (low value of the call).

Multiplying the previous positions by 40, the strategy involves integer quantities of assets only and we still have an AO.

3.a $\mu = \dfrac{\mathbb{E}(S_1) - S}{S}$ then $\mathbb{E}(S_1) = pS^u + (1-p)S^d = S(1+\mu)$

hence $\begin{cases} p(S^u - S^d) + S^d = S(1+\mu) \\ p^*(S^u - S^d) + S^d = S(1+r) \end{cases}$, we deduce $\mu - r = (p - p^*)\frac{S^u - S^d}{S}$

while $\sigma = \sqrt{p(1-p)}\frac{S^u-S^d}{S}$. Therefore: $\Pi = \frac{p-p^*}{\sqrt{p(1-p)}}$.

b. For the given option, let $F^u = F_1(\omega_1)$, $F^d = F_1(\omega_0)$, and μ_F be the expected return of the option. From $F = \dfrac{\mathbb{E}(F_1)}{1+\mu_F} = \dfrac{\mathbb{E}^*(F_1)}{1+r}$, we get like for the equity: $\mu_F - r = (p-p^*)\frac{F^u - F^d}{F}$. The option volatility being $\sqrt{p(1-p)}\frac{|F^u-F^d|}{F}$, we deduce: $\Pi_F = \Pi$ if $F^u > F^d$, $\quad$ ($F^u = F^d$ excluded for an option)

$-\Pi$ if $F^u < F^d$, for example for a put.

Π is the risk premium for the risk factor linked to the equity price.
Options on this equity share this same risk factor.

Using the risk-neutral probability P^* (hence p^*) allows to price the options, without having to know (or estimate) the equity's risk premium, Π, and the actual probability p.

Exercise 11.

1. S_1 takes 2 values s_1^1 and s_1^2. We have $S_1^{-1}(\{s_1^1\}) = \{\omega_1, \omega_2\}$ and $S_1^{-1}(\{s_1^2\}) = \{\omega_3, \omega_4\}$.
Hence $\mathcal{F}_1 = \{\emptyset, \{\omega_1,\omega_2\}, \{\omega_3,\omega_4\}, \Omega\}$.

$\mathcal{F}_2$ is made of: $\emptyset$, Ω, and all singletons $\{\omega_i\}$, couples $\{\omega_i, \omega_j\}$, and triplets $\{\omega_i, \omega_j, \omega_k\}$.

2. The random variable Y is $\mathcal{F}_1$-measurable iff for any Borel set A, $Y^{-1}(A) \in \{\emptyset, \{\omega_1,\omega_2\}, \{\omega_3,\omega_4\}, \Omega\}$.
This is equivalent to: $Y(\omega_1) = Y(\omega_2)$ and $Y(\omega_3) = Y(\omega_4)$.

3. Let X $\mathcal{F}_2$-measurable. We look for Z $\mathcal{F}_1$-measurable s.t. for any Y $\mathcal{F}_1$-measurable, $\mathbb{E}(XY) = \mathbb{E}(ZY)$.

Let Y $\mathcal{F}_1$-measurable, we set $y = Y(\omega_1) = Y(\omega_2)$ and $y' = Y(\omega_3) = Y(\omega_4)$.

$\mathbb{E}(XY) = \sum_i P(\{\omega_i\})X(\omega_i)Y(\omega_i) = y[P(\{\omega_1\})X(\omega_1) + P(\{\omega_2\})X(\omega_2)] + y'[P(\{\omega_3\})X(\omega_3) + P(\{\omega_4\})X(\omega_4)]$,

must be equal to $\mathbb{E}(ZY)$, $\forall y, y'$.

$Z(\omega_1) = Z(\omega_2) =$ is denoted by z and $Z(\omega_3) = Z(\omega_4)$ by z'.

We have $\mathbb{E}(ZY) = zy[P(\{\omega_1\}) + P(\{\omega_2\})] + z'y'[P(\{\omega_3\}) + P(\{\omega_4\})]$.

Hence $Z(\omega_1) = Z(\omega_2) = \dfrac{P(\{\omega_1\})X(\omega_1) + P(\{\omega_2\})X(\omega_2)}{P(\{\omega_1\}) + P(\{\omega_2\})}$ and $Z(\omega_3) = Z(\omega_4) = \dfrac{P(\{\omega_3\})X(\omega_3) + P(\{\omega_4\})X(\omega_4)}{P(\{\omega_3\}) + P(\{\omega_4\})}$.

We get $\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)Y)$,

with $\mathbb{E}(X|\mathcal{F}_1)(\omega_1) = \mathbb{E}(X|\mathcal{F}_1)(\omega_2) = \dfrac{\mathbb{E}(X \mathbb{1}_{\{\omega_1,\omega_2\}})}{P(\{\omega_1,\omega_2\})}$ and same for $\{\omega_3, \omega_4\}$.

i.e. to calculate $\mathbb{E}(XY)$ for Y $\mathcal{F}_1$-measurable, no need to have the finer information of the $X(\omega_i)$, the averages of $X(\omega_1)$ and $X(\omega_2)$ and of $X(\omega_3)$ and $X(\omega_4)$ are sufficient.

Generalisation:

if $\{B_1, B_2, ..., B_n\}$ is a complete system of events (a partition of Ω such that $\cup_i B_i = \Omega$), with $P(B_i) \neq 0$ for all i, and $\mathcal{B}$ the sub-σ-algebra generated by this complete system (made of unions of B_i, and $\emptyset$), then for any $X \in L^1(\Omega, \mathcal{F}, P)$, $\mathbb{E}(X|\mathcal{B}) = \sum_{i=1}^{n} \dfrac{\mathbb{E}(X\mathbb{1}_{B_i})}{P(B_i)} \mathbb{1}_{B_i}$. Otherwise stated, on B_i, $\mathbb{E}(X|\mathcal{B})$ is equal to $\dfrac{\mathbb{E}(X\mathbb{1}_{B_i})}{P(B_i)}$.

Exercise 13.
1. Note: $(\mathcal{F}_n)$ is the "natural filtration" associated to the stochastic process $(X_n)_{n\in\mathbb{N}^*}$.
(X_n) is $(\mathcal{F}_n)$-adapted.

(M_n) is $(\mathcal{F}_n)$-adapted and is in L^2.
$\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(M_n + X_{n+1}|\mathcal{F}_n) = M_n + \mathbb{E}(X_{n+1}|\mathcal{F}_n) = M_n$ (X_{n+1} independent of $\mathcal{F}_n$ and centered).

2. Let $X_n = 1$ if success (proba p), 0 else. The X_n are independent and $N_n = X_1 + ... + X_n$.
A martingale has a constant expectation. $\forall n$, $E(N_n - n\alpha) = n(p-\alpha)$, then α can only be p.

For $n \in \mathbb{N}$, let $\mathcal{F}_n = \sigma\{X_1, ..., X_n\}$. N_n is $\mathcal{F}_n$-mes.
$\mathbb{E}(N_{n+1}|\mathcal{F}_n) = \mathbb{E}(N_n + X_{n+1}|\mathcal{F}_n) = N_n + \mathbb{E}(X_{n+1}|\mathcal{F}_n) = N_n + \mathbb{E}(X_{n+1}) = N_n + p$.
Then $(N_n - np)_{n\geq 1}$ is a martingale.

Note that we could have used the first question with $X_n - p$ replacing X_n.

Exercise 15.
$\forall n \in \mathbb{N}$,
· X_n is $\mathcal{F}_n$-measurable:
for $k \leq n$, Δ_{k-1}, M_{k-1}, M_k are $\mathcal{F}_{k-1}$ or $\mathcal{F}_k$-measurable, hence $\mathcal{F}_n$-measurable.
· X_n is integrable: $|\Delta_{k-1}(M_k - M_{k-1})| \leq c(|M_k| + |M_{k-1}|)$ and the M_k are integrable.
· $\mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = \mathbb{E}(\Delta_n(M_{n+1} - M_n)|\mathcal{F}_n) = \Delta_n \mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n)$ as Δ_n is $\mathcal{F}_n$-measurable.
$= 0.$ Then $\mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n = 0$.

Exercise 16.
1. For any $1 \leq k \leq n$, $\{\tau = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$, then $\{\tau \leq n\} = \{\tau = 1\} \cup ... \cup \{\tau = n\} \in \mathcal{F}_n$.
Therefore $\{\tau > n\} \in \mathcal{F}_n$.

2. $\mathbb{E}(M_T) = \mathbb{E}(\sum_{k=1}^{T} M_T \mathbb{1}_{\{\tau=k\}}) = \sum_{k=1}^{T} \mathbb{E}(\mathbb{E}(M_T|\mathcal{F}_k)\mathbb{1}_{\{\tau=k\}})$
(note that the conditioning does not change anything when $k = T$).
$= \mathbb{E}\Big(\sum_{k=1}^{T} M_k \mathbb{1}_{\{\tau=k\}}\Big) = \mathbb{E}(M_\tau)$.

3. τ is a bounded stopping time. Indeed for $n \in \mathbb{N}^*$, $\{\tau = n\} = \{M_1 < G\} \cap ... \cap \{M_{n-1} < G\} \cap \{M_n \geq G\}$.
Then $\mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0$.

Exercise 17.

Equity price at right top of the tree:

$$\begin{array}{ccc} & & Su^N \\ & Su^{N-1} & \\ Su^{N-2} & & Su^{N-1}d \\ & Su^{N-2}d & \\ & & Su^{N-2}d^2 \end{array}$$

Corresponding price for the call:

$$\begin{array}{ccc} & & Su^N - K \\ & C_{N-1}^{N-1} & \\ C_{N-2}^{N-2} & & Su^{N-1}d - K \\ & C_{N-1}^{N-2} & \\ & & Su^{N-2}d^2 - K \end{array}$$

with $C_{N-1}^{N-1} = [p^*(Su^N - K) + (1-p^*)(Su^{N-1}d - K)]e^{-r\Delta t}$

$$= [p^* Su^N + (1-p^*)Su^{N-1}d - K]e^{-r\Delta t} = Su^{N-1} - Ke^{-r\Delta t}$$

$$C_{N-1}^{N-2} = Su^{N-2}d - Ke^{-r\Delta t},$$

$$C_{N-2}^{N-2} = Su^{N-2} - Ke^{-2r\Delta t}.$$

Corresponding price for the delta: Δ_{N-2}^{N-2} $\quad \Delta_{N-1}^{N-1}$, Δ_{N-1}^{N-2} . . .

$$\Delta_{N-1}^{N-1} = \frac{Su^N - K - [Su^{N-1}d - K]}{Su^N - Su^{N-1}d} = 1, \text{ same for } \Delta_{N-1}^{N-2},$$

while $\Delta_{N-2}^{N-2} = \dfrac{C_{N-1}^{N-1} - C_{N-1}^{N-2}}{Su^{N-1} - Su^{N-2}d} = 1.$

Interpretation: all these nodes corresponding to cases where the call will be exercised at T.
The bank already holds 1 equity to be able to deliver it at T.

At the right bottom of the tree, the call will not be exercised, it is worth 0 at any node, and the delta is 0 as well. No equity needed anymore in the hedging portfolio.

Exercise 18.

1. We have $\forall K, T,\ (S_T - K)^+ + K = Max(S_T, K) = (K - S_T)^+ + S_T$.
Taking the expectation under the risk-neutral probability and dividing by e^{rT}, we get the the call-put parity relationship at time 0, using $S_0 = e^{-rT}\mathbb{E}^*(S_T)$.

2. 1 call - Δ_{call} U.A. is risk-free
(locally: precisely $\{1 \text{ option } -\Delta_n \text{ U.A.}\}$ is risk-free between $n-1$ and n),
1 call + some risk-free position is equivalent to 1 put + 1 U.A. (see proof of the call-put parity),
then 1 put + 1 U.A. $-\Delta_{call}$ U.A. is risk-free, from which we get that 1 put $-(\Delta_{call} - 1)$ U.A. is risk-free, ie $\Delta_{put} = \Delta_{call} - 1$.

Exercise 19.

1. Mentioned in the lecture:
For all $n \in \mathbb{N}$, Z_n is obviously $\mathcal{F}_n$-measurable and in L^1, while
$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(Z|\mathcal{F}_{n+1})|\mathcal{F}_n) = \mathbb{E}(Z|\mathcal{F}_n) = Z_n$.

2. $\mathbb{E}^*(Y) = \mathbb{E}(ZY) = \mathbb{E}(Y\mathbb{E}(Z|\mathcal{F}_k))$ as Y is $\mathcal{F}_k$-measurable (by definition of the conditional expectation) $= \mathbb{E}(YZ_k)$.

3. Let $V = \dfrac{1}{Z_{n-1}}\mathbb{E}(XZ_n|\mathcal{F}_{n-1})$. To prove $\mathbb{E}^*(X|\mathcal{F}_{n-1}) = V$, we have to prove that V is $\mathcal{F}_{n-1}$-measurable and that for all Y bounded $\mathcal{F}_{n-1}$-measurable, we have $\mathbb{E}^*(YV) = \mathbb{E}^*(YX)$.
The first property is clear as Z_{n-1} and $E(XZ_n|\mathcal{F}_{n-1})$ are $\mathcal{F}_{n-1}$-measurable.
Let Y $\mathcal{F}_{n-1}$-measurable, YV is $\mathcal{F}_{n-1}$-measurable, then
$\mathbb{E}^*(YV) = \mathbb{E}(YVZ_{n-1}) = \mathbb{E}(Y\mathbb{E}(XZ_n|\mathcal{F}_{n-1})) = \mathbb{E}(\mathbb{E}(YXZ_n|\mathcal{F}_{n-1}))$ from Y $\mathcal{F}_{n-1}$-measurable
$= \mathbb{E}(YXZ_n) = \mathbb{E}^*(YX)$ as YX is $\mathcal{F}_n$-measurable.

4. Let $n \geq 1$, $\mathbb{E}^*(M_n - M_{n-1}|\mathcal{F}_{n-1}) = \dfrac{1}{Z_{n-1}}\mathbb{E}\left((M_n - M_{n-1})Z_n \,\middle|\, \mathcal{F}_{n-1}\right)$ as $M_n - M_{n-1}$ is $\mathcal{F}_n$-measurable..
The result is a $\mathcal{F}_{n-1}$-measurable r.v., we denote it by X_{n-1}.

For $n \geq 1$, let $M'_n = M_n - \sum_{k=1}^{n} X_{k-1}$. We have to prove that $(M'_n)_{n \in \mathbb{N}^*}$ is a $(\mathcal{F}_n)$-martingale under P^*.

For $n \geq 1$, $M'_n - M'_{n-1} = M_n - M_{n-1} - X_{n-1} = M_n - M_{n-1} - \mathbb{E}^*(M_n - M_{n-1}|\mathcal{F}_{n-1})$, from the previous result. Hence $\mathbb{E}^*(M'_n - M'_{n-1}|\mathcal{F}_{n-1}) = 0$.

Exercise 20. **1.** Let $s \leq t$. B_s and B_t are centered, then $\mathbb{E}(B_s B_t) = Cov(B_s, B_s + B_t - B_s) = \mathbb{E}(B_s^2)$ as B_s and $B_t - B_s$ are independent. And $\mathbb{E}(B_s^2) = s$.

2. 2 main properties of the conditional expectation that are used:
- X $\mathcal{B}$-measurable $\Rightarrow \mathbb{E}(X|\mathcal{B}) = X$
- X independent of $\mathcal{B} \Rightarrow \mathbb{E}(X|\mathcal{B}) = \mathbb{E}(X)$

$\forall t \geq 0$, $B_t \sim \mathcal{N}(0,t)$ therefore the r.v. are integrable: Gaussian variables have moments of any order and $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow \mathbb{E}(e^{\lambda X}) = e^{\lambda m + \frac{\lambda^2}{2}\sigma^2}$ (Laplace transform) then $\mathbb{E}(e^{\lambda B_t}) = \mathbb{E}(e^{\frac{\lambda^2}{2}t})$ ie $\mathbb{E}(e^{\lambda B_t - \frac{\lambda^2}{2}t}) = 1$.

$s \leq t$: $\cdot$ $\mathbb{E}(B_t|\mathcal{F}_s) = \mathbb{E}(B_s + B_t - B_s|\mathcal{F}_s) = B_s + \mathbb{E}(B_t - B_s) = B_s$,

$\cdot$ $\mathbb{E}((B_t - B_s)^2|\mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2) = t - s = \mathbb{E}(B_t^2|\mathcal{F}_s) - B_s^2$ then $\mathbb{E}(B_t^2 - t|\mathcal{F}_s) = B_s^2 - s$.

$\cdot$ $\mathbb{E}(e^{\lambda B_t - \frac{\lambda^2}{2}t}|\mathcal{F}_s) = \mathbb{E}(e^{\lambda(B_t - B_s)}|\mathcal{F}_s)e^{\lambda B_s - \frac{\lambda^2}{2}t}$ since B_s is $\mathcal{F}_s$-measurable
$= e^{\lambda B_s - \frac{\lambda^2}{2}s}$ since $\mathbb{E}(e^{\lambda(B_t - B_s)}|\mathcal{F}_s) = \mathbb{E}(e^{\lambda(B_t - B_s)}) = e^{\frac{\lambda^2}{2}(t-s)}$.

3. a For $0 \leq k \leq n-1$, $\dfrac{\mathbb{E}[(X_k)^2]}{(t_{k+1} - t_k)^2} = \mathbb{E}\left(\left[\dfrac{(B_{t_{k+1}} - B_{t_k})^2}{t_{k+1} - t_k} - 1\right]^2\right) = \underbrace{\mathbb{E}([(X^*)^2 - 1]^2)}_{M}$ with $X^* \sim \mathcal{N}(0,1)$.

b. Let $Y = \sum_{k=0}^{n-1}(B_{t_{k+1}} - B_{t_k})^2$. Then $\mathbb{E}[(Y - T)^2] = \mathbb{E}\left[\left(\sum_{k=0}^{n-1} X_k\right)^2\right] = \sum_{k=0}^{n-1} \mathbb{E}[(X_k)^2]$.

Indeed, if $j < k$, $\mathbb{E}(X_j X_k) = 0$ since X_j and X_k are independent.

Therefore $||Y - T||_{L^2}^2 = M \sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \leq \delta M \sum_{k=0}^{n-1}(t_{k+1} - t_k) = \delta M T$ where $\delta = |\{t_k\}|$.

Exercise 21.

1. (L_t) is a martingale, then $\forall t \geq 0$, $\mathbb{E}(L_t) = \mathbb{E}(L_0) = 1$. Therefore P^* is a probability.

2. If $Y \in L^1(\Omega, \mathcal{F}, P^*)$ is $\mathcal{F}_t$-measurable, then $\mathbb{E}^*(Y) = \mathbb{E}(Y L_T) = \mathbb{E}(Y \mathbb{E}(L_T|\mathcal{F}_t)) = \mathbb{E}(Y L_t)$.

3.a Let $u \in \mathbb{R}$, we have (take off "i" below to replace *characteristic function* by *generating function*):

$\mathbb{E}^*(e^{iu(W_t - W_s)}) = \mathbb{E}(e^{-\lambda B_t - \frac{\lambda^2}{2}t}e^{iu[B_t - B_s + \lambda(t-s)]})$ as $W_t - W_s$ is $\mathcal{F}_t$-measurable
$= \mathbb{E}(e^{(iu-\lambda)(B_t - B_s)}e^{-\lambda B_s})e^{-\frac{\lambda^2}{2}t + iu\lambda(t-s)}$
$= \mathbb{E}(e^{(iu-\lambda)(B_t - B_s)})\mathbb{E}(e^{-\lambda B_s - \frac{\lambda^2}{2}s})e^{-\frac{\lambda^2}{2}(t-s) + iu\lambda(t-s)}$, using that $B_t - B_s$ and B_s are independent
$= e^{\frac{(iu-\lambda)^2}{2}(t-s)}e^{-\frac{\lambda^2}{2}(t-s) + iu\lambda(t-s)} = e^{-\frac{u^2}{2}(t-s)}$.

b. For $s \leq t$, from **a.**, $W_t - W_s \sim \mathcal{N}(0, t-s)$ under P^*.

$(W_t)_{t \leq T}$ is a stochastic process with continuous paths, and $W_0 = 0$.
$(B_t)_{t \leq T}$ and $(W_t)_{t \leq T}$ have the same natural filtration, denoted by $(\mathcal{F}_t)_{t \leq T}$.

We want to prove that for $s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ under P^*.

Writing $W_t - W_s = B_t - B_s + \lambda(t-s)$, we see easily the independence, but under P.

According to the lemma (proved below), to prove that $W_t - W_s$ is independent of $\mathcal{F}_s$ under P^*, it is sufficient to prove:

$\forall u \in \mathbb{R},\ \mathbb{E}^*(e^{iu(W_t-W_s)}|\mathcal{F}_s) = e^{-\frac{u^2}{2}(t-s)}$ (*).

Let Y $\mathcal{F}_s$-measurable. For $u \in \mathbb{R}$, we compare $\mathbb{E}^*(e^{iu(W_t-W_s)}Y)$ to $\mathbb{E}^*(e^{-\frac{u^2}{2}(t-s)}Y)$, using the same steps as in 3.a., with $\mathbb{E}(e^{-\lambda B_s})$ replaced by $\mathbb{E}(e^{-\lambda B_s}Y)$:

$$\mathbb{E}^*(e^{iu(W_t-W_s)}Y) = \mathbb{E}(e^{-\lambda B_t - \frac{\lambda^2}{2}t} e^{iu[B_t - B_s + \lambda(t-s)]}Y) = \mathbb{E}(e^{(iu-\lambda)(B_t-B_s)}e^{-\lambda B_s}Y)e^{-\frac{\lambda^2}{2}t + iu\lambda(t-s)}$$
$$= \mathbb{E}(e^{-\lambda B_s - \frac{\lambda^2}{2}s}Y)e^{-\frac{u^2}{2}(t-s)} \text{ (comparing to 3.a.)}$$
$$= \mathbb{E}^*(e^{-\frac{u^2}{2}(t-s)}Y), \quad \text{i.e. (*)}.$$

Note: proof of the lemma: we have: $\forall B \in \mathcal{B},\ \mathbb{E}(e^{iuX}\frac{\mathbb{1}_B}{P(B)}) = \mathbb{E}(e^{iuX})$.

Then X has same law under P than under the probability with density $\frac{\mathbb{1}_B}{P(B)}$ with respect to P (the characteristic functions are the same). Then for any $f : \mathbb{R} \to \mathbb{R}$ Borelian and bounded, $\mathbb{E}(f(X)\frac{\mathbb{1}_B}{P(B)}) = \mathbb{E}(f(X))$ i.e. $\mathbb{E}(f(X)\mathbb{1}_B) = \mathbb{E}(f(X))\mathbb{E}(\mathbb{1}_B)$, which proves the independence.

4. $S_T \geq K \iff S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T} \geq K \Leftrightarrow \sigma W_T \geq ln\frac{K}{S_0} - (r - \frac{\sigma^2}{2})T$

Then $P^*(S_T \geq K) = P^*(W_1 \leq d_2) = N(d_2)$ where $d_2 = \dfrac{ln\frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

Exercise 22. 1. $\displaystyle\int_0^T f(t)dB_t = \sum_{k=0}^{n-1} a_k(B_{t_{k+1}} - B_{t_k})$.

The $B_{t_{k+1}} - B_{t_k}$ are independent and $B_{t_{k+1}} - B_{t_k} \sim \mathcal{N}(0, t_{k+1} - t_k)$.

Then $\displaystyle\int_0^T f(t)dB_t \sim \mathcal{N}\Big(0, \sum_{k=0}^{n-1} a_k^2(t_{k+1} - t_k)\Big)$ ie $\displaystyle\mathcal{N}\Big(0, \int_0^T f^2(t)dt\Big)$.

2. See lecture notes.

Exercise 23.

We consider a simple process $(H_t)_{0\leq t\leq T}$ such that $\displaystyle H_t(\omega) = \sum_{k=0}^{n-1} H^k(\omega)\mathbb{1}_{[t_k,t_{k+1}[}(t)$

with $t_0 = 0 < t_1 < ... < t_n = T$, and for $0 \leq k \leq n-1$, $H^k \in L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_{t_k}$-measurable.

$\star$ continuity : P as in ω:

$\displaystyle\Big(\int_0^t H_s dB_s\Big)(\omega) = \sum_{k=0}^{n-1} H^k(\omega)(B_{t_{k+1}\wedge t}(\omega) - B_{t_k\wedge t}(\omega))$ and $t \mapsto B_t(\omega)$ is continuous.

$\star$ The 2 processes are $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted:

For $0 \leq t \leq T$: adding t in the subdivision, with $t = t_N$, we get $\displaystyle\int_0^t H_u dB_u = \sum_{k=0}^{N-1} H^k(B_{t_{k+1}} - B_{t_k})$, $\mathcal{F}_t$-measurable, as sum of r.v. $\mathcal{F}_t$-measurable (for each k, $t_k \leq t_N = t$). Same for the 2nd process.

$\star$ For $s \leq t \leq T$, we want to prove: $\displaystyle\mathbb{E}\Big(\int_0^t H_u dB_u|\mathcal{F}_s\Big) = \int_0^s H_u dB_u$.

We add s and t in the subdivision (and rename the times), getting $t_0 = 0 < t_1 < ... < t_N = T$, then, with $\displaystyle M_n = \int_0^{t_n} H_u dB_u$, it is sufficient to prove that (M_n) is a $(\mathcal{F}_{t_n})$-martingale. Obtained from:

$$\forall 0 \le n \le N-1,\ \mathbb{E}(M_{n+1}-M_n|\mathcal{F}_{t_n}) = \mathbb{E}(H^n(B_{t_{n+1}}-B_{t_n})|\mathcal{F}_{t_n}) = \underbrace{H^n}_{\mathcal{F}_{t_n}-\text{meas}}\ \mathbb{E}(\underbrace{B_{t_{n+1}}-B_{t_n}}_{\text{indep of } \mathcal{F}_{t_n}}|\mathcal{F}_{t_n}) = 0.$$

The conclusion is straightforward as s and t belong to the subdivision ($s = t_m$ for some m and $t = t_N$).

$\star$ on the same way: $\mathbb{E}(M_{n+1}^2|\mathcal{F}_{t_n}) = M_n^2 + \mathbb{E}[(M_{n+1}-M_n)^2|\mathcal{F}_{t_n}]$

indeed double product: $\mathbb{E}[M_n(M_{n+1}-M_n)|\mathcal{F}_{t_n}] = M_n\mathbb{E}[M_{n+1}-M_n|\mathcal{F}_{t_n}] = 0$ (martingale).

But $\mathbb{E}[(M_{n+1}-M_n)^2|\mathcal{F}_{t_n}] = (H^n)^2\mathbb{E}[(\Delta B_n)^2|\mathcal{F}_{t_n}] = (H^n)^2(t_{n+1}-t_n) = \int_{t_n}^{t_{n+1}} H_s^2 ds.$

2. We deduce that for $0 \le t \le T$, $\mathbb{E}\Big[\Big(\int_0^t H_s dB_s\Big)^2\Big] = \mathbb{E}\Big(\int_0^t H_s^2 ds\Big)$ (in particular for $t = T$), as a martingale has a constant expectation (we take the 2nd one).

Interpretation: with $I(H) = \int_0^T H_t dB_t$ for H simple process,

we get: $\mathcal{E} \to L^2(\Omega)$

$H \mapsto I(H)$ with $||I(H)||_{L^2(\Omega)} = ||H||_{L^2(\Omega\times]0,T[)}$

isometry from $\mathcal{E}$ equipped with the norm $L^2(\Omega\times]0,T[, \mathcal{F}\times\mathcal{B}_{\mathbb{R}^+}, P\times dt)$ in $\underbrace{L^2(\Omega,\mathcal{F},P)}_{\text{complete space}}$.

Allows to extend, by density, to $\bar{\mathcal{E}}$

which contains $\Big\{H_.(.)$ measurable, $(\mathcal{F}_t)$ adapted s.t. $\mathbb{E}\Big(\int_0^T H_t^2 dt\Big) < +\infty\Big\}$.

Exercise 24. 1. $f(t,x) = e^{\frac{t}{2}}cosx$. We have $df(t,B_t) = \frac{1}{2}X_t dt - e^{\frac{t}{2}}sinB_t dB_t - \frac{1}{2}X_t dt$,

then $X_t = 1 - \int_0^t e^{\frac{s}{2}}sinB_s dB_s$.

2. $X_t = \int_0^t e^{\frac{s}{2}}cosB_s dB_s$.

3. $f(t,x) = (x+t)e^{-x-\frac{t}{2}}$. We have $\frac{\partial f}{\partial x}(t,x) = e^{-x-\frac{t}{2}} - f(t,x)$ and $\frac{\partial^2 f}{\partial x^2}(t,x) = -2e^{-x-\frac{t}{2}} + f(t,x)$, then $X_t = \int_0^t e^{-B_s-\frac{s}{2}}(1-s-B_s)dB_s$.

Note that $\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0$ in the 3 cases.

Exercise 25.

A. 1. See in chapter II. the computation of a future price when there is a continuous dividend on the U.A.: we proved that the portfolio (B) containing $e^{(r-\delta)(T-t)}$ stocks at time t, and in which all the dividends continuously paid are immediately reinvested in the stock, will contain exactly 1 stock at time T. We deduced that $F(t,T) \overset{(*)}{=} S_t e^{(r-\delta)(T-t)}$.

Here we get $F(t,T) = S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} e^{(r-\delta)(T-t)} = S_0 e^{(r-\delta)T} e^{(\mu-r+\delta-\frac{\sigma^2}{2})t+\sigma B_t} = F(0,T)e^{(\mu-r+\delta-\frac{\sigma^2}{2})t+\sigma B_t}$.

2. We deduce $dF(t,T) = F(t,T)((\mu-r+\delta)dt + \sigma dB_t)$, by comparaison with the equation for (S_t) or using the Ito lemma (to compute $dG(t,S_t)$ when $G(t,x) = xe^{(r-\delta)(T-t)}$).

B. 1. Like in the case with no dividend, the price at time t depends on t, S_t, and not on S_s, $s<t$, since the future variations of the UA price is function only of S_t (Markov process), denoted by $F(t,S_t)$ where $F : [0,T]\times\mathbb{R}^+ \to \mathbb{R}$, $(t,x) \mapsto F(t,x)$.

F is again of class $C^{1,2}$.

2. See course notes page 37: for any financial asset whose price at time t can be written $F(t, S_t)$ with F $C^{1,2}$, F satisfies the PDE ("parabolic equation"):

$$\frac{\partial F}{\partial t}(t,x) + (r-\delta)x\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 F}{\partial x^2}(t,x) = rF(t,x)$$

3. A particular solution F is identified by the "boundary condition", that sets the value of $F(T,\cdot)$ (payoff of the option at T as a function of S_T).

4. Forward contract with maturity T, future price K (note that $K = F(t_0, T)$ with t_0 the inception date of the contract).

Value of the contract at t (it delivers at T: 1 stock against the payment of K)
$= S_t e^{-\delta(T-t)} - Ke^{-r(T-t)}$, denoted by $F(t, S_t)$.

We check that $F(t,x) = xe^{-\delta(T-t)} - Ke^{-r(T-t)}$ satisfies the equation indeed.

Exercise 26. Black-Scholes model when the stock pays a continuous dividend yield at a constant annualised rate of δ (see previous exercise).

For any financial asset whose price at time t can be written $F(t, S_t)$ with F $C^{1,2}$, F satisfies the PDE:

$$\frac{\partial F}{\partial t}(t,x) + (r-\delta)x\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 F}{\partial x^2}(t,x) = rF(t,x)$$

The given function F corresponds to the price of a European call with maturity T and strike price K, then F satisfies the previous equation, and $F(T,x) = (x-K)^+$.

We have $F(t,x) = C(t, xe^{-\delta(T-t)})$, where C is the call price funtional when the U.A. pays no dividend.

The call on the stock paying the continuous dividend is equivalent to a call on the portfolio B described in exercise 25, which is now an underlying asset paying no dividend.

Exercise 27. 1. $C_0 + Ke^{-rT} = P_0 + S_0$ and $C_0 = S_0 N(d_1) - Ke^{-rT}N(d_2) \Rightarrow$

$P_0 = S_0 N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S_0 = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$.

2. The price at time t can be written $F(t, S_t)$ (Markov).

We consider locally a portfolio constituted of -1 option and $\Delta_t = \frac{\partial F}{\partial x}(t, S_t)$ U.A..

Let V_t be the value of the portfolio at time t: $V_t = -F(t, S_t) + \Delta_t S_t$.

The variation of the portfolio value between t and $t+dt$ is: $dV_t = -dF(t, S_t) + \Delta_t dS_t$ with

$$dF(t,S_t) = \left[\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2\right] dt \ + \frac{\partial F}{\partial x}(t,S_t)dS_t.$$

Then $dV_t = -\left[\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2\right] dt$ contains terms in dt only and none in dB_t.

The portfolio is then risk-free (no randomness), then $dV_t = rV_t dt = r\left[-F(t,S_t) + \frac{\partial F}{\partial x}(t,S_t)S_t\right] dt$.

We get the PDE satisfied by F: $\quad \frac{\partial F}{\partial t}(t,S_t) + rS_t\frac{\partial F}{\partial x}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2 = rF(t,S_t).$

It is independent of μ, as is the boundary condition.

Option pricing can therefore be done as if investors were risk-neutral, hence the formula.

3. $F_0 = e^{-rT} \mathbb{E}^*(S_0^2 e^{(2r-\sigma^2)T+2\sigma W_T}) = S_0^2 e^{(r+\sigma^2)T} \mathbb{E}^*(e^{-\frac{(2\sigma)^2}{2}T+2\sigma W_T}) = S_0^2 e^{(r+\sigma^2)T}$.

4.a. $dF(t,S_t) = \left[\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2\right] dt \ + \frac{\partial F}{\partial x}(t,S_t) dS_t$.

b. From a., the portfolio $\begin{cases} -1 \text{ option} \\ \Delta_t \text{ UA} \end{cases}$ is risk-free between t and $t+dt$ for $\Delta_t = \frac{\partial F}{\partial x}(t,S_t)$.

Δ_t is the quantity of UA to be held at time t by the option seller when he wants to be hedged. The hedging portfolio has to be adjusted dynamically.

c. $s_t = \sigma \frac{\Delta_t S_t}{F_t}$, then the option volatility is $\sigma \frac{|\Delta_t| \ S_t}{F_t}$.

d. For a call, $\Delta_t = N(d_1(t,S_t)) \geq 0$ and
$F_t = S_t N(d_1(t,S_t)) - Ke^{-r(T-t)} N(d_2(t,S_t)) \leq S_t N(d_1(t,S_t)) = S_t \Delta_t$, then $s_t \geq \sigma$.

e. If S_t is small, the put has a high probability to be exercised, then its price is close to $Ke^{-r(T-t)} - S_t$, and its volatility is low (as S_t variations are small compared to $Ke^{-r(T-t)}$ and to the put price).