Probabilistic Methods in Finance
2021-2022

10 May 2022

English or French can be used.
Please mention your degree on your sheet, and do not copy the questions.

For all the exercises, we make the usual technical assumptions on the market (about taxes, transaction costs, short sales...).

I. We consider the usual **binomial model** with 2 basis assets on 1 period:
- the risk-free asset is worth 1 at $t = 0$ and $1 + r$ at $t = 1$,
- the risky asset is a stock paying no dividend;
it is worth S at $t = 0$ and S^u or S^d at $t = 1$, with $S^d < S(1 + r) < S^u$.

1. We consider a portfolio of options written on the stock. Explain why the delta of the portfolio will be obtained as a linear combinaison of the deltas of the different options.

2. We consider a portfolio made of 3 European call options and 3 European put options on the stock, each of them having the exercise price $K = \frac{2S^u+S^d}{3}$ and the maturity $T = 1$.
How many stocks should be added to the portfolio at time 0 to get a risk-free portfolio between 0 and 1? You will suggest a value and check that it works.

II. General model (not binomial)

A N-periods model with $(d + 1)$ financial assets is built on a finite probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_n)_{0\leq n\leq N}$, making the vector of prices at time n, $\mathcal{S}_n = (S_n^0, S_n^1, ...S_n^d)$, being $\mathcal{F}_n$-measurable.
We consider a European option on these assets, given by its payoff $F_T \geq 0$ at its maturity T.
The price of the asset 0 at time $0 \leq n \leq N$ is assumed to be $S_n^0 = e^{rn\Delta t}$, with $\Delta t = \frac{T}{N}$.
We assume that there exists a unique Equivalent Martingale Measure P^* in that model.
Prove that the discounted option price process is a martingale under P^* (full proof expected).

III. We consider the **Black-Scholes model**. The risky asset is a stock paying no dividend, whose price is assumed to follow the process $dS_t = S_t(\mu dt + \sigma dB_t)$ with $(B_t)_{t\geq 0}$ a Brownian motion, and $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^{+*}$. The risk-free rate is denoted by $r > 0$.

1. Using the Ito lemma for $f(x) = \ln x$ on the open set $\mathbb{R}^{+*}$, write $\frac{S_t}{S_0}$ as an exponential.

2. Compute $\mathbb{E}(S_t)$ for any $t > 0$.

3. Explain what change of probability allows to have r instead of μ in the previous result.

4. Can the process $(S_t)_{t\geq 0}$ be a martingale under the new probability, or should this process be modified to get this property?

5. For a given n in $\mathbb{N}^*$, compute the price at 0 of a derivative product paying $(S_T)^n$ at T.

6. Compute the delta at time 0 of the above derivative.

7. Reproduce the arguments leading to an EDP in that model.

Correction

I. (5 pts)

1. Δ is the unique number making of 1 option - Δ U.A. a risk-free portfolio.
Portfolio with α_i options i, $1 \le i \le N$.
1 portfolio - $\sum_i \alpha_i \Delta_i$ U.A. is risk-free, hence the coefficient delta is linear.

2. $\Delta_C = \frac{1}{3}$, $\Delta_P = -\frac{2}{3}$. Hence buy 1 stock.
Checking: the portfolio 3 call + 3 put + 1 stock is worth at $t = 1$:
either $3(S^u - K) + S^u = 2S^u - S^d$, either $3(K - S^d) + S^d = 2S^u - S^d$,
both are equal, then it is risk-free.

II. (4 pts)

Unique P^* $\Rightarrow$ option replicable by a self-financing portfolio strategy Θ. The value of the portfolio at time n is $V_n^\Theta = \Theta_n \cdot \mathcal{S}_n$. The self-financing property implies: $\tilde{V}_n^\Theta = \tilde{V}_0^\Theta + \sum_{k=0}^{n-1} \Theta_{k+1} \cdot (\tilde{S}_{k+1} - \tilde{S}_k)$.

$\forall n \in \mathbb{N}$, $\cdot$ $\tilde{V}_n^\Theta$ is $\mathcal{F}_n$-measurable: for $k \le n-1$, Θ_{k+1} is $\mathcal{F}_k$-measurable then $\mathcal{F}_n$-measurable, while $\tilde{S}_{k+1} - \tilde{S}_k$ is $\mathcal{F}_{k+1}$-measurable then $\mathcal{F}_n$-measurable. $\cdot$ $\tilde{V}_n^\Theta \in L^1$ (as Ω is finite).
$\cdot$ $\mathbb{E}^*(\tilde{V}_{n+1}^\Theta - \tilde{V}_n^\Theta | \mathcal{F}_n) = \mathbb{E}^*(\Theta_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n) | \mathcal{F}_n) = \sum_{i=0}^{d} \theta_{n+1}^i \mathbb{E}^*(\tilde{S}_{n+1}^i - \tilde{S}_n^i | \mathcal{F}_n)$ as θ_{n+1}^i is $\mathcal{F}_n$-measurable for each i. Then $\mathbb{E}^*(\tilde{V}_{n+1}^\Theta - \tilde{V}_n^\Theta | \mathcal{F}_n) = 0$.

III. (14,5 pts)

1. $d[\ln S_t] = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt = (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t$,
then $\forall t \ge 0$, $\ln S_t = \ln S_0 + \int_0^t (\mu - \frac{\sigma^2}{2})ds + \int_0^t \sigma dB_s$, hence $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$.

2. $\forall t \in [0, T]$, $\mathbb{E}(S_t) = S_0 e^{\mu t}$ as $\mathbb{E}(e^{\sigma B_t - \frac{\sigma^2}{2}t}) = 1$.

3. We introduce the process $(W_t)_{t \in [0,T]}$ with $W_t = B_t + \lambda t$ for $t \ge 0$, where $\lambda = \frac{\mu - r}{\sigma}$.
We know that $(W_t)_{t \in [0,T]}$ is a Brownian motion under P^*, proba with density $e^{-\lambda B_T - \frac{\lambda^2}{2}T}$ w.r.t. P.
Under this new probability, we have $\mathbb{E}^*(S_t) = S_0 e^{rt}$ (from $S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}$).

4. No, as $\mathbb{E}^*(S_t)$ is not constant.

5. The price at 0 is obtained as $F_0 = e^{-rT}\mathbb{E}^*((S_T)^n)$.
$e^{-rT}\mathbb{E}^*(S_0^n e^{n(r - \frac{\sigma^2}{2})T + n\sigma W_T}) = S_0^n \mathbb{E}^*(e^{n\sigma W_T - \frac{n^2\sigma^2}{2}T}) e^{-rT + n(r - \frac{\sigma^2}{2})T + \frac{n^2\sigma^2}{2}T} = S_0^n e^{(n-1)(r + n\frac{\sigma^2}{2})T}$.

6. $\Delta_0 = \frac{\partial F}{\partial x}(0, S_0) = n S_0^{n-1} e^{(n-1)(r + n\frac{\sigma^2}{2})T}$.

7. We consider locally a portfolio constituted of $\left\{ \begin{array}{l} -1 \text{ option} \\ \Delta_t \text{ U.A.} \end{array} \right.$ where $\Delta_t = \frac{\partial F}{\partial x}(t, S_t)$.

Its value at time t is: $V_t = -F(t, S_t) + \Delta_t S_t$. The variation of the portfolio value between t and $t + dt$ is: $dV_t = -dF(t, S_t) + \Delta_t dS_t = \left[\frac{\partial F}{\partial t}(t, S_t) + \frac{\partial^2 F}{\partial x^2}(t, S_t) \frac{\sigma^2}{2}(S_t)^2 \right] dt$ from the Ito lemma.

dV_t contains terms in dt only and none in dS_t. The portfolio is then risk-free between t and $t + dt$: its return can only be r, leading to $dV_t = rV_t dt = r\left[-F(t, S_t) + \frac{\partial F}{\partial x}(t, S_t) S_t \right] dt$.

We get:
$$\frac{\partial F}{\partial t}(t, S_t) + rS_t \frac{\partial F}{\partial x}(t, S_t) + \frac{\partial^2 F}{\partial x^2}(t, S_t) \frac{\sigma^2}{2}(S_t)^2 = rF(t, S_t).$$

As S_t can take any value of $]0, +\infty[$, it can be replaced in the previous equation by $x \in]0, +\infty[$, leading to the PDE satisfied by the price function F.