PROBABILISTIC METHODS IN FINANCE

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Finance, but mathematical tools not sufficient

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Chapter I. Introduction. Preliminaries

Aim = Option pricing.
Model in discrete time (Cox-Ross-Rubinstein model), or continuous time (Black-Scholes model).
Pricing by arbitrage.
Very simple market:

one risk-free asset (ie deterministic return, one value only at the end of any period:
discrete time: $(1+r)^n$ after n periods, continuous time: e^{rt} at time t),
one risky asset = underlying asset of the option. S_t denotes its price at time t.

Options belong to derivative products, we will be quick on description and use (forwards/futures, options), can be discussed in the tutorial.

A derivative is determined by its payoff, i.e. what it pays at time T (expiration or maturity date), generally $f(S_T)$.

The market will be complete, i.e. any asset is replicable

(example in discrete time: 2 states of the world at the end of each period).

Maths tools: martingales, for continuous time: Brownian motion.

1 Derivative products: Description and use

A derivative security is a security whose cash flows (hence value) depend on the values of other more basic underlying variables, which may be the prices of traded securities, prices of commodities, stock indices, exchange rates, or any observable variable (temperature,...).
Derivative securities are also known as *contingent claims*.

1.1 Forward contracts

A **forward contract** is an agreement to buy or sell an asset (called the underlying asset, "U.A.") at a certain future time T (called the maturity) for a certain price (called the delivery price).

Ex1: a wheat producer can specify in advance the price at which he will sell a given part of its production. This cancels his price risk (low price when the harvest is ready).

Ex2: A company using oil for its production can fix the price it will pay for it.

One of the parties to the contract assumes a long position and agrees to buy the underlying asset on T for the specified price.
The other party assumes a short position and agrees to sell the asset on T for the same price.
The contract must specify: the quantity of the asset to be delivered, the quality, place, way of delivery.

A very wide range of commodities and financial assets form the underlying assets.
At the beginning, it was for seasonal products, products that are stored to satisfy the demand: agricultural products...
Then any storable product (coffee, sugar, wood, live cattle...)
Now there exists contracts on all sort of products, even non storable like electricity.
Financial products, interest rates, stock indices ($\Rightarrow$ cash delivery), currencies, bonds...
commodities: gold, silver, copper, aluminum...

Ex: FRA (Forward Rate Agreement) = Forward on an interest rate, eg between a bank and its client. It allows to fix now, the rate for a loan that the client will take in the future.

Contracts have been created on anything whose value fluctuates, to provide an hedge.
Eg: weather derivatives, on temperature, quantity of snow, rain (example: for a beer producer),...
A longevity swap allows a pension scheme to remove the risk that members live longer than expected.

Payoff: Let S_T denote the price of the underlying asset on the date of expiration T and K the delivery price. The terminal payoff for the long position is $S_T - K$.

If the contract is exchanged on a market:
At the time the contract is entered into, the delivery price is chosen so that the value of the forward contract to both parties is zero: it costs nothing to take either a long or a short position.
Later the contract can have a positive or negative value depending on movements in the price of the underlying asset: if this price rises after the initiation of the contract, the value of a long position becomes positive (because it gives you the opportunity to buy - forward - at a better price than the current market forward price).

1.2 Future contract

Forward contracts are OTC = private contract between 2 counterparties, difficult to sell back + counterparty risk (default for delivery or for payment).
Future contracts are the same products, but exchanged on a market.
$\Rightarrow$ standardization (to get liquidity): in amount, in term (ex: Mar, Jun, Sep, Dec), in quality.

1st future contract = on the CBOT for the wheat.
One of the key role of the exchange is to organize trading so that contract defaults are minimized. When an investor enters a contract through a broker, the broker will require the investor to deposit funds in what is termed a margin account.
1. initial margin computed to cover the biggest loss possible for one day (depends on the underlying asset)
2. at the end of each trading day, the margin account is adjusted to reflect the investor's gain or loss.

Equivalent to terminate the contract every evening.
$F(t, T)$ forward price at date t for the maturity T.

You enter a contract at time 0. At time 1, the new forward price is $F(1, T)$, if for example $F(1, T) < F(0, T)$, you could enter the same contract with a lower price $F(1, T)$
$\Rightarrow$ potential loss $= F(0, T) - F(1, T) =$ margin call.
If negative, money is put back in your account.

At maturity, you have paid $F(0, T) - F(T, T) = F(0, T) - S_T$ which is what you "owe" (can be negative) at time T, since you have to pay $F(0, T)$ to receive the underlying asset (in case of cash delivery, after the last margin call, you owe nothing).
Note that interest is paid on the margins.
The Clearinghouse acts as an intermediary between buyers and sellers. Members = brokers. It determines the close price depending on all the positions.
+ it organizes the exchange (daily price movement limits on the exchanged volumes..)

The future price reflects the investors expectations about future supply and demand for the product. It gives some insight for the decisions of production, storage...

Note that most contracts are terminated before maturity. You can buy futures on the live cattle and terminate them before receiving the animals...
Sometimes cash settlement anyway.

1.3 Options

The underlying assets include those of futures contract, and derivatives themselves.
A *call option* gives the holder the right to buy the underlying asset by a certain date (called the *maturity*) for a certain price K (exercise price or strike price).
A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price.

American options can be exercised at any time up to T (the maturity is called the *expiration date*).
European options can only be exercised on the maturity itself.
(these terms do not refer to the location of the option!)

Payoff: We consider a European call with maturity T and exercise price K.
Let S_T denote the price of the underlying asset at time T.

The terminal payoff from the long position in this call is $max(S_T - K, 0) = (S_T - K)^+$ (where, for any $x \in \mathbb{R}$, x^+ denotes $max(x, 0)$). Indeed, at T, if $S_T \geq K$, the call is exercised, and its owner receives, in value: $S_T - K$, else the call is not exercised and the payoff is 0.

For the corresponding put, the payoff is $(K - S_T)^+$.

Note that the cash-flows for a call buyer are:

- premium at the time the call is bought, $(S_T - K)^+$ at time T.

An option gives the holder the right to do something. The holder does not have to exercise this right. Whereas it costs nothing to enter into a forward contract, an investor must pay to purchase an option contract. But he can choose the strike price of the contract.

Example (exercise 1): Consider a US company due to receive M euros at a known future time T (because it exports in Europe). It costs are in dollars and it will have to change euros against dollars.
If this exchange takes place at time T, the company doesn't know how much dollars it will get because the future rate (S_T = number of dollars for 1€ at T) is not known today.
The company is therefore exposed to a foreign exchange risk.

It has two solutions to hedge this risk:
1. A forward contract locks in the exchange rate for a future transaction. The company will assume a *short position*: agrees to sell the asset (each euro) at time T for the given price.
(currency future on EUR: contract size = $125,000$€).
2. Foreign currency options are an interesting alternative. The company can hedge its risk by buying M put options on euro which mature at time T.
At time T, the company receive M euros and $(K - S_T)^+$ for each put, where S_T is the value of one euro at time T.
The resulting cash-flow is

$$M[S_T + (K - S_T)^+] = \begin{cases} MK \text{ if } S_T \leq K \text{ (the puts are exercised)} \\ MS_T \text{ if } S_T \geq K \text{ (the puts are not exercised)} \end{cases} = max(K, S_T) \ \$$$

This guarantees that the value of the euro will not be less than the exercise price K, while allowing the company to benefit from any favorable exchange-rate movements.

Whereas a forward contract locks in the exchange rate for a future transaction, an option provides a type of insurance (against losses on the exchange).
Of course, insurance is not free. It costs nothing to enter into a forward transaction, while options require a premium to be paid up front.

Note the difference: **option pricing** is about finding the initial value of the option contract (for a given exercise price, that is chosen in the contract), while **future pricing** means computing the fair delivery price that ensures an initial value of 0 for the future contract.

2 Rates and discounting

An **interest rate** is the cost of borrowing or the price paid for the rental of funds (usually expressed as a percentage of the rental of $100 per year, ie annualized). The rate of return (or return) is the reward that investors demand for accepting delayed payment.
1 euro today is worth more than 1 euro tomorrow because the euro today can be used/ invested today.
It should be better to have it today than in the future, as you can just keep it, so an interest rates should be positive.

But not in particular cases (and currently very frequent!):

Ex: in June 2014, the European Central Bank (ECB) cut its deposit rate to -0.1%, to encourage eurozone banks to lend to small firms rather than to accumulate cash (the hope is to boost the economy). Instead of earning interest on money left with the ECB, banks are charged by the central bank to park their cash with it.

Such policies act on the short end of the yield curve.
Some regulations (e.g. for insurance companies) act on the remaining of the curve: companies are not allowed to keep large amounts of cash, then it has to be invested in some secure assets.

Ex: in August 2019, German government rates up to maturity 30 years were all negative.

A **bond** is a debt security that promises to pay a certain stream of payments (coupons and principal) in the future, until a final date called maturity: the basic bond pays its owner a fixed-interest payment (coupon) every period (ex: every year) until the maturity date, when a specified final amount (face value or par value) is repaid.

Note: an interest rate depends on the maturity, for derivatives pricing we will assume the rates to be constant most of the time.

2.1 Future and present value

1. Future value $\rightarrow$ capitalization $\rightarrow$ compounding interest rates
r interest rate, supposed to be constant.

Definition: the future value in n years of a capital M is the value that one obtains when investing this capital on n years.

Ex: once per year.
Consider an amount M invested for t years at an interest rate of r_d per year (discrete rate for one year). After n years, you get: $M(1 + r_d)^n$.

We consider the general case where an interest is paid several times per year and compounded: the already paid interests are themselves invested. We have m periods in 1 year, the interest is compounded m times per year. The length of one period is $1/m$ (eg, $m = 12$: compounded monthly, $m = 24$ for the saving account "livret A" in France, i.e. interest compounded half-monthly).

We want to compute what should be the interest rate on each period, knowing that after 1 year the total interest has to correspond to the annual rate r_d, as announced by the bank.

Ex: the bank promises a rate of 10% for money invested for 1 year. But an interest is paid each month, so you wonder what the rate for a period of one month is. The answer is: 0,797%. But this number cannot be easily compared to other rates. The habbit is to annualise any rate to facilitate comparison. For that you multiply by 12 to know what the rate would be over a one year period (without any compounding). You get 9,569% (values checked in the tutorial, exercise **2.1**).

General case: let r_m be the annualised rate for 1 period, i.e. the interest on M invested over 1 period is $r_m M\times$ period length $= r_m \frac{M}{m}$, meaning that over one period, the wealth gets multiplied by $1+\frac{r_m}{m}$.
On one year, the interest rate (obtained by compounded the intermediate interests) is r_d.

Then r_m is obtained by $\boxed{\left(1+\frac{r_m}{m}\right)^m = 1+r_d}$, where r_d is the announced annual rate for one year.

Exercise **2.2**: prove that you get too much if you use r_d instead of r_m in above formula (because of the compounding).

An amount M being invested at time 0, let $M(t)$ be the value of the investment at time t.

For any $k\in\mathbb{N}$, the future value of M after k sub-periods (hence k interest payments), is:

$$M(\tfrac{k}{m}) = M(1+\tfrac{r_m}{m})^k = M(1+r_d)^{\frac{k}{m}} = Me^{\frac{k}{m}\ln(1+r_d)} = Me^{r\frac{k}{m}} \text{ where } \underline{r=\ln(1+r_d)}.$$

So $M(t) = M(1+r_d)^t = Me^{rt}$ holds not only for t integer, but also for any $t=\frac{k}{m}$, $k\in\mathbb{N}$, while $M(t)$ is constant on $[\frac{k}{m};\frac{k+1}{m}[$.

We observe that after each interest payment, the value of the investment belongs to the curve $t\mapsto e^{rt}$, where t is the time (in years), curve that is independent of m.

Most mathematical models for option pricing (ex Black-Scholes) are in continuous time.
Hence a continuous interest rate is used: the interests are assumed to be continuously paid and compounded, instead of discretely. That modeling can be seen as the limit when m goes to infinity in the above discrete compounding (first approach).

When the number of periods m gets large, the future value dynamics is close to $t\mapsto Me^{rt}$, with $r=\ln(1+r_d)$, which is the future value dynamics for a continuous interest rate. We have also:

Th: When $m\to+\infty$, r_m converges toward $r=\ln(1+r_d)$.

Proof: we have $(1+\frac{r_m}{m})^m = e^r$, hence $r_m = m(e^{\frac{r}{m}}-1) \sim m(1+\frac{r}{m}-1) = r$, ie $r_m\to r=\ln(1+r_d)$.
(we used $\forall x>0, \forall y\in\mathbb{R}$, $x^y = e^{y\ln x}$ and $e^x\sim 1+x$ when $x\to 0$).

Second approach for a continuous rate: (modeling directly in continuous time)

$M(t)$ wealth at t.
Continuous interest rate: at any period of time (even very short) an interest is paid.
In the continuous compound model, it is assumed that the instantaneous rate of change of M is proportional to M: $\quad \forall t\geq 0,\ \dfrac{dM(t)}{dt} = rM(t) \quad (*)$

Another writing is: "$dM(t) = rM(t)dt$". This is a notation for (*), that can be interpreted as follows: for an infinitely small change in t, dt, the corresponding change in the wealth at time t is $dM(t) = M(t+dt) - M(t) = rM(t)dt$. This variation corresponds to the interest paid between

t and $t+dt$. This means in fact that for h infinitely small, $M(t+h) - M(t) \sim rM(t)h$, hence $\lim_{h \to 0} \dfrac{M(t+h) - M(t)}{h} = rM(t)$ i.e. (*).

Note that this is consistent with what we have in discrete time: on one period of length $\frac{1}{m}$, the change in the wealth corresponds to the paid interest: $\Delta M = r_m M \frac{1}{m}$, therefore the rate of change in the wealth on one period is: $\frac{\Delta M}{1/m} = r_m M$, proportional to the wealth.

From (*) we get: $\forall t \geq 0$, $M'(t) = rM(t)$, then the relative rate of change of the function M is constant:

$\forall t \geq 0$, $\frac{M'(t)}{M(t)} = r$, which implies: $\forall t \geq 0$, $[\ln M]'(t) = r$.

Integrating between 0 and t, we obtain $\ln M(t) - \ln M(0) = rt$ thus $M(t) = M(0)e^{rt}$.

Conclusion: 1 euro today is worth at t : $\begin{cases} (1+r)^t & \text{if } r = \text{ discrete interest rate} \\ e^{rt} & \text{if } r = \text{ continuous interest rate} \end{cases}$

2. Present value $\rightarrow$ Pricing of a future cash flow
reverse of Future Value.

An asset is hold for the future cash flows it will give. We need to calculate the present value of a future cash flow.

a. Deterministic cash flow (ie known at the time of pricing)

= "risk-free asset": can have only one value in the future

present value (today, at time 0) of 1 euro at $t = \begin{cases} \dfrac{1}{(1+r)^t} & \text{if } r = \text{ discrete interest rate} \\ e^{-rt} & \text{if } r = \text{ continuous interest rate} \end{cases}$

A zero-coupon bond (for example in dollar) with maturity date T is a contract which guarantees the holder 1\$ to be paid at time T.

Its price = discount factor.

Price of a 0-coupon bond with maturity T (a bond that pays no coupons and pays 1 euro at T) = discount factor at t for time T, denoted by $B(t,T)$.
$\rightarrow$ Discounting (= calculating what dollars received in the future are worth today)

allows to: bring back a future flow at date 0, compare flows at different times.

Obviously, the rate used to discount a cash-flow paid at time t is the rate that prevails, at the pricing time 0, for the maturity t. We will denote it by $r(0,t)$.

b. General case: stochastic cash flow
Imagine now that the flow is not known at the date of the pricing: its value at future date of payment is random.
Present value of a stochastic cash flow C: should we take: $\dfrac{E(C)}{(1+r)^t}$?

Imagine you can choose between two flows to be received in one year:

- first cash flow is worth 500 in any case,

- second cash flow (lottery) is worth $\begin{cases} 1000 & \text{with proba } 1/2 \\ 0 & \text{with proba } 1/2 \end{cases}$

Both flows have the same expectation but you certainly prefer the first one.
That means that you are "risk averse".

If you have to pay today to receive these flows in 1 year, you will be ready to pay more for the first one (which is worth $\frac{500}{1+r}$).
An interpretation is that your preferences (the values you give for a future cash flow) can be represented by a utility function U which is concave ($U(x)$ is the value attributed to the certain flow x):

To win 1 million is much better than 0 but to win 2 millions is not as much better than 1 million. A concave utility function means risk aversion: you prefer to receive $E(C)$ for sure than to receive C. Indeed, since U is concave, the inequality of Jensen states: $E(U(C)) \leq U(E(C))$, i.e.: the expected utility of a flow is less than the utility of its expected value.
$\dfrac{E(C)}{(1+r)^t}$ is the PV of the cash flow $E(C)$ at t (deterministic cash flow), then the PV today of the cash flow C at t is less than $\dfrac{E(C)}{(1+r)^t}$.

Generally the utility function of the investor is not estimated, only summarized by a discount rate taking it into account with the investment risk as well: the Present Value of a stochastic cash flow C is then expressed as:

$$\begin{cases} \dfrac{E(C)}{(1+\mu)^t} & \text{if } \mu = \text{ discrete interest rate} \\ E(C)e^{-\mu t} & \text{if } \mu = \text{ continuous interest rate} \end{cases}$$

We have most often $\mu > r$ and $\mu - r$ increases with the risk on the asset.

The method of Discounted Cash Flows (DCF) is the classical way to evaluate companies/projects in corporate finance.

For quoted assets, the prices settle at levels that reflect an average of the preferences of the investors. The implied utility function is the one of the market (notion of "representative investor"). When the market risk aversion increases, an equity price generally goes down even if nothing has changed for the company itself.

μ depends on the risk of the cash flow. Eg: stock.
The riskier the stock is (ie the more dispersed its future values are), the larger μ:
the investors require compensation for the risk they assume by buying this stock.

More precisely, the value of μ depends on the systematic risk of the investment.

See the Capital Asset Pricing Model (course on Portfolio Choice):
An investor should not require a higher expected return for bearing nonsystematic risk, as this risk can be almost completely eliminated by holding a well-diversified portfolio. But he generally requires a higher expected return than the risk-free interest rate for bearing positive amounts of systematic risk.

3 Arbitrage methods

We make the following assumptions on the financial markets for derivatives pricing ("usual assumptions"):

1. There are no transactions costs.

2. All trading profits are subject to the same tax rate (or no taxes).

3. The market participants can borrow and lend money at the same risk-free rate of interest. This interest rate is denoted by r. A government bond is considered as a risk-free investment. Its return equals therefore r. We call it a risk-free asset.

4. Short selling is allowed. This is a trading strategy that yields a profit when the price of a security goes down. It involves selling securities that are not owned and buying them back later (reverse of buy/sell).

5. Assets are divisible: one can hold α assets, with $\alpha \in \mathbb{R}$.

6. **NAO**: there are no arbitrage opportunities
An arbitrage opportunity is a situation where you can make a profit without any risk, by choosing a given portfolio (ie a combination of assets) that requires no initial investment, and yields an amount in the future that is non-negative under all possible circumstances ("states of the world"), and not identically zero.

Arbitrage is defined as a strategy that allows to make a profit out of nothing without taking any risk.
NAO: no sure gain at T can be made when investing 0 at time 0.

Mathematical formulation:
The uncertainty in the market on a period $\mathcal{T} = [0, T_f]$ or $\{0, 1, ..., T_f\}$ can be described as 'randomness' interpreted in the context of some probability space $(\Omega, \mathcal{F}, P)$. The unknown future is just one of many possible outcomes, called states of the world (or economy).

One elementary event ω = one possible state of the world
= { realised prices between 0 and T_f, for any economical data }.

Ω = set of elementary events ω.

P a probability measure defined on a σ-algebra (or σ-field) $\mathcal{F}$ of Ω.

Seen from time 0, the prices at any future time $t \in \mathcal{T}$ are random variables.

(for some reminders in probability theory, see for example Jacod, J., Protter, P. (2000) Probability Essentials. Springer).

Definition: An arbitrage opportunity (at time 0) is a portfolio, whose value at time t is V_t, with

$$V_0 = 0 \quad \text{and } \exists T > 0 \text{ such that} \quad \begin{cases} P(V_T \geq 0) = 1 \\ P(V_T > 0) > 0 \end{cases}$$

NAO assumes that this does not exist ("no free-lunch"): the idea is that there are enough investors such that the prices reach an equilibrium immediately. If there exists an AO, some product is undervalued, the demand for it will increase, its price also. The AO disappears.

Since the market participants take advantage of arbitrage opportunities as they occur, arbitrage opportunities disappear quite quickly: indeed an arbitrage opportunity means that an asset is not expensive enough when taking into account the price of other assets. Then the investors buy this asset (and sell the other ones), its price increases and the arbitrage opportunity disappears.

That is why we can make this assumption of NAO.

Note: short selling is done through a broker who borrows the security from another client and sell it in the market, depositing the sale proceeds to the investor's account. An investor with a short position must pay to his broker any income, such as dividends or interest, that would normally be received on the securities that have been shorted.

We are going to prove some relationships involving derivatives prices, by showing that if they were not satisfied arbitrage opportunities would exist (arbitrage pricing).
These relationships come from links between prices of the derivatives and of their underlying asset and between derivatives prices themselves (redundancy of markets). Ex: relationship between European call and put (same underlying asset, strike price and maturity).

Consequence of the assumption of NAO

If two portfolios (a portfolio is a list of financial positions) have same value at a future time T (equality of random variables), they must have same value at any earlier time, t (with no addition/substraction to the portfolio, eg dividends stay in the portfolio).

Proof by contradiction: If this were not true, an investor could make a riskless profit by **short selling** the most expensive one and buying the cheapest portfolio.
Indeed, let V_t^i the value at t of portfolio i, for $i = 1$ or 2. We have $V_T^1 = V_T^2$.
If for example $V_t^1 > V_t^2 > 0$:

at t we sell short one portfolio 1, we buy one portfolio 2.
We have $V_t^1 - V_t^2$ \$ (can be invested at the risk-free rate).

at T, we sell the portfolio 2 for V_T^2\$, we use this cash to buy one portfolio 1 and reimburse the short sale. Our gain is at least $V_t^1 - V_t^2$ \$ (more if we invested this cash).

Some alternative definitions are given on exercice 5.

Some applications:

1. A risk-free portfolio can only have a return equal to r.

2. Call-put parity
Without any a priori model of the evolution of the underlying asset, we can derive an important relationship between the prices of European call and put on a non-dividend paying stock.
We consider European call and put with a same strike price K and same maturity T on a non-dividend paying stock. Let C_t be the call price at time t and P_t the put price.

Value at t of the 0-coupon bond maturing at T: $$B(t,T) = \begin{cases} \frac{1}{(1+r)^{T-t}} & \text{if } r \text{ is a discrete rate} \\ e^{-r(T-t)} & \text{if } r \text{ is a continuous rate} \end{cases}$$

Consider the two following portfolios:

	at t	at T
Portfolio A:	1 call and K 0-coupon bond with maturity T (or an amount of cash equal to $Ke^{-r(T-t)}$)	$(S_T - K)^+ + K$
		$= \max(S_T, K)$
Portfolio B:	1 put and 1 stock	$(K - S_T)^+ + S_T$

(we always assume that cash is invested at the risk-free rate)

Both are worth $max(S_T, K)$ at expiration of the options

(since the options are European, they cannot be exercised prior to the expiration date).

The portfolios must therefore have the same value at time t, then:

$$C_t + KB(t,T) = P_t + S_t \qquad \text{eg} \quad P_t = C_t - S_t + Ke^{-r(T-t)} \quad \text{if } r \text{ is a continuous rate}$$

This relationship is known as call-put parity. It shows that the value of a European put with a certain exercise price and exercise date can be deduced from the value of a European call with the same exercise price and date (and vice versa).

Of course, if this relationship does not hold, there are arbitrage opportunities.

3. Condition on the parameters in the binomial model (see beginning of chapter III.)

4. Forward prices computation (see chapter II.)

Chapter II. Futures / forward contracts pricing

We use arbitrage arguments to provide a relationship between the forward price for the maturity T, $F(t,T)$, and the spot price at time t, S_t.
We compute the fair delivery price for a forward contract that would be exchanged on a market (or equivalently the fair delivery price for a future contract assuming that there are no margin calls).

1. Forward contracts on a security that provides no income and has no storage cost
Examples: non-dividend paying stock or 0-coupon.

For there to be no arbitrage opportunities, at time t, the relationship between the forward price and the spot price must be, for a no-income security:

$F(t,T) = \dfrac{S_t}{B(t,T)} = S_t e^{r(T-t)}$, where $r = r(t,T)$ is the continuous risk-free interest rate.

Proof 1: consider the two following portfolios:

Portfolio A: $\begin{cases} \text{one long forward contract on the security} \\ F(t,T)\ 0-\text{coupons with maturity } T \text{ (or an amount of cash equal to } F(t,T)e^{-r(T-t)}) \end{cases}$

Portfolio B: one unit of the security.

In portfolio A, the cash will grow to an amount $F(t,T)$ at time T. It can be used to pay for the security at the maturity of the forward contract. Both portfolios will therefore consist of one unit of the security at time T. It follows that they must be equally valuable at the earlier time, t: $0 + F(t,T)B(t,T) = S_t$, since when a forward contract is initiated, the delivery price specified in the contract is chosen so that the value of the contract is zero. Thus

$$F(t,T) = \frac{S_t}{B(t,T)} = S_t e^{r(T-t)}$$

Proof 2 (by contradiction):

$\star$ assume first that $F(t,T) > \frac{S_t}{B(t,T)}$.

An investor can borrow S_t dollars for a period of time $T-t$ at the risk-free rate of r (or equivalently sell short $\frac{S_t}{B(t,T)}$ 0-coupons with maturity T), buy the asset, and take a short position in the forward contract. At time T, the asset is sold under the terms of the forward contract for $F(t,T)$, and $\frac{S_t}{B(t,T)}$ is used to repay the loan. A profit of $F(t,T) - \frac{S_t}{B(t,T)}$ in \$ is therefore realized at time T.

$\star$ assume next that $F(t,T) < \frac{S_t}{B(t,T)}$.

An investor can take a long position in the forward contract and short the asset. The short position leads to a cash inflow of S_t that can be invested in 0-coupons with maturity T. At time T, the asset is purchased under the terms of the forward contract for $F(t,T)$ and used to close the short position, and a profit of $\frac{S_t}{B(t,T)} - F(t,T)$ in \$ is realized.

Note: such contracts do not normally arise in practice: forward contract exist on securities that provide an income to the holder, or that represent a cost to the holder (example: some commodities).

Below examples will be seen in tutorial, to get used with the arbitrage arguments.

2. Forward contracts on a security that provides a known cash income

dividend assumed to be known at t (generally announced in advance, OK if $T - t$ is small).

· Discrete income: ex: D_i at t_i.
We want to use the same argument as above. Portfolio A is not changed by the existence of dividends on the U.A.. But portfolio B will contain the dividends, needs to be modified.
S_t = value at t of this portfolio at T and not of the portfolio containing 1 underlying asset

Define D as the present value, using the risk-free discount rate, of income to be received during the life of the forward contract.

Ex:
$$D = \sum_i D_i e^{-r(t_i - t)}$$

Portfolio B modified: one unit of the security plus borrowings of amount D at the risk-free rate (discrete).
The income from the security is used to repay the borrowings so that the portfolio contains exactly one unit of the security at time T. Then both portfolios A and B have same content at T. As before, they need to have the same value at t to ensure that we have NAO.

The value of portfolio B at t is changed in $S_t - D$ (instead of S_t).

Using the NAO assumption as above, we get: $F(t,T) = \dfrac{S_t - D}{B(t,T)}$.

· Continuous income: (stock indices can be regarded as securities that provide known dividend yields in continuous time)

We assume that the underlying asset pays a continuous dividend proportional to the U.A. value, with an annualised rate δ: for any date s, between s and $s + ds$, 1 unit of the U.A. pays a dividend equal to $\delta S_s ds$.
Portfolio B modified: $n(t)$ units of the underlying asset; between t and T, all income continuously paid is immediately reinvested in the security. We choose $n(t)$ such that portfolio B contains exactly one unit of the security at time T.

Let $n(s)$ be the number of securities held in portfolio B at time $s \in]t, T[$.
between s and $s + ds$, each security held in portfolio B provides a dividend equal to $\delta S_s ds$,which is reinvested in δds securities. Then $dn(s) = n(s)\delta ds$: the number of securities held in portfolio B grows exponentially at rate δ: $n(s) = n(t)e^{\delta(s-t)}$. In order to get $n(T) = 1$, we take $n(t) = e^{-\delta(T-t)}$.

Portfolios A and B are therefore worth the same at time T.
From equating their values at time t, we obtain, with $r = r(t,T)$ the continuous risk-free interest rate:

$0 + F(t,T)e^{-r(T-t)} = S_t e^{-\delta(T-t)}$. Thus

$$F(t,T) = S_t e^{(r-\delta)(T-t)}$$

3. Commodity forward contracts

For a commodity forward contract, holding the security represents a cost (storage cost: from having to rent a warehouse, or from losses: part of the product may get spoiled during the storage).

Assuming that the storage cost on $[t, T]$ is known at t, the previous arbitrage argument holds, with the storage cost considered as a negative dividend. The portfolio B in the above argument must be changed in:

one unit of the security plus an amount of cash (or 0-coupon) equal to the present value of all the storage costs that will be incurred during the life of the forward contract.

Discrete cost: the value of portfolio B at t is changed in $S_t + U$ where $U = \sum_i U_i e^{-r(t_i - t)}$ sum of the discounted payments for storage. We get $F(t,T) = \dfrac{S_t + U}{B(t,T)}$.

Continuous cost: if the storage costs incurred at any time are proportional to the price of the commodity (with the rate u), the forward price is $F(t,T) = S_t e^{(r+u)(T-t)}$.

Remark: the previous results hold for investment commodities, i.e. for commodities that are held solely for investment (ex: gold). Some other commodities are held primarily for consumption (ex: oil). For the 2nd type, there can be some incentive to keep the commodity in inventory to avoid shortage and our previous arguments have to be modified (see below).

4. General result (in continuous time)

The relationship between forward prices and spot prices can be summarized in terms of what is known as the cost of carry. This measures (the interest that is paid to finance the asset) plus (the storage cost) minus (the income earned on the asset). Cost of carry: $c = r + u - \delta$. Then

$$F(t,T) = S_t e^{c(T-t)}$$

Example: $c = r - \delta$ for a stock.

Conclusion: when the cost of carry between t and T is known at t, the forward price can be precisely determined from the spot price. We used the fact that the forward contract can be replicated by a position in its underlying asset that is "carried" between t and T. The forward contract is "redundant".

Notes:

- the probability distribution of S_T is not involved (no need of a model for it).

- we made no difference between forwards and futures for the pricing, but future and forward prices can differ, specially for long (in time) contracts. Indeed, taxes and transaction costs are missing in our calculations, liquidity can differ (futures more liquid, easier to trade...), interest rates are random (it matters if margin calls are taken into account as a future contract is terminated every day through the margin call, so rates at intermediary dates are involved).

In the general case, the cost of carry between t and T is not known at time t, the relationships between forward and spot prices are more complex to establish, the cost of carry has to be modeled. Then forward prices as observed on the market give some information on cost of carry expectations (example on dividends for an equity).

An upward sloping forward curve is said to be "in contango": it is the situation where the price of a security for future delivery is higher than the spot price, and a far future delivery price higher than a nearer future delivery. The opposite market condition to contango is known as backwardation. For example, a contango is normal for a (non-perishable) commodity which has a positive cost of carry.

For a **consumption** asset, some investors can prefer to hold the asset (ex: oil).
Indeed, users of a consumption asset may obtain a benefit from physically holding the asset (as inventory) prior to T (maturity) which is not obtained from holding the futures contract. These benefits include the ability to profit from temporary shortages, and the ability to keep a production process running.

Then arbitrage arguments can only be used to give an upper bound to the futures prices. But, because of this preference for the spot position, the futures prices can stay lower than expected, as if the UA was providing an income (cf formula $F(t,T) = S_t e^{(r-\delta)(T-t)}$).
Therefore futures prices (observed value of $T \to F(t,T)$) show an implied return from holding of the asset, called the "convenience yield" . It measures the benefit from physically holding the asset. We get $(F(t,T) = S_t e^{(r+u-y)(T-t)}$, with y=convenience yield).

The convenience yield reflects the market's expectations concerning the future availability of the commodity. The greater the possibility that shortages will occur during the life of the futures contract, the higher the convenience yield. In particular, the convenience yield is inversely related to inventory levels. Low inventories tend to lead to high convenience yields.

Example: currency forward
The underlying variable is, for example for a \$-based investor, the current price in \$ of one unit of the foreign currency (for example the euro).
A foreign currency has the property that the holder of the currency can earn interest at the risk-free interest rate prevailing in the foreign country (case of a security that provides an income). This interest can be regarded as a dividend yield. We denote by r_f the value of this foreign risk-free interest rate with continuous compounding.
Consider the two following portfolios:
Portfolio A: one long forward contract on the security and $F(t,T)$ 0-coupons in \$, with maturity T
Portfolio B: 1 0-coupon in the currency.

The value of portfolio B at time t is $B_f(t,T)$ in the currency (with $B_f(t,T)$ price at t of the 0-coupon in the foreign currency), or $S_t B_f(t,T)$ \$ where S_t is the exchange rate at time t.
Both portfolios will become worth the same as one unit of the foreign currency at time T. Then $F(t,T) = S_t \dfrac{B_f(t,T)}{B_\$(t,T)}$ or

$$F(t,T) = S_t e^{(r-r_f)(T-t)}$$

This is called the *interest rate parity relationship.* It involves the interest rate differential (domestic minus foreign $r - r_f$).

Exercise: find an arbitrage opportunity if this relationship doesn't hold.
See Exercise 7. to explain the name *interest rate parity relationship.*

Chapter III. Option pricing in discrete time: one-period binomial model

Why a model is needed to price an option :

Roughly speaking, the volatility of a stock price is a measure of how uncertain we are about future stock price movements.
As volatility increases, the chance that the stock will do very well or very poorly increases.
For the owner of a stock (spot or future), these two outcomes tend to offset each other. However, this is not so for the owner of a call or put. The owner of a call benefits from price increases but has limited downside risk in the event of price decreases since the most that he can lose is the price of the option (equivalent result for a put).

Therefore, the values of both calls and puts increase as volatility increases.

Whereas the forward price $F(t,T) = S_t e^{c(T-t)}$ is independent of the distribution of the possible values for S_T, an option price depends on this distribution. Therefore, we need to make a supplementary assumption on the evolution of the stock value to price an option.

The objective of this chapter is to present the main ideas related to option theory within the very simple framework of discrete-time models, with the example of Cox, Ross and Rubinstein's model.

One period binomial model

2 dates 0 and 1. At $t = 1$, the risk-free asset is worth $1 + r$ for any state of the world. We assume that the risky asset can take 2 values:

$$S \nearrow S^u, \quad S \searrow S^d \qquad \text{with } S^d < S^u$$

That means that there are 2 states of the world ω_0 and ω_1:
the risky asset is worth S_1 at $t = 1$, with $S_1(\omega_1) = S^u$ and $S_1(\omega_0) = S^d$.
Let $\Omega = \{\omega_0, \omega_1\}$. We assume $P(\{\omega_1\}) = p$ and $P(\{\omega_0\}) = 1 - p$. P is a probability on $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the σ-algebra $\{\emptyset, \{\omega_0\}, \{\omega_1\}, \Omega\}$.

We make the assumptions of chapter 1 on the market made of the basis assets and their derivatives (in particular, short selling of the risk-free asset is equivalent to borrowing at rate r).

Due to the **NAO** assumption, we have

$$S^d < S(1 + r) < S^u \tag{1}$$

Proof: we argue by contradiction.
If for example $S(1 + r) \leq S^d$, at time $t = 0$, you find the following "free-lunch":
borrow S \$ at rate r, buy the stock. At time $t = 1$, you get $S^u - S(1 + r)$ \$ or $S^d - S(1 + r)$ \$.
Both are non-negative values, and the first one is positive. This is an AO.
($S(1 + r) \leq S^d$ implies that everybody will buy stocks instead of saving money on the bank account, whereas $S(1 + r) \geq S^u$ nobody will buy the stock.)

This strategy corresponds to the portfolio $\begin{cases} +1 \text{ U.A.} \\ -S \text{ risk} - \text{free asset} \end{cases}$

Exercise: write the argument when $S^u \leq S(1+r)$.
It involves a short sale of the U.A., the corresponding portfolio is $\begin{cases} -1 \text{ U.A.} \\ +S \text{ risk} - \text{free asset.} \end{cases}$

We consider a European option on the risky asset with maturity $T = 1$. Such a derivative product is determined by F_1, its payoff at T: let $F_1(\omega_1) = F^u$, $F_1(\omega_0) = F^d$.

F $\quad F^u$ / F^d $\quad$ ex: call with strike price $K \in]S^d, S^u[$: $\quad F^u = S^u - K$ (exercise)
and $F^d = 0$ (no exercise).

F_1 is $\mathcal{F}_1$-measurable where $\mathcal{F}_1 = \{\emptyset, \{\omega_0\}, \{\omega_1\}, \Omega\}$ is the information at date 1 (note that $\mathcal{F}_1 = \mathcal{F}$).

To price the option, we will replicate its payoff at $T = 1$ with a portfolio consisting of basis assets. Here, any such derivative asset is replicable (attainable):

there exists a portfolio portfolio consisting of basis assets (α risk-free asset, Δ risky asset) whose value at $t = 1$ is F_1, ie $\alpha(1+r) + \Delta S_1 = F_1$

($\alpha, \Delta \in \mathbb{R}$, cf pricing assumptions; ex: $\alpha < 0$ means borrowing at rate r, $\Delta < 0$ means short sale of the risky asset).

Called replicating portfolio.

Proof: looking for (α, Δ) s.t. $\begin{cases} \alpha(1+r) + \Delta S^u = F^u \\ \alpha(1+r) + \Delta S^d = F^d \end{cases}$

$\exists$ a unique solution: $\Delta = \dfrac{F^u - F^d}{S^u - S^d}$ and $\alpha = \dfrac{F^u - \Delta S^u}{1+r}$.

NAO assumptions $\Rightarrow$ the asset price is:

$$\begin{aligned} F = \alpha + \Delta S &= \frac{F^u}{1+r} + \Delta\Big(S - \frac{S^u}{1+r}\Big) \\ &= \frac{1}{(S^u - S^d)(1+r)}\Big(F^u(S^u - S^d) + (F^u - F^d)[S(1+r) - S^u]\Big) \\ &= \frac{1}{(S^u - S^d)(1+r)}\Big(F^u[S(1+r) - S^d] + F^d[S^u - S(1+r))\Big] \end{aligned}$$

We get $\qquad F \overset{(*)}{=} \dfrac{1}{1+r}\Big[p^* F^u + (1-p^*) F^d\Big]$ with $p^* = \dfrac{S(1+r) - S^d}{S^u - S^d}$.

$p^* \in]0, 1[\quad$ since $S^d < S(1+r) < S^u$.

Notes: **1.** We are able to price any derivative asset. Would not be the case if the risky asset had 3 possible values at time 1 (3 equations, 2 unknowns).
General result: as many assets needed as the number of states of the world.

2. The same equation (*) would be obtained by considering the portfolio: $\begin{cases} -1 \text{ option} \\ \Delta \text{ U.A.} \end{cases}$.

This portfolio is risk-free, indeed it can take only one value at time 1,
as its "two" possible values, $\Delta S^u - F^u$ and $\Delta S^d - F^d$ are equal with our choice of Δ.

Hence its return can only be r, that leads to the equation (*).

3. You will check easily that for a call with strike K with $S^d < K < S^u$ (the other calls are always, or never, exercised, so they are not options), the delta belongs to $]0,1[$.
For a put with strike K, the delta belongs to $]-1,0[$, indeed $\Delta = \dfrac{0-(K-S^d)}{S^u - S^d}$.

Let $P^*(\{\omega_1\}) = p^*$ et $P^*(\{\omega_0\}) = 1-p^*$.
P^* is a probability on $(\Omega, \mathcal{F}_1)$, equivalent to P (ie > 0 on the same events).

Equation (*) can be written: $\boxed{F = \dfrac{1}{1+r}\mathbb{E}^*(F_1)}$

expectation under P^* of the future value, discounted at the risk-free rate.

I.e., when P is replaced by P^*, we can do as if the investors were risk neutral (indifferent between receiving F_1, which is random, or receiving $\mathbb{E}(F_1)$, which is certain, at $t=1$): then we get rid of the question of determining the correct discount rate for the flow F_1.

P^* is called **risk-neutral probability**, and P real or historical probability.

Therefore to price the derivatives, you just have to "set yourself in the (imaginary, fictitious) risk-neutral universe": you attribute the probability p^* at state ω_1, and $1-p^*$ at state ω_1 and you price all the assets as if the investors were risk-neutral. = Risk-neutral valuation
We will refer to a world where everyone is risk neutral as a *risk neutral* world. In such a world investors require no compensation for risk, and the expected return on all securities is the risk-free rate.

Notes: **1.** use the formula for the 2 basis assets. We have in particular $S = \frac{1}{1+r}\mathbb{E}^*(S_1)$: setting the probability of an up movement equal to p^* is equivalent to assuming that the return on the stock equals the risk-free rate.

2. we obtain a price formula that does not depend on the real probability! In fact, the price depends on S, and S depends itself on the real probability. The real probability and the risk-aversion are embedded in the spot price of the U.A., and thanks to the replication, we get the derivative price as a function of S.

The option pricing formula does not involve the probabilities of the stock price moving up or down. This is surprising and seems counterintuitive. It is natural to assume that as the probability of an upward movement in the stock price increases, the value of a call option on the stock increases and the value of a put option decreases. This is not the case. The key reason for that is that we are not valuing the option in absolute terms. We are calculating its value in terms of the price of the underlying stock. The probabilities of future up or down movements are already incorporated into the price of the stock.

The strategies in this model are static. In a model with several periods, dynamic strategies are used. Fo this purpose, we need some concepts like *filtrations* to describe the information structure, and particular stochastic processes (*martingales*) to model the prices.

Chapter IV. Mathematical tools

$(\Omega, \mathcal{F}, P)$ is a given probability space. We will use the following definitions:

• **Stochastic process**:

$(X_t)_{t\in\mathcal{T}}$ family of random variables on $(\Omega, \mathcal{F}, P)$ with values in $(E, \mathcal{E})$ measurable space (will most often be $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ie $\mathbb{R}$ equipped with the Borel σ-algebra),
with $\mathcal{T} = \{0, ..., T\}$ in discrete-time or $[0, T]$ in continuous-time.

Ex: X_t an asset price at time t, then $(X_t)_{t\geq 0}$ is a stochastic process.

• **Filtration** on $(\Omega, \mathcal{F}, P)$: non-decreasing family of sub-σ-algebras of $\mathcal{F}$: $(\mathcal{F}_t)_{t\in\mathcal{T}}$, $s \leq t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$.
$(\mathcal{F}_t)$ = "information structure". $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ is called a "filtered probability space".

used to model a flow of information, for example corresponding to the observations of an asset price: the σ-algebra $\mathcal{F}_t$ usually models the events which can be observed up to time t.
Eg: with S_s the asset price at time s,
let, for $t \geq 0$, $\mathcal{F}_t = \sigma\{S_s, s \leq t\}$, i.e. smallest σ-algebra that contains all pre-images of measurable subsets of $\mathbb{R}$ for times s up to t (or all sets of the form $\{a \leq S_s \leq b\}$ for $0 \leq s \leq t$, $a, b \in \mathbb{R}$).
$(\mathcal{F}_t)_{t\in\mathcal{T}}$ is obviously non-decreasing. It is called the **natural filtration** of the process $(S_t)_{t\in\mathcal{T}}$.
It records the "past behaviour" of the process, and only that information.

More generally, $\mathcal{F}_t$ = information available to investors at time t,
which consists of assets prices before and at time t,
i.e., for N assets: $\mathcal{F}_t = \sigma\{S^i_s, s \leq t, 1 \leq i \leq N\}$, where S^i_s is the price at time s of asset i.

In this set-up, we will always assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Then "$Z$ $\mathcal{F}_0$-measurable" means that Z is constant.

As time passes, an observer knows more and more information, that is, finer and finer partitions of Ω.

Example: just one risky asset is observed, in discrete time, price taking discrete values. Then its price dynamics can be described in a tree, a node being a set $\{\omega | X_t(\omega) = a\}$ (i.e. the pre-image $X_t^{-1}(\{a\})$) for some time t and some value a.

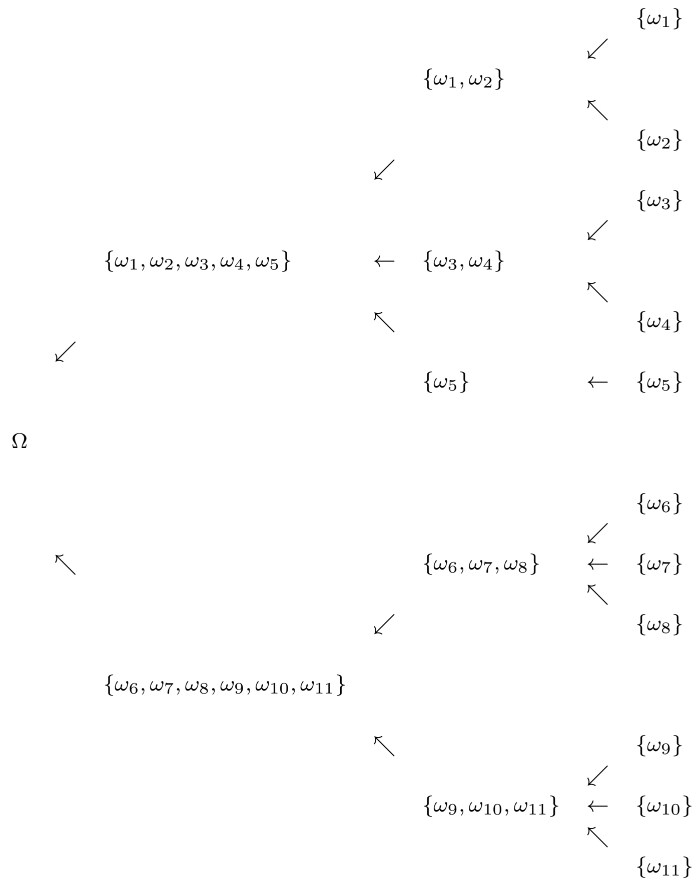
A state of the world ω is a whole path in the tree (not a terminal node).

Note that this tree corresponds to a family $(\mathcal{P}_t)_{t\in\mathcal{T}}$ of partitions of Ω, satisfying:

$\mathcal{P}_0 = \{\Omega\}$, $\mathcal{P}_T = \{\{\omega_1\}, ..., \{\omega_N\}\}$, and $\forall A \in \mathcal{P}_{t+1}$, $\exists A^- \in \mathcal{P}_t$ such that $A \subset A^-$.

The filtration describing the information structure is then given by $(\mathcal{F}_t)$ such that for any $t \in \mathcal{T}$, $\mathcal{F}_t$ is generated by $\mathcal{P}_t$, or, equivalently: $\mathcal{F}_t = \sigma\{S_s, s \leq t\}$.
It is the information available to an observer of the assets prices $S.$ up to time t.

For example, below: $\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}\}\}$.
$\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}, \{\omega_6, \omega_7, \omega_8\}, \{\omega_9, \omega_{10}, \omega_{11}\}\}$.



This tree describes how the information is revealed:

At $t = 0$, the available information is: we are in Ω.

At $t = 1$, we know if we have event $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ or event $\{\omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}\}$, and so on...

At $t = 3$, we know exactly which element of our universe of possible states has been realised.

• A stochastic process $(X_t)_{t \in \mathcal{T}}$ is called **adapted to the filtration** $(\mathcal{F}_t)_{t \in \mathcal{T}}$ iff for all $t \in \mathcal{T}$, X_t is $\mathcal{F}_t$-measurable.
X $\mathcal{F}_t$-measurable means: observable at time t given the information $\mathcal{F}_t$. Note that a stochastic process is always adapted to its natural filtration.

Ex: $\mathcal{F}_t^1$ σ-algebra generated by the r.v. representing the prices of all the assets until time t,

$\mathcal{F}_t^2$ same, but for quoted assets only.

X_t price at t of a non quoted asset, then (X_t) is adapted to filtration $(\mathcal{F}_t^1)$, but not to $(\mathcal{F}_t^2)$, as for a given t, X_t is not $\mathcal{F}_t^2$-measurable.

In the computations at different times, the conditional expectation will be an important tool:

for $X \in L^1(\Omega, \mathcal{F}, P)$ (real-valued r.v. $\mathcal{F} = \mathcal{F}_T$-measurable such that $\mathbb{E}(|X|) < +\infty$),

$\mathbb{E}(X|\mathcal{F}_t)$ is the expectation of X given the information available at time t.

In particular, if X is $\mathcal{F}_t$-measurable, $\mathbb{E}(X|\mathcal{F}_t) = X$ and,

if X is $\mathcal{F}_t$-measurable and bounded, $\mathbb{E}(XY|\mathcal{F}_t) = X\mathbb{E}(Y|\mathcal{F}_t)$ for any $Y \in L^1$:

ie X is considered as a constant at time t (i.e. known).

Reminder on the **conditional expectation**:

Definition : $\mathcal{B}$ sub-σ-algebra of $\mathcal{F}$, $X \in L^1(\Omega, \mathcal{F}, P)$ (i.e. $\mathbb{E}(|X|) < +\infty$),

$\mathbb{E}(X|\mathcal{B})$ is the unique integrable r.v. $\mathcal{B}$-measurable such that

$\forall Y$ $\mathcal{B}$-mesurable and bounded, $\mathbb{E}(XY) = \mathbb{E}(Y\mathbb{E}(X|\mathcal{B}))$.

"Conditional expectation of Y given $\mathcal{B}$".

For $X \in L^2(\Omega, \mathcal{F}, P)$, $\mathbb{E}(X|\mathcal{B})$ is the unique integrable r.v. $\mathcal{B}$-mesurable such that

$\forall Y \in L^2(\Omega, \mathcal{F}, P)$, $\mathbb{E}(XY) = \mathbb{E}(Y\mathbb{E}(X|\mathcal{B}))$.

Properties :

· linearity

For any $\mathcal{B}$ sub-σ-algebra of $\mathcal{F}$ and $X \in L^1$ (all the inequalities and equalities below hold P-almost surely only, except first one):

· $\mathbb{E}(\mathbb{E}(X|\mathcal{B})) = \mathbb{E}(X)$,

· X $\mathcal{B}$-measurable $\Rightarrow \mathbb{E}(X|\mathcal{B}) = X$,

· X independent of $\mathcal{B} \Rightarrow \mathbb{E}(X|\mathcal{B}) = \mathbb{E}(X)$,

· for $Y \in L^1$ $\mathcal{B}$-measurable s.t. $XY \in L^1$, $\mathbb{E}(XY|\mathcal{B}) = Y\mathbb{E}(X|\mathcal{B})$,

· for $X, Y \in L^2$, if Y $\mathcal{B}$-measurable, then $\mathbb{E}(XY|\mathcal{B}) = Y\mathbb{E}(X|\mathcal{B})$

· $\mathcal{A}$ sub-σ-algebra of $\mathcal{B} \Rightarrow \mathbb{E}(\mathbb{E}(X|\mathcal{B})|\mathcal{A}) = \mathbb{E}(\mathbb{E}(X|\mathcal{A})|\mathcal{B}) = \mathbb{E}(X|\mathcal{A})$ ("tower property")

· positivity: $X \geq 0 \Rightarrow \mathbb{E}(X|\mathcal{B}) \geq 0$,

then if $X, Y \in L^1$, $X \leq Y \Rightarrow \mathbb{E}(X|\mathcal{B}) \leq \mathbb{E}(Y|\mathcal{B})$.

See also exercise 11.

• **Martingale**:

$(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ filtered probability space.

1. In discrete-time filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$

Def: $(\mathcal{F}_n)$**-martingale** in discrete-time: $(M_n)_{n\in\mathbb{N}}$ $(\mathcal{F}_n)_{n\in\mathbb{N}}$-adapted process with values in $\mathbb{R}$, s.t.

$\forall n \in \mathbb{N}$, $M_n \in L^1(\Omega, \mathcal{F}, P)$ and $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$ P-a.s. (depends on the probability P).

Prop: $*$ (M_n) $(\mathcal{F}_n)$-martingale then $\forall m \geq n$, $\mathbb{E}(M_m|\mathcal{F}_n) = M_n$ P-a.s. (from the "tower property").

$*$ $\mathbb{E}(M_n)$ is constant (cf $\mathbb{E}(\mathbb{E}(X|\mathcal{B}) = \mathbb{E}(X))$.

martingale = mathematical object with good properties (applications in finance and in game theory)

Example of use (tutorial):

$(X_n)_{n\in\mathbb{N}^*}$ sequence of independent r.v. in L^2 such that $\forall n \in \mathbb{N}^*$, $\mathbb{E}(X_n) = 0$ (fair play).

$\mathcal{F}_n$ generated by $\{X_1, ..., X_n\}$ ($(\mathcal{F}_n)$ = "natural filtration").

$M_n = X_1 + ... + X_n$ (gain after n games). $(M_n)_{n\in\mathbb{N}^*}$ is an $(\mathcal{F}_n)$-martingale.

Then expected gain after n games: $\mathbb{E}(M_n) = \mathbb{E}(M_1) = 0$.

"Strategy": we can stop at a given date, ex: after gain G
ie replace n by a r.v. function of the obtained "gain".

Def: $(\mathcal{F}_n)$-stopping time: $\tau : (\Omega, \mathcal{F}, P) \mapsto \mathbb{N} \cup \{\infty\}$ such that: $\forall n \in \mathbb{N}$, $\{\tau = n\} \in \mathcal{F}_n$.

It can be proved that: $\mathbb{E}(M_\tau) = \mathbb{E}(M_1) = 0$, for any bounded stopping time τ (see tutorial): i.e. if the player decides to play until a date defined by its gain, its expected gain is still 0.

2. In continuous time filtration $(\mathcal{F}_t)_{t\geq 0}$

Def: $(\mathcal{F}_t)_{t\geq 0}$**-martingale**: $(M_t)_{t\geq 0}$ $(\mathcal{F}_t)_{t\geq 0}$-adapted process with values in $\mathbb{R}$ such that $\forall t \geq 0,\ M_t \in L^1(\Omega, \mathcal{F}, P)$ and if $\forall s \leq t,\ \mathbb{E}(M_t|\mathcal{F}_s) = M_s$ P-a.s.

Ex: <u>For any $X \in L^1(\Omega, \mathcal{F}, P)$, $(\mathbb{E}(X|\mathcal{F}_t))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale</u> (same in discrete time).

Proof: $(\mathbb{E}(X|\mathcal{F}_t))_{t\geq 0}$ obviously $(\mathcal{F}_t)$-adapted.
$|\mathbb{E}(X|\mathcal{B})| \leq \mathbb{E}(|X|\ |\mathcal{B})$ (from positivity: X and $-X \leq |X|$ or using Jensen with $\varphi(x) = |x|$).
$\mathbb{E}(|X|\ |\mathcal{B})$ is integrable then $|\mathbb{E}(X|\mathcal{B})|$ is integrable.

For $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$ thus $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(X|\mathcal{F}_s)$.

See classical exercices 13. to 15.

Chapter V. Option pricing in discrete time: general N periods model, with d risky assets

I. Market model

A discrete-time financial model is built on a finite probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_n)_{0\leq n\leq N}$. The horizon N will correspond to the maturity of the options.
Again, for $n \leq N$, $\mathcal{F}_n$ is the σ-algebra of events up to time n.
We will always assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F} = \mathcal{P}(\Omega)$, and $\forall \omega \in \Omega$, $P(\{\omega\}) > 0$.

The market consists in $(d+1)$ financial assets, whose prices at time n are given by the non-negative random variables $S_n^0, S_n^1, ...S_n^d$, $\mathcal{F}_n$-measurable (investors know past and present prices but obviously not the future ones).
The vector $\mathcal{S}_n = (S_n^0, S_n^1, ...S_n^d)$ is the vector of prices at time n, $(\mathcal{S}_n)_{0\leq n\leq N}$ is $(\mathcal{F}_n)_{0\leq n\leq N}$-adapted.
The asset 0 is the risk-free asset, its return over one period is constant and equal to r, taking $S_0^0 = 1$, we get $S_n^0 = (1+r)^n$ or, more conveniently (then r does not depend on n), $e^{rn\Delta t}$ after n periods, where Δt is the length of 1 period.
The coefficient $\frac{1}{S_n^0}$ corresponds to the discount factor between time 0 and time n.
Remark: even if the interest rate was random, asset 0 would be considered as risk-free as S_n^0 is known (unique value) as soon as $r_k, k \leq n$ is known.
The assets 1 to d are risky assets. We will use below their discounted prices $\frac{S_n^i}{S_n^0}$.
We do the usual technical assumptions on the market regarding transaction costs, taxes, short selling allowed - including for the risk-free asset -, assets divisible ("frictionless market").

II. Strategies

A portfolio strategy is defined by a stochastic process $\Theta = ((\theta_n^0, \theta_n^1, ...\theta_n^d))_{0\leq n\leq N}$, where $\theta_n^i \in \mathbb{R}$ denotes the quantity of asset i in the portfolio at time n.
This process is supposed to be predictable, i.e., Θ_0 is $\mathcal{F}_0$-measurable, and, for $1 \leq n \leq N$, Θ_n is $\mathcal{F}_{n-1}$-measurable.
This assumption means that the positions in the portfolio at time n are decided with respect to the information available at time $n-1$ and kept until time n when new quotes are available for the assets.
The value of the portfolio at time n is the scalar product $V_n^\Theta = \Theta_n \cdot \mathcal{S}_n = \sum_{i=0}^{d} \theta_n^i S_n^i$.

We define the notion of self-financing portfolio strategy, meaning that at any time n, once the new prices $S_n^0, S_n^1, ...S_n^d$ are quoted, the investor readjusts (or rebalances) his position from Θ_n to Θ_{n+1} without bringing or consuming any wealth external to the portfolio.
Def: the portfolio strategy Θ is said to be self-financing iff $\forall 0 \leq n \leq N-1$, $\Theta_n \cdot \mathcal{S}_n = \Theta_{n+1} \cdot \mathcal{S}_n$.
The self-financing condition at time n, $\Theta_n \cdot \mathcal{S}_n = \Theta_{n+1} \cdot \mathcal{S}_n$, is equivalent to:

$$\Theta_{n+1} \cdot \mathcal{S}_{n+1} - \Theta_n \cdot \mathcal{S}_n = \Theta_{n+1} \cdot (\mathcal{S}_{n+1} - \mathcal{S}_n) \quad \text{i.e.} \quad V_{n+1}^\Theta - V_n^\Theta = \Theta_{n+1} \cdot (\mathcal{S}_{n+1} - \mathcal{S}_n)$$

Then the variation of value of the portfolio between time n and time $n+1$ is due to the assets price moves only.

Note: this notion is obviously useless when one period only is considered like when deriving forward prices, the call-put parity relationship, or when replicating the option in the 1-period binomial model: the portfolios involved in our arguments were static strategies hence obviously self-financing strategies, it is useful for dynamic strategies only.

We write the self-financing condition on the discounted prices w.r.t. time 0:
Y_n being an asset price (or a vector of asset prices) at time n, let $\tilde{Y}_n = \frac{Y_n}{S_n^0}$ (e.g. $e^{-rn\Delta t} Y_n$).
Ex: $\tilde{S}_n^0$ is constant, equal to 1.

For any n, $\Theta_n \cdot \mathcal{S}_n = \Theta_{n+1} \cdot \mathcal{S}_n \Longleftrightarrow \Theta_n \cdot \tilde{\mathcal{S}}_n = \Theta_{n+1} \cdot \tilde{\mathcal{S}}_n$

$\Longleftrightarrow \Theta_{n+1} \cdot \tilde{\mathcal{S}}_{n+1} - \Theta_n \cdot \tilde{\mathcal{S}}_n = \Theta_{n+1} \cdot [\tilde{\mathcal{S}}_{n+1} - \tilde{\mathcal{S}}_n]$ ie $\tilde{V}_{n+1}^{\Theta} - \tilde{V}_n^{\Theta} = \Theta_{n+1} \cdot (\tilde{\mathcal{S}}_{n+1} - \tilde{\mathcal{S}}_n)$.

Therefore, the portfolio strategy Θ is self-financing iff:

$$\forall 1 \le n \le N, \quad \tilde{V}_n^{\Theta} = \tilde{V}_0^{\Theta} + \sum_{k=0}^{n-1} \Theta_{k+1} \cdot (\tilde{\mathcal{S}}_{k+1} - \tilde{\mathcal{S}}_k)$$

Knowing $\tilde{S}_k^0 = 1$ for any k, we can see that a self-financing strategy is determined by its initial value V_0^{Θ} $(= \tilde{V}_0^{\Theta})$ and the quantity of risky assets at any time (the proceeds of the trading in the risky assets are put in cash).

III. No arbitrage opportunity property

Short-selling and borrowing are allowed, but we add a contraint on the sign of the portfolio's values at all times:

Definition: A strategy Θ is admissible if it is self-financing and if $V_n^{\Theta} \ge 0$ for any $n \in \{0, 1, ..., N\}$.
i.e. the investor must be able to pay back his debts (in risk-free or risky asset) at any time.

In this multi-periods framework, the notion of arbitrage opportunity is formalised as follows:
Def: an arbitrage opportunity is:
an admissible strategy with zero initial value and non-zero final value.
Hence an arbitrage opportunity is a a self-financing strategy such that:

$$V_0^{\Theta} = 0, \quad \forall n \le N,\ V_n^{\Theta} \ge 0 \text{ (including at time } N), \quad \text{and } P(V_N^{\Theta} > 0) > 0.$$

> Th1: **The market is without arbitrage** (NAO property, or "viable market") **iff there exists a probability** P^* equivalent to P such that **the discounted prices of the** $d+1$ **basis assets are martingales under** P^*.
>
> P^* is called an *Equivalent Martingale Measure* (EMM) or a *Risk-neutral probability* as we will have the risk-neutral valuation under this probability.

Reminder: two probability measures P_1 and P_2 are equivalent if and only if for any event $A \in \mathcal{F}$, $P_1(A) = 0$ iff $P_2(A) = 0$. Here, P^* equivalent to P means that, for any $\omega \in \Omega$, $P^{\star}(\{\omega\}) > 0$.

Lemma:

> if there exists an EMM P^*, then, for any self-financing strategy Θ, $(\tilde{V}_n^{\Theta})$ is a martingale under P^*.

Indeed, we have: $\forall n \le N$, $\tilde{V}_n^{\Theta} = \tilde{V}_0^{\Theta} + \sum_{k=0}^{n-1} \Theta_{k+1} \cdot (\tilde{S}_{k+1} - \tilde{S}_k)$.

Thus (exercise 15.), $(\tilde{V}_n^\Theta)$ is a martingale under P^* (sum of d martingale transforms: $(\tilde{S}_n^i)$ by (θ_{n+1}^i), for $i = 1, ..., d$, with all the θ_n^i bounded as Ω is finite).

Proof of Th1: $\boxed{\Leftarrow}$

$\mathbb{E}^*$ being the expectation under probability measure P^*, for any self-financing strategy Θ, we have $\tilde{V}_0^\Theta = \mathbb{E}^*(\tilde{V}_N^\Theta)$ then the value at time 0 of the strategy is: $\boxed{V_0^\Theta = \frac{1}{S_N^0}\mathbb{E}^*(V_N^\Theta)}$.

If the strategy is admissible and its initial value is zero, then $\mathbb{E}^*(V_N^\Theta) = 0$, with $V_N^\Theta \geq 0$ P a.s. then P^* a.s.. Hence $V_N^\Theta = 0$ since $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$ (like for P). Hence there is no arbitrage opportunity.

(otherwise stated: when an EMM P^* exists, $[V_N^\Theta \geq 0$ P a.s. and $P(V_N^\Theta > 0) > 0] \Rightarrow$ same for P^*, then $V_0^\Theta = \mathbb{E}^*(V_N^\Theta) > 0$, hence Θ cannot be an AO).

$\boxed{\Rightarrow}$ The proof of the converse implication is more tricky (the probability P^* is built using the convex sets separation theorem). We will build P^* in the example of the binomial tree only.

IV. Complete markets and option pricing

A European option with maturity T is given by its payoff at time T (after N periods):

$F_N = f(\mathcal{S}_1, ..., \mathcal{S}_N) \geq 0$, with f measurable (borelian). F_N is $\mathcal{F}_N$-measurable.

Note that the filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$ will be generally the natural filtration of the prices process.
Reminder: X is $\sigma(\mathcal{S}_1, ..., \mathcal{S}_N)$-measurable iff $X = f(\mathcal{S}_1, ..., \mathcal{S}_N)$ with f measurable.

For instance, $F_N = (S_N^1 - K)^+$ for a call on the asset 1 with strike price K. In this example, F_N is a function of S_N only. There are some options dependent on the whole path of the underlying asset, i.e. F_N is a function of $S_0, S_1, ..., S_N$, they are said to be path-dependent (e.g.: Asian options involve the average of the stock prices observed during a certain period of time before maturity).

Def: an option with payoff F_N $\mathcal{F}_N$-measurable is said to be replicable (or attainable) if there exists an admissible portfolio strategy Θ whose value at time N is: $V_N^\Theta = F_N$ P a.s. (the corresponding portfolio is called a replicating portfolio).

Remark: In a market without arbitrage, we just need to find a self-financing strategy worth F_N at maturity to say that F_N is replicable (as $F_N \geq 0$). Indeed, as an EMM exists, this strategy will always be admissible:

if Θ is a self-financing strategy and if P^* is a probability equivalent to P under which discounted prices are martingales, then $(\tilde{V}_n^\Theta)$ is also a martingale under P^* (like in proof of Th1). Hence, for $n \leq N$, $\tilde{V}_n^\Theta = \mathbb{E}^*(\tilde{V}_N^\Theta | \mathcal{F}_n)$. Therefore, if $V_N^\Theta \geq 0$ P a.s. (which is the case for an option), then, for any n, $\tilde{V}_n^\Theta \geq 0$ P a.s., hence the strategy is admissible.
(retain: Θ self-financing with $V_N^\Theta \geq 0$ P^* a.s. $\Rightarrow$ Θ admissible).

Def: the market is complete if any option written on the risky assets is replicable.

Note: to assume that a financial market is complete is a rather restrictive assumption that does not have such a clear economic justification as the no-arbitrage assumption. The interest of complete markets is that it allows us to derive a simple theory of derivatives pricing and hedging.

The Cox-Ross-Rubinstein model is a very simple example of complete market modeling.
The following theorem gives a precise characterisation of complete markets without arbitrage.

> Th2: **A market without arbitrage is complete** if and only if there exists **a unique probability P^* equivalent to P under which discounted prices are martingales**.

The probability P^* will appear to be the computing tool whereby we can derive closed-form pricing formulae and hedging strategies.

Proof: $\boxed{\Rightarrow}$ We assume that the market is without arbitrage and complete. Then there exists some EMM (Th1).
Consider P_1 and P_2 two probability measures under which discounted prices are martingales. We want to prove they are equal.

For any $\forall B \in \mathcal{F}_N$, $1\!\!1_B$ is $\mathcal{F}_N$-measurable, then replicable: can be written as $1\!\!1_B = V_N^\Theta$, where Θ is an admissible strategy that replicates the payoff $1\!\!1_B$. Since Θ is self-financing, we know that $\frac{1\!\!1_B}{S_N^0} = \tilde{V}_N^\Theta = V_0^\Theta + \sum_{k=0}^{n-1} \Theta_{k+1} \cdot (\tilde{\mathbb{S}}_{k+1} - \tilde{\mathbb{S}}_k)$.

$(\tilde{V}_n^\Theta)$ is a martingale under both P_1 and P_2. It follows that, for $i = 1$ or 2, $\mathbb{E}_i(\tilde{V}_N^\Theta) = \mathbb{E}_i(V_0^\Theta) = V_0^\Theta$, the last equality coming from the fact that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Then $P_i(B) = \mathbb{E}_i(1\!\!1_B) = \mathbb{E}_i(S_N^0 \tilde{V}_N^\Theta) = S_N^0 V_0^\Theta$, where Θ is a replicating strategy. Then $P_i(B)$ is uniquely determined (if two strategies exist, they must have same value at 0, by NAO).
Therefore $P_1(B) = P_2(B)$ and since B is arbitrary, $P_1 = P_2$ on the whole σ-algebra $\mathcal{F}_N$ (assumed to be equal to $\mathcal{F}$).

$\boxed{\Leftarrow}$ By contradiction (admitted): if there exists a random variable $F_N \geq 0$, $\mathcal{F}_N$-measurable, which is not replicable, it is possible to build another probability equivalent to P under which discounted prices are martingales.

Conclusion: pricing and hedging derivatives in complete markets

The market is assumed to be without arbitrage and complete. We denote by P^* the unique probability under which the discounted prices of financial assets are martingales. Let F_N be an $\mathcal{F}_N$-measurable, non-negative random variable and Θ be an admissible strategy replicating the derivative hence defined, i.e. $V_N^\Theta = F_N$.

The sequence $(\tilde{V}_n^\Theta)$ is a P^*-martingale, then for $n = 0, 1, ..., N$, $\tilde{V}_n^\Theta = \mathbb{E}^*(\tilde{V}_N^\Theta | \mathcal{F}_n)$.
In particular, $V_0^\Theta = \tilde{V}_0^\Theta = \mathbb{E}^*(\frac{F_N}{S_N^0})$ (as stated in the proof of theorem 1).
We get the same results as in the one-period model: $\boxed{V_0^\Theta = \frac{1}{S_N^0}\mathbb{E}^*(F_N)}$.

Hence we have the risk-neutral valuation under P^* like in the one-period model.
P^* is called again the risk-neutral probability.

Here again, the computation of the option price does not require the knowledge of the probability P, but of P^* only.

Note that the value of an admissible strategy replicating F_N is completely determined by F_N

at any time. V_n^Θ corresponds to the price of the option: that is the wealth needed at time n to replicate F_N at time N by following the strategy Θ.
If, at time 0, an investor sells the option for $\mathbb{E}^*(\frac{F_N}{S_N^0})$, he can follow a replicating strategy Θ in order to generate an amount F_N at time N. His portfolio at time $n = 1^-, ..., N^-$ is: $\begin{cases} -1 \text{ option} \\ \theta_n^1 \text{ assets } 1 \\ ... \\ \theta_n^d \text{ assets } d \end{cases}$

This strategy allows him to be perfectly hedged.

V. Example: N periods binomial model (Cox-Ross-Rubinstein)

1. Model

Pricing of an option with maturity T on a risky asset. The U.A. price is modeled in discrete time.
We divide $[0, T]$ in N sub-periods with duration $\Delta t = \frac{T}{N}$.
We assume that the risky asset price changes only at the discrete times Δt, $2\Delta t$,..., $(N-1)\Delta t$ (dynamic model in discrete time).

2 basis assets:

- a risk-free asset whose price is 1 at time 0. Its return is r (continuous interest rate chosen as we will take N large later). Then this asset is worth $S_n^0 = e^{rn\Delta t}$ after n periods,
- a non-dividend-paying stock whose price is S_n after n periods (corresponds to time $n\frac{T}{N}$). The initial price S_0 is denoted by S.

The most elementary model is to suppose that, given S_n, S_{n+1} has two possible values.
The price process is then described by a tree. For programming purpose, we need a recombining tree ($n + 1$ possible values at time n, 2^n for the non-recombining tree).

We will assume that, with $0 < d < u$, at each period: $S_{n+1} = uS_n$ or dS_n, then the tree is recombining.

$$\begin{array}{ccc} & & Su^2 \\ & Su & \\ S & & Sdu \\ & Sd & \\ & & Sd^2 \end{array}$$

It is equivalent to assume that the rate of return on the asset over each period of time interval Δt is worth $u - 1$ or $d - 1$.

To calculate the option price at the initial node of the tree we will apply repeatedly the principles established for one period: the option will be replicated by a portfolio rebalanced at each period. Then the dynamics of the U.A. price matters: the moves of the U.A. give the relevant "events" (no need to consider other assets).

Definition of Ω: an elementary event ω corresponds to a path in the tree (observation of the risky asset prices $S_1, ..., S_N$, or of $(X_1, ..., X_N)$ where $X_n = \frac{S_n}{S_{n-1}}$, e.g.: $\omega = (u, d, d, ..., u)$).
$\Omega = \{d, u\}^N$: each N-tuple represents the successive values of X_n.

Information at time n: path until the current node. It corresponds to the natural filtration of the U.A. price process. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geq 1$, $\mathcal{F}_n = \sigma(S_1, ..., S_n)$.

We assume $\mathcal{F}_N = \mathcal{F}$, ie no other uncertainty than the risky asset price.

No assumption on the real probability of each path, as the probability of d and u was not involved in the one-period option price formula. In particular, the probabilities of each move on the sub-periods are not necessarily constant in the tree.

Note: the knowledge of a probability on $\mathcal{F}$ is equivalent to the knowledge of the law of $(X_1, ..., X_N)$.

The market is assumed frictionless.

2. Self-financing strategies (same as in the general case)

A portfolio strategy is defined by a predictable stochastic process $\Theta = ((\theta_n^0, \theta_n))_{0 \leq n \leq N}$:
at time n, there are θ_n^0 risk-free asset, θ_n risky asset, with θ_n^0, θ_n $\mathcal{F}_{n-1}$-measurable for $1 \leq n \leq N$.

The value of the portfolio at time n is $V_n^\Theta = \Theta_n \cdot \mathcal{S}_n$ (scalar product), with $\mathcal{S}_n = (S_n^0, S_n)$.

Y_n being an asset price at time n, the discounted price is computed as $\tilde{Y}_n = e^{-rn\Delta t} Y_n$.

The portfolio strategy Θ is self-financing iff $\forall 0 \leq n \leq N-1$, $\Theta_n \cdot \tilde{\mathcal{S}}_n = \Theta_{n+1} \cdot \tilde{\mathcal{S}}_n$ or

iff $\forall 1 \leq n \leq N$, $\quad \tilde{V}_n^\Theta = \tilde{V}_0^\Theta + \sum_{k=0}^{n-1} \theta_{k+1}(\tilde{S}_{k+1} - \tilde{S}_k) \qquad$ (indeed $\tilde{S}_{k+1}^0 = \tilde{S}_k^0$ for any k).

i.e. a self-financing strategy is determined by its initial value V_0^Θ and its quantity of risky asset at any time (the proceeds of the trading in the risky asset are put in cash).

3. Risk-neutral probability

We assume that the market is without arbitrage, then there exists a probability P^* equivalent to P such that $(\tilde{\mathcal{S}}_n)_{0 \leq n \leq N}$ is a martingale under P^* ($(\tilde{S}_n^0)_{0 \leq n \leq N}$ is constant).

We have for $0 \leq n \leq N-1$, $\mathbb{E}^*(\tilde{S}_{n+1} | \mathcal{F}_n) = \tilde{S}_n$, with $\tilde{S}_{n+1} = e^{-r\Delta t} \tilde{S}_n X_{n+1}$ and $\tilde{S}_n$ $\mathcal{F}_n$-measurable.
Therefore $\mathbb{E}^*(X_{n+1} | \mathcal{F}_n) = e^{r\Delta t}$ and $\mathbb{E}^*(X_{n+1}) = e^{r\Delta t}$.
It gives $P^*(X_{n+1} = u)u + [1 - P^*(X_{n+1} = u)]d = e^{r\Delta t}$ hence $P^*(X_{n+1} = u) = \frac{e^{r\Delta t} - d}{u - d}$.

We get $\boxed{\forall 0 \leq n \leq N-1, \; P^*(X_{n+1} = u) = p^*, \text{ where } p^* = \frac{e^{r\Delta t} - d}{u - d}}$.

The condition $P^*(X_{n+1} = u) \in]0, 1[$ (as P^* is equivalent to P) implies $d < e^{r\Delta t} < u$, which makes sense as it is needed to avoid the existence of arbitrage opportunities on the sub-periods (which would lead to AO between 0 and T, using the risk-free asset). And we find on each mesh the same coefficient $p^* = \frac{e^{r\Delta t} - d}{u - d}$ (corresponding to what we found in the 1-period tree ($\frac{S(1+r) - S^d}{S^u - S^d}$).

We build the tree with these new values p^* (for the up move) and $1 - p^*$ (for the down move) at each step. This leads to the following definition: let P^* the probability defined by:

for a given path ω, $P^*(\omega) = p^{*k}(1 - p^*)^{N-k}$, where k is the number of moves $\times u$ in the path.
P^* is equivalent to P.

We assume that the successive moves are independent under P^*.

Under P^*, the r.v. (X_n) are i.i.d. with law: $X_n = \begin{cases} u \text{ with probability p}^* \\ d \text{ with probability } 1 - p^* \end{cases}$.

Then for $0 \leq n \leq N-1$, $\mathbb{E}^*(\tilde{S}_{n+1}|\mathcal{F}_n) = e^{-r\Delta t}\tilde{S}_n\mathbb{E}^*(X_{n+1}|\mathcal{F}_n) = e^{-r\Delta t}\tilde{S}_n\mathbb{E}^*(X_{n+1}) = \tilde{S}_n$, thus $(\tilde{S}_n)_{0\leq n\leq N}$ is a martingale under P^*. And P^* is the unique probability equivalent to P having that property.

4. Option pricing

Consequence: the market is complete. Then, as in the general model, to price any derivative product given by its payoff $F_N = f(S_1, ..., S_N) \geq 0$ at time T (after N periods), we just need to replicate it with a self-financing strategy Θ. We have that $(\tilde{V}_n^\Theta)_{0\leq n\leq N}$ is a martingale under P^*, and then, the value at time 0 of the derivative is:

$$F_0 = V_0^\Theta = e^{-rT}\mathbb{E}^*(V_N^\Theta) = e^{-rT}\mathbb{E}^*(F_N).$$

Note that for a standard option (non path-dependent), F_N is a function of S_N only. We compute the price using the law of S_N under P^*, given by:

$P^*(S_N = Su^k d^{N-k}) = C_N^k p^{*k}(1-p^*)^{N-k}$ for $0 \leq k \leq N$.

The replicating strategy is built through a recursive backward procedure: starting at the expiration date and working backwards, using the result on one period (=what is done by programs).

On any mesh between time $N-1$ and N, we can replicate the payoff at N (see one-period model). The replicating strategy is then computed by backward induction. The part in cash is uniquely determined (self-financing strategy).

Note that the discounted price of the option is a martingale under P^* (indeed it is $(\tilde{V}_n^\Theta)$).

Th: the option price at time $t = n\Delta t$ is given by: $\boxed{F_n = e^{-r(T-t)}\mathbb{E}^*(F_N|\mathcal{F}_n)}$.

expectation under P^* of the future value, discounted at risk-free rate.

I.e.: replacing P by P^*, we can compute prices as if the investors were risk-neutral.

P^* is called the **risk-neutral probability** and P real or historical probability (ie as anticipated).

We still have the principle of risk-neutral valuation, like in the 1-period model:

the **option price is equal to its expected payoff in a risk-neutral world** (the tree is built under P^*), **discounted at the risk-free rate.**

The real probability does not appear in the price formula.

5. Delta hedging

The price is obtained by replicating the option.
At each node, by holding a given number of U.A., we know how to replicate the option on the next period.
This number depends on the date n and on the state of the world at that date. It is called the delta.

For $n \in \{1, ..., N\}$, let S_n^k be the risky asset price at time n after k up moves, and F_n^k the corresponding option price.

At time $n-1$ after k up moves:

$$S_{n-1}^k \nearrow S_n^{k+1}, \quad S_{n-1}^k \searrow S_n^k \qquad\qquad F_{n-1}^k \nearrow F_n^{k+1}, \quad F_{n-1}^k \searrow F_n^k$$

Prices of time $n-1$ are known, then, for the U.A. price or for the option price, the 2 possible values are known as well. We compute the quantity of U.A. to be held until time n (included) as: $\Delta_n^k = \dfrac{F_n^{k+1} - F_n^k}{S_n^{k+1} - S_n^k}$ (value known at $n-1$). It corresponds to the coefficient θ_n of the above replicating strategy Θ. The other coefficient θ_n^0 is chosen in order to have a self-financing strategy Θ. The strategy Θ is determined by $(\Delta_1, \Delta_2, ..., \Delta_n)$.
The predictability is clear from above computation.

The option can then be replicated by holding a portfolio that contains delta U.A. and some cash, which implies buying and selling continuously, to maintain the correct quantity of U.A. (self-financing portfolio strategy).
This portfolio is worth exactly as the option at any time / state of the world.
The option price does not depend on the real probability because it is a replication price.

The bank that sells the option is exposed to a risk: it has to pay F_N at T, potentially large amount. Generally, it does not speculate: as soon as the option is sold, with the premium F, it builds the replicating portfolio, called *hedging portfolio*: the premium is invested in that portfolio, then the correct quantity of U.A. is maintained during the whole life of the option (dynamic hedge).
For the bank, the final outcome is then null: the risk is cancelled.

The bank becomes indifferent to the U.A. price variations (up or down moves), indeed:

$$\text{the portfolio} \begin{cases} -1 \text{ option} \\ \Delta_n^k \text{ U.A.} \end{cases} \text{is risk-free between } n-1 \text{ and } n.$$

which is the <u>main property of the delta</u>.

This trading is equivalent to building (manufacturing) the option.
In fact, the bank sells at price $F+$ margin and it is this service that the buyer pays with the margin. Moreover: theoretically, there is a perfect replication, but not in reality, then part of the margin exists to cover the residual risk.

So the writer of the option (for example a bank that has sold a put on a currency) can cover his position by purchasing the underlying so that the loss in the short position in the option is offset by the long position in the stock. This construction of a risk-free hedge is referred to as *delta hedging.*

Chapter VI. Option pricing in continuous time: Black-Scholes model

More complex.

Cox, Ross, Rubinstein (1979) Option pricing: a simplified approach. J of Financial Economics.
Abstract: "*This paper presents a simple discrete-time model for valuing options. The fundamental economic principles of option pricing by arbitrage methods are particularly clear in this setting. Its development requires only elementary mathematics, yet it contains as a special limiting case the celebrated Black-Scholes model, which has previously been derived only by much more difficult methods.*"

Mathematical models of financial derivatives, Y.K. Kwok, Springer (2nd ed 2008) (pages 1-130).

Baxter, M. and Rennie, A. Financial calculus. Cambridge University Press, 1996.

I. Brownian motion and Ito processes

$(\Omega, \mathcal{F}, P)$ probability space.

The risky asset price is now modeled in continuous time, and taking continuous values.
(real world: discrete values, ex: cents, and discrete variations, only when market open...).

Stochastic process: sequence of r.v. $(X_t)_{t\in\mathbb{R}^+}$. For a given ω, $t \mapsto X_t(\omega)$ = path.

Def: $(X_t)_{t\geq 0}$ is said to be continuous, or to have continuous paths
iff P-as in ω, $t \mapsto X_t(\omega)$ is continuous.

1. Brownian motion

In 1828, the botanist Robert Brown observed the motion of a pollen particle suspended in water under a microscope and described it as a continuous jittery motion: the particles moved in an irregular, random fashion. Ceaseless (incessant) changes of direction. The phenomenon is now known as Brownian motion.
1905: Einstein argued that the jiggling of the pollen grains seen in Brownian motion was due to molecules of water hitting the tiny pollen grains.
1923: Wiener (and Einstein) proposes a rigorous mathematical study of Brownian motion (in particular proves the existence).

> <u>Existence theorem</u> (admitted): $(\Omega, \mathcal{F}, P)$ probability space.
> There exists a stochastic process $(B_t)_{t\geq 0}$ with real values and continuous paths (P a.s. $t \mapsto B_t$ is continuous), called <u>Brownian motion</u> s.t. $B_0 = 0$ and:
> $$\forall 0 \leq s \leq t, \begin{cases} B_t - B_s \sim \mathcal{N}(0, t-s) \\ B_t - B_s \text{ is independent of } \sigma(B_u, u \leq s) \text{ (the past)} \end{cases}$$

Note: $\forall t \geq 0$, $B_t \sim \mathcal{N}(0, t)$ (as it is $B_t - B_0$).

In fact, for any division in sub-intervals, B_t is the sum of some independent Normal r.v..
Consistency: $t_1 < t_2 < t_3$,
$B_{t_2} - B_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$, $B_{t_3} - B_{t_2} \sim \mathcal{N}(0, t_3 - t_2)$ and independent r.v.;
allows indeed $B_{t_3} - B_{t_1} \sim \mathcal{N}(0, t_3 - t_1)$.

Properties: Let (B_t) be Brownian motion. Let $\mathcal{F}_t = \sigma(B_s, s \le t)$.

$(B_t)_{t\ge 0}$, $(B_t^2 - t)_{t\ge 0}$ and, for $\lambda \in \mathbb{R}$, $(e^{\lambda B_t - \frac{\lambda^2}{2}t})_{t\ge 0}$ are $(\mathcal{F}_t)$-martingales with continuous paths.

Proof (tutorial): 2 main properties of the conditional expectation are used: for $X \in L^1$,

· X $\mathcal{B}$-measurable $\Rightarrow \mathbb{E}\,(X|\mathcal{B}) = X$

· X independent of $\mathcal{B} \Rightarrow \mathbb{E}\,(X|\mathcal{B}) = \mathbb{E}\,(X)$

The 3 processes are obviously $(\mathcal{F}_t)$-adapted.

$\forall t \ge 0$, $B_t \sim \mathcal{N}(0,t)$ therefore the r.v. are integrable: Gaussian variables have moments of any order and $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow \mathbb{E}\,(e^{\lambda X}) = e^{\lambda m + \frac{\lambda^2}{2}\sigma^2}$

(see characteristic function, generating function, or Laplace transform of the Gaussian law)

then $\mathbb{E}\,(e^{\lambda B_t}) = e^{\frac{\lambda^2}{2}t}$ or $\underline{\mathbb{E}\,(e^{\lambda B_t - \frac{\lambda^2}{2}t}) = 1}$.

Let $s \le t$. We use below: $(*)$ $B_t - B_s$ is independent of $\mathcal{F}_s$ and $(**)$ $B_t - B_s \sim \mathcal{N}(0, t-s)$.

· $\mathbb{E}\,(B_t|\mathcal{F}_s) = \mathbb{E}\,(B_t - B_s + B_s|\mathcal{F}_s) \stackrel{*}{=} \mathbb{E}\,(B_t - B_s) + B_s \stackrel{**}{=} B_s$ (uses B_s $\mathcal{F}_s$-measurable),

· $\mathbb{E}\,((B_t - B_s)^2|\mathcal{F}_s) \stackrel{*}{=} \mathbb{E}\,((B_t - B_s)^2) \stackrel{**}{=} t - s$ and

$= \mathbb{E}\,(B_t^2|\mathcal{F}_s) - B_s^2$ from the martingale property of (B_t) and $\mathcal{F}_s$-measurability of B_s.

Then $\mathbb{E}\,(B_t^2 - t|\mathcal{F}_s) = B_s^2 - s$.

· $\mathbb{E}\,(e^{\lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)}|\mathcal{F}_s) \stackrel{*}{=} \mathbb{E}\,(e^{\lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)}) \stackrel{**}{=} 1$.

$e^{\lambda B_s - \frac{\lambda^2}{2}s}$ is $\mathcal{F}_s$-measurable, then $\mathbb{E}\,(e^{\lambda B_t - \frac{\lambda^2}{2}t}|\mathcal{F}_s) = e^{\lambda B_s - \frac{\lambda^2}{2}s}$.

Note that $\underline{Cov(B_s, B_t) = s \wedge t}$ indeed if $s \le t$, $\mathbb{E}\,(B_s B_t) = \mathbb{E}\,(B_s \mathbb{E}\,(B_t|\mathcal{F}_s)) = \mathbb{E}\,(B_s^2) = s = s \wedge t$.

The **Markov property** for a process $(X_t)_{t\ge 0}$ on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_{t\ge 0}))$ means that at any given time t (present), its future behavior is independent of the past:

for $h \ge 0$, the law of X_{t+h} (future) depends on X_t only (present) and not on the $X_s, s < t$ (past).

Rigorous definition:

Def: $(X_t)_{t\ge 0}$, an $(\mathcal{F}_t)_{t\ge 0}$-adapted process with values in $\mathbb{R}^k$, is a Markov process with respect to $(\mathcal{F}_t)$ iff for any Borelian bounded function $f : \mathbb{R}^k \to \mathbb{R}$, we have:

$$\forall s \le t, \quad \mathbb{E}\,(f(X_t)|\mathcal{F}_s) = \mathbb{E}\,(f(X_t)|\sigma(X_s))$$

If no filtration is mentioned, it may be assumed to be the natural one generated by $(X_t)_{t\ge 0}$.

In finance: the assets prices are frequently modeled as Markov processes (ex: binomial model).

It is an efficient-market hypothesis (**EMH**): the current price reflects already all the available information (including the information embedded in past prices).

Note that Markovian prices would not be compatible with the possibility of "chartism" (or technical analysis): indeed this assumption implies the same dynamics after S_t, is it after an upward move or after a downward move, while chartists use past values of stock prices (and volume) to try to forecast future stock prices. They try to detect presence of geometric shapes in historical price charts, ex: head-and-shoulders.

According to the weak-form efficient-market hypothesis, such forecasting methods are valueless, since prices follow a random walk or are otherwise essentially unpredictable.

Interpretation of EMH: there are many investors; if the price was forecast to increase with a high probability, everybody would buy, the price would increase instantaneously, deleting the pattern: the current price contains reflects already this anticipation).

Note: in our both models (discrete and continuous), the U.A. price is Markovian and the price of an option depends only on the U.A. price at that date, not on its past values (except obviously for path-dependent options, i.e. whose payoff depends on the past values of the U.A. price, for exemple options involving an average of the U.A. prices).

Th: **The Brownian motion is a Markov process**.

idea: $B_{s+h} = B_s + X_h$, with $X_h \sim \mathcal{N}(0,h)$ independent of $\mathcal{F}_s = \sigma(B_u, u \le s)$.

Proof: $f : \mathbb{R} \to \mathbb{R}$ Borelian bounded function, $s \le t$, we have $\mathbb{E}\left(f(B_t)|\mathcal{F}_s\right) = \mathbb{E}\left(f(B_s + B_t - B_s)|\mathcal{F}_s\right)$ with B_s $\mathcal{F}_s$-measurable and $B_t - B_s$ independent of $\mathcal{F}_s$.

To compute this term, we need the following result:

Lemma: for $\mathcal{B}$ sub-σ-algebra of $\mathcal{F}$,
let X, Y r.v. with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with X $\mathcal{B}$-measurable and Y independent of $\mathcal{B}$.
For any $f : \mathbb{R}^2 \to \mathbb{R}$ Borelian bounded (or non-negative), we have:
$\mathbb{E}\ [f(X,Y)|\mathcal{B}] = \varphi(X)\ P$-as, with φ Borelian, defined on E by $\varphi(x) = \mathbb{E}\ [f(x,Y)]$,
i.e. can be computed as if X was a constant.

This is written: $\quad \mathbb{E}\ [f(X,Y)|\mathcal{B}] = \mathbb{E}\ [f(x,Y)]_{/x=X}$.

Proof of the lemma: by definition, $\varphi(x) = \int_F f(x,y)dP_Y(y)$ where P_Y is the distribution of Y.
The measurability of φ results from the Fubini theorem* (recalled below), as f is bounded.
Then $\varphi(X)$ is $\mathcal{B}$-measurable.
Let Z $\mathcal{B}$-measurable bounded, let $P_{X,Z}$ denote the law of (X,Z).
Y and (X,Z) being independent, we have:
$\mathbf{E}\ [f(X,Y)Z] = \int\int f(x,y)z\ dP_{X,Z}(x,z)dP_Y(y) = \int \left(\int f(x,y)dP_Y(y)\right) z\ dP_{X,Z}(x,z)$
$= \int \varphi(x)z\ dP_{X,Z}(x,z) = \mathbf{E}\ [\varphi(X)Z]$. $\square$

The Fubini theorem* states that it is possible to compute a double integral (assumed to be finite when the integrand is replaced by its absolute value) by using an iterated integral, and that one may switch the order of integration:
$f : (E \times F, \mathcal{E} \otimes \mathcal{F}) \to \mathbb{R}$ measurable and s.t. $\int |f| dP_{X,Y} < \infty$,
then $x \mapsto \int f(x,y)dP_Y(y)$ is measurable and integrable w.r.t. dP_X.

Back to the theorem's proof: we compute $\mathbb{E}\left(f(B_s + B_t - B_s)|\mathcal{F}_s\right)$ as if B_s was constant:

$$\mathbb{E}\left(f(B_t)|\mathcal{F}_s\right) = \mathbb{E}\left(f(x + B_t - B_s)|\mathcal{F}_s\right)_{/x=B_s} = \mathbb{E}\left(f(x + B_t - B_s)\right)_{/x=B_s}$$

$\mathbb{E}\left(f(B_t)|\sigma(B_s)\right)$ is computed on the same way as:

$\mathbb{E}\left(f(x + B_t - B_s)|\sigma(B_s)\right)_{/x=B_s} = \mathbb{E}\left(f(x + B_t - B_s)\right)_{/x=B_s}$, both are then equal.

Quadratic variation of the Brownian motion

Th: For a given $T > 0$, let $t_0 = 0 < t_1 < ... < t_n = T$ a subdivision of $[0, T]$.

We have: $$\sum_{k=0}^{n-1}(B_{t_{k+1}} - B_{t_k})^2 \xrightarrow{L^2} T \quad \text{when} \quad \max_k |t_{k+1} - t_k| \to 0.$$

Proof: see exercise 20.

Note that $\max_k |t_{k+1} - t_k|$ is the length of the longest of the subintervals of the partition, it is called the norm (or step, or mesh) of the partition, or 'subdivision step', sometimes denoted by $|\{t_k\}|$.

The quadratic variation of the Brownian motion (or of any stochastic process) over $[0, T]$ is defined as the **limit in probability** of the sum above, and denoted by $[B, B]_T$).
Here the convergence is in L^2, which implies convergence in probability. We get:

The **quadratic variation of the Brownian motion** over $[0, T]$ is T.

For a class C^1 function f, we have on contrary: $\sum_{k=0}^{n-1}[f(t_{k+1}) - f(t_k)]^2 \longrightarrow 0$ when $|\{t_k\}| \to 0$.

Ie class C^1 implies vanishing of the quadratic variation of the function.
So, for a given ω, how smooth can $t \mapsto B_t(\omega)$ be?

It can be proved that, as soon as a continuous process is of bounded variation on $[0, T]$, then it has a quadratic variation on $[0, T]$ equal to 0 (see Appendix on the EPI).

So, it is not the case for a Brownian motion.

In fact, a Brownian motion has very irregular paths (between any two times, a Gaussian variable is drawn):

for almost every $\omega \in \Omega$, the Brownian motion path $t \mapsto B_t(\omega)$ is continuous,
but for almost every ω, $t \mapsto B_t(\omega)$ is **nowhere differentiable**.

Precisely, the set below contains an event with probability 1:

$$\{\omega \in \Omega \mid \forall t \geq 0, \text{ either } \overline{\lim}_{h \to 0^+} \frac{B_{t+h}(\omega) - B_t(\omega)}{h} = +\infty, \text{ either } \underline{\lim}_{h \to 0^+} \frac{B_{t+h}(\omega) - B_t(\omega)}{h} = -\infty\}$$

(while $\overline{\lim} = \underline{\lim} < +\infty$ is needed to have differentiability at t)

In the option pricing model in continuous time, the instantaneous variation of the U.A. price dS_t will involve the instantaneous variation of the Brownian motion dB_t.
In discrete time the value of a self-financing strategy has been expressed as $\sum_k \theta_{k+1}(S_{k+1} - S_k)$.
In continuous time this is replaced by $\int \theta_t dS_t$, which involves an integral $\int \theta_t dB_t$.

But for a given ω, $t \mapsto B_t(\omega)$ is not of bounded variation, hence we cannot define the integral $\int \theta_t(\omega) dB_t(\omega)$ as a Stieltjes integral.

$\int \theta_t dB_t$ cannot be defined pathwise (i.e. for each ω separately).

Therefore Ito integral = new concept.

We define it on an interval $[0, T]$ (T will be the maturity of the options in the applications).

Problem: defining $\int_0^T H_t dB_t$ with (H_t) stochastic process, cannot be done ω by ω.

Reminder: the Riemann integral is defined first for step functions, then extended to a larger class of functions (the Riemann-integrable functions) by approximation: the integral of a function f is defined to be the limit of the integrals of step functions which converge (in a certain sense) to f.

We do the same to define the Ito integral. Step functions are replaced by simple processes, which are random step functions. The integral is then extended to larger classes of processes by approximation.

• def: (H_t) is a simple process (or elementary process) if for some finite sequence of times $t_0 = 0 < t_1 < ... < t_n = T$, we have: $\forall t \in [0,T]$, P-as in ω,

$$H_t(\omega) = \sum_{k=0}^{n-1} H^k(\omega)\, \mathbb{1}_{[t_k,t_{k+1}[}(t) \text{ where for } 0 \le k \le n-1,\ \underline{H^k \in L^2(\Omega,\mathcal{F},P)} \text{ and is } \underline{\mathcal{F}_{t_k}\text{-measurable}}.$$

To ensure $\int_a^b dB_t = B_b - B_a$, we take the following definition for H simple process as above:

def: $$\int_0^T H_t dB_t = \sum_{k=0}^{n-1} H^k (B_{t_{k+1}} - B_{t_k}).$$

Note that for a simple process, $H_t(\omega)$ is evaluated at the left-hand point of the interval in which t falls. This is a key component in the definition of the stochastic integral and it makes the resulting theory suitable for financial applications. In particular, if we interpret $H_t(\omega)$ as a trading strategy and the stochastic integral as the gains or losses from this trading strategy, then evaluating $H_t(\omega)$ at the left-hand point is equivalent to imposing the non-anticipativity of the trading strategy, a property that we had in discrete time as well.

The function I: $\mathcal{E} = \{\text{simple processes}\} \to \mathrm{L}^2(\Omega)$

$$H \longmapsto I(H) = \int_0^T H_t dB_t \quad \text{satisfies: } \underline{||I(H)||_{L^2(\Omega)} = ||H||_{L^2(\Omega\times]0,T[)}}$$

i.e. it is an **isometry** from $\mathcal{E}$, equipped with norm $L^2(\Omega\times]0,T[, \mathcal{F}\times\mathcal{B}_{]0,T[}, P\times dt)$, to $\underbrace{L^2(\Omega,\mathcal{F},P)}_{\text{complete space}}$.

Proof of $\mathbb{E}\left[\left(\int_0^t H_s dB_s\right)^2\right] = \mathbb{E}\left(\int_0^t H_s^2 ds\right)$ for H simple process: see exercise 23.

• extension by taking limits for other processes:

- first extension to processes $H_.(.)$ measurable, $(\mathcal{F}_t)$ adapted s.t. $\mathbb{E}\left(\int_0^T H_s^2 ds\right) < +\infty$. Indeed, such a process is the limit in $L^2(\Omega\times]0,T[, \mathcal{F}\times\mathcal{B}_{]0,T[}, P\times dt)$ of a sequence of simple processes; $\int_0^T H_s dB_s$ is then defined as the limit in $L^2(\Omega,\mathcal{F},P)$ of the sequence of corresponding integrals.

We admit that this limit is the same for any sequence of simple processes converging to $H_.(.)$.

(- next extension: to processes $H_.(.)$ measurable, $(\mathcal{F}_t)$ adapted s.t. $\int_0^T H_s^2 ds < +\infty$ P-as).

Example: compute $\int_0^T B_t dB_t$ (question 2 of exercise 22.).

For $n \in \mathbb{N}$, we set $t_k = \frac{kT}{n}$ for $k = 0,...,n$, and $B_t^n = \sum_{k=0}^{n-1} B_{t_k} \mathbb{1}_{[t_k,t_{k+1}[}(t)$.

$(B_t^n)_{0\le t\le T}$ is a simple process as $B_{t_k} \in L^2(\Omega,\mathcal{F},P)$ and is $\mathcal{F}_{t_k}$-measurable.

$\star$ When $n \to +\infty$, $(B^n_t)_{0\le t\le T}$ converges in $L^2(\Omega\times]0,T[,\mathcal{F}\times\mathcal{B}_{]0,T[},P\times dt)$ toward $(B_t)_{0\le t\le T}$, indeed we have, for a given n (note that the dates $t_0, t_1, ..., t_n$ depend on n):

$$\mathbb{E}\left(\int_0^T [B^n_t - B_t]^2 dt\right) = \mathbb{E}\left(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} [B^n_t - B_t]^2 dt\right) = \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\left([B_{t_k} - B_t]^2\right) dt$$
$$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t-t_k)dt \le \frac{T}{n}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} 1dt = \frac{T^2}{n} \to 0.$$

$\star$ Then, with $\Delta B_k = B_{t_{k+1}} - B_{t_k}$, $\displaystyle\int_0^T B_t dB_t = \lim_{n\to+\infty} (\text{in L}^2) \sum_k B_{t_k}\Delta B_k$ (by definition).

But $B_T^2 = B_0^2 + \sum_k (B^2_{t_{k+1}} - B^2_{t_k}) = \sum_k [(B_{t_k} + \Delta B_k)^2 - B^2_{t_k}] = \sum_k [2B_{t_k}\Delta B_k + (\Delta B_k)^2]$

$$= 2\int_0^T B^n_t dB_t + \sum_k (B_{t_{k+1}} - B_{t_k})^2 \quad \text{converges to } 2\int_0^T B_t dB_t + T \text{ when } |\{t_k\}| \to 0.$$

We get $\boxed{B_T^2 = 2\int_0^T B_t dB_t + T}$. That is **denoted** as "$d(B_t^2) = 2B_t dB_t + dt$ ",

while for a C^1 function f, we have: "$d(f(t)^2) = 2f(t)df(t)$" i.e. $(f^2)'(t) = 2f(t)f'(t)$, hence $[f(T)]^2 = 2\displaystyle\int_0^T f(t)df(t)$ (assuming $f(0) = 0$).

When using the Ito integral, the "**differential calculus rules**" (giving the way to differentiate $t \mapsto f(B_t)$) are modified by an **additional term** linked to the fact that the quadratic variation of the Brownian motion on $[0, T]$ is not 0.

2. Ito processes

$(B_t)_{t\ge 0}$ Brownian motion. $\forall t \ge 0$, $\mathbb{E}(B_t) = 0$

but we want to model processes whose expectation increases with time (equity price).

$\to$ we build more general processes from (B_t).

def: Ito process: $(X_t)_{t\le T}$ with real values s.t. its variation between t and $t + dt$ can be written:
$dX_t = a(t, X_t)dt + b(t, X_t)dB_t$, with $a, b : [0, T] \times \mathbb{R} \to \mathbb{R}$ C^0.

To be understoood as a notation for the equation:

$$\forall t \le T,\ X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s \ P\text{-as.}$$

Note that for a and b having good properties (that we assume), this equation has a unique solution.
$a(t, X_t)$ is called the drift (instantaneous trend).

An Ito process is a Markov process, heuristic proof:
(B_t) is a Markov process, then dB_t, interpreted as $B_{t+dt} - B_t$, is independent of $\mathcal{F}_t$.
From $X_{t+dt} - X_t = a(t, X_t)dt + b(t, X_t)dB_t$, we deduce that the law of X_{t+dt} depends only on X_t and not on the past.

Ex: geometric Brownian motion (model for the equity price in Black-Scholes model):
$dX_t = X_t(\mu dt + \sigma dB_t)$. (X_t) is an Ito process.
Note: we do not deduce that $d(\ln X_t) = \mu dt + \sigma dB_t$ indeed $d(\ln X_t) \ne \frac{dX_t}{X_t}$:

we cannot integrate in a usual way, an additional term should arise.

3. Ito lemma

$(X_t)_{t\leq T}$ Ito process: $dX_t = a(t, X_t)dt + b(t, X_t)dB_t$.

For a given ω, $t \mapsto X_t(\omega)$ is a function g. If g was C^1, we would write, for any C^1 function f: $d[f(g(t))] = f'(g(t))dg(t)$, meaning:

$$\forall t \leq T,\ f(g(t)) = f(g(0)) + \int_0^t f'(g(s))g'(s)ds.$$

But $t \mapsto X_t(\omega)$ is not C^1 and $f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s$ is **wrong for an Ito process**.

• $f : \mathbb{R} \to \mathbb{R}$ C^2 ; the variation of $f(X_t)$ between t and $t + dt$ is (we expand $d[f(X_t)]$ in a Taylor series up to second-order term):

$$d[f(X_t)] = f(X_{t+dt}) - f(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 + o[(dX_t)^2].$$

The last term contains a term of 1st order (heuristic proof): $(dX_t)^2 = [a(t, X_t)dt + b(t, X_t)dB_t]^2$ contains:

terms in $dtdB_t$ and $(dt)^2$, negligible compared with 1st order terms (in dt and dB_t)

a term in $(dB_t)^2$; but $\mathbb{E}[(dB_t)^2] = dt$, since $B_{t+dt} - B_t \sim \mathcal{N}(0, dt)$,

and $V[(dB_t)^2]$ is in $(dt)^2$, negligible compared with dt (cf $\mathbb{E}((dB_t)^4) = 3(dt)^2$.

Then we identify $(dB_t)^2$ to its expectation, dt.

We get that $(dX_t)^2$ is equivalent to $[b(t, X_t)]^2dt$.

We obtain that the variation of $f(X_t)$ between t and $t + dt$ is:

$$d[f(X_t)] = f'(X_t)dX_t + \frac{1}{2}f''(X_t)[b(t, X_t)]^2dt$$

The correct meaning is the integrated version:

Th: Ito Formula.

$(X_t)_{t\leq T}$ Ito process: $dX_t = a(t, X_t)dt + b(t, X_t)dB_t$. $f : \mathbb{R} \to \mathbb{R}$ class C^2;

we have: $\forall t \leq T,\ f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)[b(s, X_s)]^2ds$ P-as.

i.e. there is an additional term.

Notes: 1. the rigorous proof of this result relies on the same steps as for the computation of $\int_0^T B_tdB_t$ (which corresponds to a particular case of the lemma).

2. the Ito Formula proves that $(f(t, X_t))_{t\leq T}$ is still an Ito process (sum of terms in dt and in dB_t).

Examples of use: $dB_t = 0 \cdot dt + 1 \cdot dB_t$ (the Brownian motion is obviously an Ito process), then:

⋆ using $f(x) = x^2$, we get $d(B_t)^2 = 2B_tdB_t + \frac{1}{2}2dt$ (see previous page for a rigorous proof).

⋆ definition: **geometric Brownian motion**: stochastic process for which:

$\exists\ \mu, \sigma \in \mathbb{R}$, with $\sigma > 0$ such that $dS_t = S_t(\mu dt + \sigma dB_t)$, with $S_0 > 0$ known.

In the Black Scholes model, the stock price is assumed to be a geometric Brownian motion.

The instantaneous return satisfies: $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ then $\frac{dS_t}{S_t} \sim$ "$\mathcal{N}(\mu dt, \sigma^2 dt)$".

Compare to $\frac{dS_t}{S_t} = rdt$ for a risk-free asset (return = r, deterministic): a (Gaussian) randomness is added.

We compute $d[\ln S_t]$ (Ito on an open set, since ln not defined on $\mathbb{R}$).
$d[\ln S_t] = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt = (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t$, then $\forall t \geq 0$, $\ln S_t = \ln S_0 + \int_0^t (\mu - \frac{\sigma^2}{2})ds + \int_0^t \sigma dB_s$
hence $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$.

In fact, we saw Ito on $\mathbb{R}$ only $\Rightarrow$ better to start with the solution (and use the unicity of the solution of the Stochastic Differential Equation $dS_t = S_t(\mu dt + \sigma dB_t)$, for S_0 given):
let $X_t = (\mu - \frac{\sigma^2}{2})t + \sigma B_t$ and $S_t = S_0 e^{X_t}$. Ito lemma applied to $dX_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t$ with $x \mapsto S_0 e^x$:
$dS_t = S_t dX_t + \frac{\sigma^2}{2} S_t dt = S_t(\mu dt + \sigma dB_t)$.

Properties of the geometric Brownian motion: continuous process, with positive values,
lognormal: each r.v. $\ln S_t$ is gaussian. Indeed $\ln S_t = \ln S_0 + \mathcal{N}((\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$.

More general Ito Formula:
• $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ $C^{1,2}$ (twice differentiable in x, once in t, and continuous derivatives in (t,x)), we have:

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)[b(t, X_t)]^2 dt.$$

II. Black-Scholes model assumptions
= basis model in continuous time.

financial market consisting of 2 basis assets and their derivative products:
- one risk-free asset, whose value grows at the interest rate $r > 0$ constant: price at time t: e^{rt}.
- the other asset is risky (equity paying no dividend), with price S_t at time t.

Let $(B_t)_{t \in [0,T]}$ be a Brownian motion,
we assume $\underline{dS_t = S_t(\mu dt + \sigma dB_t)}$, μ, σ constants, $\sigma > 0$, ie $\underline{(S_t) \text{ geometric Brownian motion}}$.

We have: $\dfrac{dS_t}{S_t} = \mu dt + \sigma dB_t \quad \sim$ "$\mathcal{N}(\mu dt, \sigma^2 dt)$ ":
μ is interpreted as the instantaneous expected rate of return, and σ^2 as its instantaneous variance.
The successive shocks dB_t are independent and Gaussian.
σ measures the sensitivity to these shocks, called "volatility".

$\underline{\forall t \geq 0,\ S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}}$. Then $\ln S_t$ is normal (S_t lognormal) and (S_t) is a Markov process.

$\mathbb{E}(S_t) = S_0 e^{\mu t}$ grows like the price of an asset with constant instantaneous return rate μ.

For a given time length δ (ex 1 day), the log-returns $\ln \frac{S_{(j+1)\delta}}{S_{j\delta}}$ are i.i.d., with law $\mathcal{N}((\mu - \frac{\sigma^2}{2})\delta, \sigma^2\delta)$.

It is then easy to test the validity of the model on real data.
If δ small, $\ln \frac{S_{(j+1)\delta}}{S_{j\delta}} = \ln\left(1 + \frac{S_{(j+1)\delta} - S_{j\delta}}{S_{j\delta}}\right) \simeq \frac{S_{(j+1)\delta} - S_{j\delta}}{S_{j\delta}}$ = successive returns of the equity.

Test = are the successive returns i.i.d. Gaussian?

Conclusion: not perfect.

Note about μ: $\mu - r$ corresponds to a premium to compensate for the risk of the asset. The higher σ, the riskier the asset and the higher μ (generally). The higher the level of risk aversion of the investors, the higher μ will be.

We make the usual assumptions on the financial markets for derivatives pricing:

1. There are no transactions costs and no taxes.

2. Short selling is allowed, with no limit. In particular, with the risk-free asset:
The market participants can borrow and lend money at the same risk-free rate of interest r.

3. The assets are divisible.
Then one can hold α assets, with $\alpha \in \mathbb{R}$.

4. Trading takes place continuously in time: the quantities of assets being held can change at any time.

5. **NAO**: there are no arbitrage opportunities.

III. Partial differential equation approach

Pricing of a European option with maturity T and payoff $h(S_T)$ at T (with $\mathbb{E}\left(h(S_T)^2\right)$ finite).

1. The option price at time t can be written $F(t, S_t)$

The price at time t depends on t, S_t, and not on S_s, $s < t$, since the future variations of the U.A. price are functions of S_t only (Markov process), denoted by $F(t, S_t)$ where $F : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$, $(t,x) \mapsto F(t,x)$.

We assume that F has class $C^{1,2}$ (that is satisfied when h is C^2, with $\mathbb{E}\left[h(S_T)^2\right]$ finite; or for $h(x) = (x-K)^+$, or $(K-x)^+$:

for the call and the put with strike K, h is not differentiable at K,
but F will have class $C^{1,2}$ as $P(S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T} = K) = 0$).

2. Partial differential equation satisfied by F

We consider locally a portfolio constituted of

$$\begin{cases} -1 \text{ option} \\ n_t \text{ U.A.} \end{cases}$$

where we choose n_t s.t. the portfolio is risk-free between t and $t + dt$ (short sale if $n_t < 0$).

Let V_t be the value of the portfolio at time t: $V_t = -F(t, S_t) + n_t S_t$.

Between t and $t + dt$, the quantity of U.A. remains equal to n_t (this is linked to some notion of self-financing strategy like in discrete time). The variation of the portfolio value between t and $t + dt$ is: $dV_t = -dF(t, S_t) + n_t dS_t$ with (Ito lemma):

$$dF(t,S_t) = \left[\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2\right] dt \; + \frac{\partial F}{\partial x}(t,S_t) dS_t$$

We take $n_t = \frac{\partial F}{\partial x}(t, S_t)$. Then dV_t contains terms in dt only and none in dS_t. No randomness: the portfolio is risk-free between t and $t+dt$; then its return can only be r: $dV_t = rV_t dt$ (else AO locally):

$$dV_t = -\left[\frac{\partial F}{\partial t}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2\right]dt = r\left[-F(t,S_t) + \frac{\partial F}{\partial x}(t,S_t)S_t\right]dt$$

Note: the portfolio is risk-free for an infinitesimally short period of time. To keep it risk-free between 0 and T, it is necessary to change continuously the proportions of the derivative security and the stock in the portfolio.

We get the PDE satisfied by the price function F:

$$\frac{\partial F}{\partial t}(t,S_t) + rS_t\frac{\partial F}{\partial x}(t,S_t) + \frac{\partial^2 F}{\partial x^2}(t,S_t)\frac{\sigma^2}{2}(S_t)^2 = rF(t,S_t)$$

Changing S_0, we see that S_t can take any value of $]0,+\infty[$, then we have:

$$\boxed{\forall x \in]0,+\infty[,\ \forall t \in]0,T[,\quad \frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 F}{\partial x^2}(t,x) = rF(t,x)\quad \text{("parabolic equation")}}$$

with: $$\boxed{\forall x \in]0,+\infty[,\qquad F(T,x) = h(x)\qquad \text{("boundary condition").}}$$

Notes: these two equations are independent of μ.
The equation is the same for any European derivative, only the boundary condition changes.
Any derivative price satisfies this PDE (precisely: any asset with a price $\mathcal{F}_T$-measurable ie whose randomness comes from a same risk factor, the BM driving the risky asset price S; in particular any derivative with U.A. = the risky asset).

Ex: forward contract, maturity T, future price K.
value at $t = S_t - Ke^{-r(T-t)}$. We check that $F(t,x) = x - Ke^{-r(T-t)}$ satisfies the equation.

Resolution of the equation: seen later (some numerical methods can be used anyway. Ex: finite difference method = discretisation of derivative functions).

3. Hedging

Let $\Delta_t = \dfrac{\partial F}{\partial x}(t,S_t)$. The portfolio $\{-1$ option, Δ_t U.A.$\}$ is locally risk-free.
Then the portfolio $\{1$ option,$-\Delta_t$ U.A.$\}$ is equivalent to cash (its annualised return between t and $t+dt$ is r).

That means also that we can build a portfolio $\begin{cases} \Delta_t \text{ U.A.} \\ + \text{ some risk free asset} \end{cases}$ replicating the option,

i.e. a portfolio consisting of the basis assets that is worth the same than the option at any time.

The seller of the option will build this portfolio, called <u>hedging portfolio</u> (in complement of the short option position).
At T, this portfolio provides exactly $h(S_T)$, i.e. what is due to the option buyer.

The seller makes money on the margin (he sells at F_0+margin) and <u>eliminates completely its risk</u>:

there is a perfect replication of the final payoff.
Indeed the hedging portfolio is continuously adjusted (dynamic hedge), i.e. continuous trading of the U.A., in order to hold the correct quantity of U.A. ($\Delta_t = \frac{\partial F}{\partial x}(t, S_t)$, the "delta" of the option) at any time.
The seller is then indifferent to the fact that the U.A. price goes up (even if he sold a call option) or down (even if he sold a put option).

Note that the delta measures the sensitivity of the option price to the variations of the U.A. price. To a change of 1 euro in the price at t, corresponds a change of Δ_t euro in the option price.

In the theoretical model, the replication is perfect.
In reality, the hedge is done discretely:

- the physical transactions are necessarily discrete,
- existence of transaction costs that limit the frequency of re-hedging.

The hedging in then not perfect and the seller's portfolio is in fact risky. The more frequently he re-hedges, the closer to the option is his portfolio, but the more transaction costs he pays.

The hedging transactions can be less frequent when the delta changes slowly.
This is measured by the gamma: $\Gamma_t = \frac{\partial^2 F}{\partial x^2}(t, S_t)$ measures the sensitivity of the delta to the variations of the U.A. price, then it measures the residual risk of the hedged position.

IV. Probabilistic approach for European options. Black Scholes formula

1. Price as an expectation

Pricing of a European option with maturity T and payoff $h(S_T)$ at T.
This other approach provides the solution of the PDE, by a probabilistic method.

The coefficients of the PDE are independent of μ (linked to the risk aversion of investors), then the price dynamics is independent of μ.

Option pricing can then be done as if investors were risk-neutral.
="set yourself in the risk-neutral universe", where any return or discount rate is equal to r. In this world, investors do not demand extra returns above the risk-free interest rate for bearing risks.

In fact, this corresponds again to a change of probability:

Girsanov theorem (particular case)

> For a given $\lambda \in \mathbb{R}$, let $L_t = e^{-\lambda B_t - \frac{\lambda^2}{2}t}$ for any $t \geq 0$. We denote by P^* the probability with density L_T with respect to P. Then $(B_t + \lambda t)_{t \in [0,T]}$ is a Brownian motion under P^*.

$(B_t + \lambda t)_{t \in [0,T]}$ Brownian motion under P^* defined by the density: $\dfrac{dP^*}{dP} = e^{-\lambda B_T - \frac{\lambda^2}{2}T}$

Proof (see also tutorial, exercise 21): $\mathbb{E}(L_T) = 1$ then P^* is a probability.

Let $W_t = B_t + \lambda t$ and $L_t = e^{-\lambda B_t - \frac{\lambda^2}{2}t}$. We know that $(L_t)_{t \geq 0}$ is a martingale under P.

$(W_t)_{t \leq T}$ is an adapted process, with continuous paths, and $W_0 = 0$.

We will prove that for $s \leq t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$ and is independent of $\mathcal{F}_s = \sigma(B_u, u \leq s) = \sigma(W_u, u \leq s)$ under P^*.

It is sufficient to prove that $\quad \forall u \in \mathbb{R},\ \mathbb{E}^*(e^{iu(W_t-W_s)}|\mathcal{F}_s) = e^{-\frac{u^2}{2}(t-s)} \quad (*)$.

Indeed, that implies that $\forall u \in \mathbb{R}$, $\mathbb{E}^*(e^{iu(W_t-W_s)}) = e^{-\frac{u^2}{2}(t-s)}$,

then $W_t - W_s$ and $\mathcal{N}(0, t-s)$ have same characteristic function, hence $W_t - W_s \sim \mathcal{N}(0, t-s)$.

And we will use below lemma to conclude.

If a r.v. X satisfies: $\forall u \in \mathbb{R}$, $\mathbb{E}(e^{iuX}|\mathcal{B}) = \mathbb{E}(e^{iuX})$ Pa.s., then X is independent of $\mathcal{B}$.

Proof: we have: $\forall B \in \mathcal{B}$, $\mathbf{E}(e^{iuX}\frac{\mathbb{1}_B}{P(B)}) = \mathbf{E}(e^{iuX})$.

Then X has same law under P than under the probability with density $\frac{\mathbb{1}_B}{P(B)}$ with respect to P (the characteristic functions are the same). Then for any $f : \mathbb{R} \to \mathbb{R}$ measurable and bounded, $\mathbf{E}(f(X)\frac{\mathbb{1}_B}{P(B)}) = \mathbf{E}(f(X))$ i.e. $\mathbf{E}(f(X)\mathbb{1}_B) = \mathbf{E}(f(X))\mathbf{E}(\mathbb{1}_B)$, which proves the independence.

To prove (*), we need to know how to compute a conditional expectation under P^*.

If $Z \in L^1(\Omega, \mathcal{F}, P^*)$ is $\mathcal{F}_t$-meas., then $\mathbb{E}^*(Z) = \mathbb{E}(ZL_T) = \mathbb{E}(Z\mathbb{E}(L_T|\mathcal{F}_t)) = \mathbb{E}(ZL_t)$.

Now we want to compute $V = \mathbb{E}^*(Z|\mathcal{F}_s)$ for $s \le t$.

V is $\mathcal{F}_s$-measurable and satisfies: for any Y $\mathcal{F}_s$-measurable in L^1, $\mathbb{E}^*(YZ) = \mathbb{E}^*(YV)$.

YZ and YV are respectively $\mathcal{F}_t$-measurable and $\mathcal{F}_s$-measurable. Hence we need:
$\mathbb{E}(YZL_t) = \mathbb{E}(YVL_s)$ for any Y $\mathcal{F}_s$-measurable, i.e.: $VL_s = \mathbb{E}(L_tZ|\mathcal{F}_s)$.

We get: $\boxed{\text{for } Z\ \mathcal{F}_t\text{-meas, and } s \le t,\ \mathbb{E}^*(Z|\mathcal{F}_s) = \frac{1}{L_s}\mathbb{E}(ZL_t|\mathcal{F}_s)}$

Now let us prove (*): we take $s \le t$ and $u \in \mathbb{R}$.

$\mathbb{E}^*(e^{iu(W_t-W_s)}|\mathcal{F}_s) = \frac{1}{L_s}\mathbb{E}(L_t e^{iu(W_t-W_s)}|\mathcal{F}_s)$

$L_t e^{iu(W_t-W_s)} = e^{-\lambda B_t - \frac{\lambda^2}{2}t + iu[B_t - B_s + \lambda(t-s)]}$. We group the terms in B_t:

let $M_t = e^{(-\lambda+iu)B_t - \frac{(-\lambda+iu)^2}{2}t}$. We know that (M_t) is a martingale under P.

We have $\mathbb{E}^*(e^{iu(W_t-W_s)}|\mathcal{F}_s) = \frac{1}{L_s}\mathbb{E}(M_t \underbrace{e^{-iuB_s - iu\lambda s - \frac{u^2}{2}t}}_{\mathcal{F}_s-\text{meas}}|\mathcal{F}_s)$

$= \frac{1}{L_s}e^{-iuB_s - iu\lambda s - \frac{u^2}{2}t}M_s = e^{\lambda B_s + \frac{\lambda^2}{2}s}e^{-iuB_s - iu\lambda s - \frac{u^2}{2}t}e^{(-\lambda+iu)B_s - \frac{(-\lambda+iu)^2}{2}s} = e^{-\frac{u^2}{2}(t-s)}$. □

Then the theorem is proved: we get that $(W_t)_{t\in[0,T]}$ is a Brownian motion under P^*, the probability with density $L_T = e^{-\lambda B_T - \frac{\lambda^2}{2}T}$ w.r.t. P.

We rewrite the dynamics of the U.A. price in order to have an instantaneous return equal to r instead of μ:

$dS_t = S_t(\mu dt + \sigma dB_t) = S_t[rdt + \sigma(dB_t + \frac{\mu-r}{\sigma}dt)]$.

Let $\lambda = \frac{\mu-r}{\sigma}$ and for $t \ge 0$, $W_t = B_t + \lambda t$.

$(W_t)_{t\in[0,T]}$ is a Brownian motion under P^*, the probability with density $e^{-\lambda B_T - \frac{\lambda^2}{2}T}$ w.r.t. P.

We have $dS_t = S_t(rdt + \sigma dW_t)$, with $W_t = B_t + \lambda t$.

In particular, for $t \ge 0$, $S_t = S_0 e^{(r-\frac{\sigma^2}{2})t + \sigma W_t}$, then $\underline{(S_t e^{-rt})_{t\le T} \text{ is a martingale under } P^*}$

i.e. the discounted asset prices are martingales under P^* (making P^* an E.M.M., with a similar definition as in discrete time).

Under P^*, we have:
$S_t = e^{-r(T-t)}\mathbb{E}^*(S_T|\mathcal{F}_t)$ = value of the future flow, discounted at the risk-free rate r.

to be compared to $S_t = e^{-\mu(T-t)}\mathbb{E}(S_T|\mathcal{F}_t)$ (using $S_t = S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}$):
under the real probability, the price is discounted at rate μ,
rate taking into account the asset risk, and the risk aversion of the investors.

Th: Price at time t of the asset paying $h(S_T)$ at T: $F_t = e^{-r(T-t)}\mathbb{E}^*(h(S_T)|\mathcal{F}_t)$.
therefore $(F_t e^{-rt})_{t\le T}$ is a martingale under P^* and $F_0 = e^{-rT}\mathbb{E}^*(h(S_T))$. We get:

$$F_0 = e^{-rT}\mathbb{E}^*[h(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T})]$$

Under P^*, the future cash-flows are discounted at rate r, as if the investors were "risk neutral". Thus the name "risk neutral probability".
By contrast, P is the "real probability, or historical or objective".

Not that the complete proof involves, like in discrete time, the fact that the market is complete: any option is replicable by a self-financing strategy. This argument is used here when establishing the PDE.

Computation easy in the standard cases: the law of $h(S_T)$ is known (S_T lognormal).

Note: there exists some numerical methods to compute an expectation = Monte-Carlo method (simulation of n r.v. with law = law of X and approximation of $\mathbf{E}(X)$ by the mean of the n r.v.).

2. Computation for a call (the put option price is then obtained from put-call parity)
Price at time t: $C_t = C(t, S_t)$. We compute C_0.

$$C_0 = \mathbb{E}^*[e^{-rT}(S_T-K)^+] = \mathbb{E}^*[S_T e^{-rT}\mathbb{1}_{\{S_T\ge K\}}] - Ke^{-rT}P^*(S_T\ge K)$$

$$S_T \ge K \Leftrightarrow S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T} \ge K \Leftrightarrow \sigma W_T \ge ln\frac{K}{S_0} - (r-\frac{\sigma^2}{2})T$$

then $P^*(S_T\ge K) = P^*(-\frac{W_T}{\sqrt{T}} \le d_2) = N(d_2)$ where $\begin{cases} d_2 = \dfrac{ln\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ N(d) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx \end{cases}$

N is the cumulative probability distribution function for a standardized normal variable.

$$\mathbb{E}^*[S_T e^{-rT}\mathbb{1}_{\{S_T\ge K\}}] = \mathbb{E}^*[S_0 e^{\sigma W_T - \frac{\sigma^2}{2}T}\mathbb{1}_{\{W_T\ge -d_2\sqrt{T}\}}] \overset{\star}{=} S_0\tilde{P}[\tilde{B}_T \ge -d_2\sqrt{T}-\sigma T] = S_0 N(\underbrace{d_2+\sigma\sqrt{T}}_{d_1})$$

$\star$ from Girsanov:
$(\tilde{B}_t = W_t - \sigma t)_{t\in[0,T]}$ is a Brownian motion under $\tilde{P}$, probability with density $e^{\sigma W_T - \frac{\sigma^2}{2}T}$ w.r.t. P^*.

then $\boxed{C_0 = S_0 N(d_1) - Ke^{-rT}N(d_2)}$

and, replacing S_0 by S_t and T by $T-t$, the price at time $t\le T$ is: (with $d_i = d_i(0, S_0)$)

$\boxed{C_t = S_t N(d_1(t,S_t)) - Ke^{-r(T-t)} N(d_2(t,S_t))}$, where

$$\begin{cases} d_1(t,x) = \dfrac{ln\frac{x}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2(t,x) = \dfrac{ln\frac{x}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \end{cases}$$

Shape :

Hedging: we compute the delta (slope of the previous curve).

$C_0 = \mathbb{E}^*[\underbrace{e^{-rT}(S_0X_T - K)^+}_{f(S_0,\omega)}]$ with $X_T = \frac{S_T}{S_0} = e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}$.

We compute the first derivative w.r.t. S_0. We can differentiate inside the expectation if:

(1) P-as in ω, $S_0 \mapsto f(S_0,\omega)$ is C^1, fine since $\{\omega | xe^{(r-\frac{\sigma^2}{2})T+\sigma W_T} = K\}$ is negligible.
(2) For S_0 in a compact set, $|f'(S_0,\omega)| \leq g(\omega)$ with $g \in L^1$.
derivative of $S_0 \mapsto f(S_0,\omega)$: $X_T 1\!\!1_{\mathbb{R}^+}(S_0X_T - K) \leq X_T$, then (2) is satisfied.

Then $\Delta_0 = \mathbb{E}^*[e^{-rT} X_T 1\!\!1_{\mathbb{R}^+}(S_0X_T - K)] = \mathbb{E}^*[e^{-rT}\frac{S_T}{S_0} 1\!\!1_{S_T \geq K}]$.

We proved $\mathbb{E}^*[S_T e^{-rT} 1\!\!1_{\{S_T \geq K\}}] = S_0 N(d_1)$. Therefore $\Delta_0 = N(d_1)$.

And $\Delta_t = N(d_1(t,S_t)) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{d_1(t,S_t)} e^{-\frac{x^2}{2}} dx \quad \in [0,1]$.

3. PDE when the U.A. pays dividends (exercise 25)

Stock paying a continuous dividend yield at rate δ (known).

We consider locally a portfolio consisting of $\begin{cases} -1 \text{ option} \\ \Delta_t \text{ U.A.} \end{cases}$, with $\Delta_t = \dfrac{\partial F}{\partial x}(t,S_t)$.

Let V_t the portfolio value at date t. We have, between t and $t+dt$:

$dV_t = -dF(t,S_t) + \Delta_t dS_t + \Delta_t \delta dS_t$
(dividend received if $\Delta_t < 0$, paid in the short sale if $\Delta_t > 0$).

The equation is then changed in:

$$\frac{\partial F}{\partial t}(t,x) + (r-\delta)x\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 F}{\partial x^2}(t,x) = rF(t,x)$$

4. Use

All the parameters are observed on the market, except the volatility σ.
2 methods to get it:

Historical volatility: under Black-Scholes assumptions, the $X_j = ln\frac{S_{(j+1)\Delta t}}{S_{j\Delta t}}$, $0 \leq j \leq n-1$ are i.i.d. with law $\mathcal{N}((\mu - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t)$.

This is used to estimate σ on past series: $\frac{1}{n-1}\sum_{j=1}^n (X_j - \bar{X})^2$ is an unbiased estimator of $\sigma^2 \Delta t$.

Implicit volatility: the call price is an increasing function of the volatility.
Observed (quoted) call prices are used to infer the volatility, by reverting the Black-Scholes formula.

= volatility as anticipated by the market.

Note that it has been assumed that r is constant. In practice the formulas are used with r equal to the risk-free rate of interest on an investment lasting for $T-t$, the life of the option.

5. Choice of u and d in the binomial tree

They need to depend on the volatility of the U.A..

We build the tree directly under the risk-neutral probability.
With $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, we have $d < e^{r\Delta t} < u$ and $p^* = \dfrac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$.
We approximate p^* by $p^* = \frac{r\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} = \frac{1}{2} + \frac{r\sqrt{\Delta t}}{2\sigma}$.

For Δt small, the return on a period, $\frac{S_{t+\Delta t} - S_t}{S_t}$, can be approximated by $\ln \frac{S_{t+\Delta t}}{S_t}$ (logarithmic return), from $\ln(1+x) \sim x$.

$\ln \frac{S_{t+\Delta t}}{S_t}$ is worth $\sigma\sqrt{\Delta t}$ or $-\sigma\sqrt{\Delta t}$. Then the expected return on a period is:

$$p^*\sigma\sqrt{\Delta t} - (1-p^*)\sigma\sqrt{\Delta t} = \sigma\sqrt{\Delta t}(1-2p^*) = \sigma\sqrt{\Delta t}\frac{r\sqrt{\Delta t}}{\sigma} = r\Delta t$$

while the expected variance is: $p^*(1-p^*)(2\sigma\sqrt{\Delta t})^2 \sim (\frac{1}{2})^2 4\sigma^2\Delta t = \sigma^2\Delta t$.

converges to BS model when $\Delta t \to 0$
(ie the discrete process converges to a geometric Brownian motion with volatility σ).

$\ln S$ moves are sums of i.i.d. Bernouilli r.v. converging to a Gaussian variable on any interval, while in the BS model, under the risk-neutral probability, the $ln\frac{S_{(j+1)\Delta t}}{S_{j\Delta t}}$ are i.i.d. with law $\mathcal{N}((r - \frac{\sigma^2}{2})\Delta t, \sigma^2\Delta t)$.