Logic and Sets
Mid-term exam 2022 (1h)

Name: QEM/MMEF

Exercise 1 (9pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. Dogs have 3 legs if and only if cows have 5 legs
2. The negation of $p \to q$ is $\neg q \to \neg p$
3. The negation of $(p \to q) \wedge (p \vee \neg q)$ is $(p \wedge \neg q) \vee (\neg p \wedge q)$
4. The negation of [$\forall n$ in the set of odd numbers $, \exists x \in \mathbb{R}, \sqrt{x} = 2n+1$] is [$\exists n$ in the set of even numbers $, \forall x \in \mathbb{R}, \sqrt{x} \neq 2n+1$]
5. The negation of $[\forall x > 0, \exists y < 0, \forall z \in \mathbb{R}, \forall t \in \mathbb{R}, f(x,y,z) = g(t)$ or $x+y+z=t]$ is $[\exists x > 0, \forall y < 0, \exists z \in \mathbb{R}, \exists t \in \mathbb{R}, f(x,y,z) \neq g(t)$ and $x+y+z \neq t]$
6. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, p(x,y,z)$ implies $\forall x \in \mathbb{R}, \exists z \in \mathbb{R}, \forall y \in \mathbb{R}, p(x,y,z)$
7. $\exists z \in \mathbb{R}, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 = z^2$.
8. $A \Delta B = (A \cup B) \setminus (A \cap B)$
9. $(A \cup A') \times (B \cup B') = (A \times B) \cup (A' \times B')$
10. $A^c \cap B^c = A \cup B$
11. $A \cup B = B \Leftrightarrow A \subseteq B$
12. $\{a, \{a\}, \{a,b\}\} = \{\{a,a\}, a, \{b,a\}\}$
13. $a \in \{\{a\}, b, \{a,b\}\}$
14. $\{a,b\} \subseteq \{a, \{a,b\}\}$
15. $\mathcal{P}(\emptyset) = \{\emptyset, \{\emptyset\}\}$
16. $\emptyset \subseteq \{a, \{b\}\}$
17. The function $f :]0, \infty[\to \mathbb{R}$ defined by $f(x) = \log(x)$ is surjective.
18. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is injective.

Exercise 2 (8pts)

1. Show by contraposition that for any function $f : X \to Y$, for any $A, B \subseteq X$, $f(A) \cap f(B) = \emptyset$ implies $A \cap B = \emptyset$. Is the converse true? Justify your answer.

2. Show by contradiction that there is no integer $a, b \in \mathbb{Z}$ such that $15a + 3b = 5$.

3. Show by induction that for any $n \geq 1$, $\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$.

4. Suppose U is the universal set and take $A, B \subseteq U$. Show that $A \cup B = B \Leftrightarrow B \cup A^c = U$.

Exercise 3 (3pts)

Write the truth tables of $(p \to \neg(p \vee q)) \to (\neg p \wedge q)$ and $((p \to q) \to (q \wedge \neg r)) \to \neg p$.