Logic and Sets
Final exam 2022 (2h)

Name: QEM/MMEF

Exercise 1 (5pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. The strong principle of induction is stronger than the least number principle (Fermat infinite descent).
2. $(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c$
3. An order relation is a binary relation which is reflexive, antisymmetric and transitive.
4. Any equivalence relation on A induces a partition of A, and conversely.
5. The inclusion relation on 2^E is a complete order on 2^E.
6. Assuming $f : E \to F$, $g : F \to G$ are bijections, $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.
7. There exists a bijection between $\{1, 2, 3, 4\}$ and $\{a, b, c, d, e\}$.
8. There exists a bijection between $\mathbb{N}$ and $\mathbb{Z}$.
9. There exists a bijection between $\mathbb{N}$ and $\mathbb{Q}$.
10. There exists a bijection between $\mathbb{Q}$ and $[0, 1]$.
11. The continuum hypothesis says that $\mathbb{R}$ is countable.
12. $2^{\mathbb{N}}$ is countable.
13. Any subset of $\mathbb{R}$ bounded from below has an infimum.
14. It is possible that a set has an infimum but no minimal element.
15. It is possible that a set has a minimal element but no infimum.
16. $f^{-1}(\{y\}) = \{f^{-1}(y)\}$ if f is a bijection.
17. The cardinality of $\mathbb{Q}$ is $\aleph_1$.
18. The set of irrational numbers is countable.
19. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
20. The projection mapping $proj_1 : A \times B \to A$ is surjective.

Exercise 2 (4 pts)

1. (1pt) Give the rank and nullity of

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 2 \end{bmatrix}$$

2. (3pts) Solve the following linear system where $a \in \mathbb{R}$. Depending on the value of a, determine the set of its solutions, if any.

$$\begin{cases} x + y - z = 1 \\ -2x - y + z = 0 \\ x + 2y + 2z = -1 \\ y + z = 0 \\ x + 2y - z = a \end{cases}$$

Exercise 3 (4pts)

Let $\mathcal{A}$ be the family of subsets of $\mathbb{R}$ which are bounded from below.

1. Write $\mathcal{A}$ in set-builder form, i.e., $\mathcal{A} = \{X \subseteq \mathbb{R} \text{ s.t.} \dots\}$.
2. Show that $\mathcal{A}$ is closed under finite unions (i.e., if $A_1, \dots, A_k \in \mathcal{A}$ then $\bigcup_{i=1}^{k} A_i \in \mathcal{A}$) and countable intersections.
3. Show that $\mathcal{A}$ is not closed under countable unions.

Exercise 4 (2 pts)

Consider two mappings $f : E \to F$ and $g : F \to G$. Show that if $g \circ f$ is injective then f is injective, and if $g \circ f$ is surjective, then g is surjective.

Exercise 5 (4pts)

Determine, if they exist, the set of lower bounds, upper bounds, the minimal and maximal elements, the infimum and supremum of the following subsets of $\mathbb{R}$:

1. $\left\{\frac{n}{1+n}, n \in \mathbb{N}\right\}$
2. $\left\{\frac{1}{x+1}, x \in \left]-1, 1\right[\right\}$

Same question for the following subsets of $\mathbb{Q}$:

1. $\left\{x \in \mathbb{Q} \ : \ x \leq \frac{1}{\pi}\right\}$
2. $\left\{1 + \frac{(-1)^n}{2n}, n \in \mathbb{N}\right\}$

Note: $0 \notin \mathbb{N}$.

Question (1pt)

Explain the Cantor-Schröder-Bernstein theorem.