MICROECOMICS 1A, 2021-22 - Exercises

QEM Master, Joint degree Erasmus Mundus, PhD

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Exercises are for class corrections (practical sessions and TA sessions). During class corrections students can be selected randomly to propose their solution to starred (*) exercises.

Double starred (**) exercises are for PhD students. They will be solved in dedicated, additional, practical sessions.

Exercises: 'Introduction' and 'Consumption'

I.1 What is microeconomics about? And why are you interested in microeconomics?

I.2 Discuss the following thoughts by Gerard Debreu on the mathematization and axiomatization of economic theory: "In its mathematical form, economic theory is open to an efficient scrutiny for logical errors... The greater logical solidity has enabled researchers to build on the work of their predecessors and to accelerate the cumulative process in which they are participating... But a Grand Unified Theory will remain out of the reach of economics, which will keep appealing to a large collection of individual theories. Each one of them deals with a certain range of phenomena that it attempts to understand and to explain. When it acquires an axiomatic form, its explicit assumptions delimit its domain of applicability and make illegitimate overstepping of its boundary flagrant" (Gerard Debreu, AER 1991)

C.1 (Ex 1.B.1 in MCWG): Suppose that $\succsim$ is rational. Prove that if $x \succ y \succsim z$, then $x \succ z$.

C.2* (Ex 1.B.2 in MCWG): Suppose that $\succsim$ is rational. Prove the following:

a) $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive;
b) $\sim$ is reflexive ($x \sim x$ for all x), transitive and symmetric (if $x \sim y$, then $y \sim x$).

C.3* Suppose that the choice structure $(\mathcal{B}, C(.))$ with $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$ and $C(\{x, y\}) = \{x\}$. Prove that if $(\mathcal{B}, C(.))$ satisfies WARP, then it must hold one of the following choice rules: $C(\{x, y, z\}) = \{x\}$ or $C(\{x, y, z\}) = \{z\}$ or $C(\{x, y, z\}) = \{x, z\}$.

C.4 (Ex 1.B.4 in MCWG): Consider a rational preference relation $\succsim$. Show that if $u(x) = u(y)$ implies $x \sim y$ and if $u(x) > u(y)$ implies $x \succ y$, then $u(\cdot)$ is a utility function representing $\succsim$.

C.5* Suppose that $\succsim$ is rational. Show that if X is finite, then there exists an utility function $u(x)$ representing $\succsim$.

C.6* (Lexicographic preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+$, $x \succsim \bar{x} \iff$ "$x_1 > \bar{x}_1$" or "$x_1 = \bar{x}_1$ and $x_2 \geq \bar{x}_2$".

a) For every $\bar{x} \in \mathbb{R}^2_+$, determine and draw the upper contour set $U(\bar{x})$.
b) Show that every $\bar{x} \in \mathbb{R}^2_+$, the indifference set $I(\bar{x})$ is a singleton.
c) Show that this preference relation is strongly monotone and strictly convex, but not continuous.

C.7* (Linear preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+$,

$$x \succsim \bar{x} \iff ax_1 + bx_2 \geq a\bar{x}_1 + b\bar{x}_2$$

with $a > 0$ and $b > 0$.

a) For every $\bar{x} \in \mathbb{R}^2_+$, determine and draw the indifference set $I(\bar{x})$ and the upper contour set $U(\bar{x})$.
c) Show that this preference relation is continuous, convex, strongly monotone, but not strictly convex.

C.8* (Leontief preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+$,

$$x \succsim \bar{x} \iff \min\{x_1, x_2\} \geq \min\{\bar{x}_1, \bar{x}_2\}$$

a) For every $\bar{x} \in \mathbb{R}^2_+$, determine and draw the indifference set $I(\bar{x})$ and the upper contour set $U(\bar{x})$.
c) Show that this preference relation is continuous, convex, monotone, but it is not strictly convex and it is not strongly monotone.

C.9 (Ex 3.C.5a in MCWG). Prove the following result: a continuous $\succsim$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one; i.e. $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$.

C.10* Let $p = (p_1, p_2) \gg 0$ be a price system and let $w > 0$ be the wealth of the consumer. Using the definition of the demand of the consumer, determine graphically the demand of the consumer in the three following cases:

a) Lexicographic preferences;
b) Linear preferences.
c) Leontief preferences.

C.11* For an utility function $u(x_1, x_2)$ representing rational preferences in $X = \mathbb{R}^2_+$ provide the graphical representations of the following solutions of an utility maximization problem (UMP) :

a) internal solution $x(p, w)$ which is a global maximizer for $u(x_1, x_2)$ everywhere quasi-concave;
b) corner solution $x(p, w)$ which is a global maximizer for $u(x_1, x_2)$ everywhere quasi-concave;
c) internal solution $x(p, w)$ which is a global maximizer for $u(x_1, x_2)$ not everywhere quasiconcave;
d) internal $x(p, w)$ which is a local but not global maximizer for $u(x_1, x_2)$.

C.12* Consider a twice continuously differentiable utility functions $u(x_1, x_2)$ representing a consumer's preference. Prove the following results:

a) convexity of preference, that is quasi-concavity of $u(x_1, x_2)$, implies that at any bundle $\bar{x} = (\bar{x}_1, \bar{x}_2)$ the marginal rate of substitution $MRS_{12}(\bar{x}) = \dfrac{\partial u(\bar{x})/\partial x_1}{\partial u(\bar{x})/\partial x_2}$ is decreasing in x_1.
b) convexity of preference guarantees that the second order conditions for the utility maximization problem (UMP) are met.

C.13* (Cobb-Douglas utility function). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$, the utility function representing Cobb-Douglas preferences takes the general form $u(x_1, x_2) = (x_1)^a (x_2)^b$ with $a > 0$ and $b > 0$

a) For every $\bar{x} \in \mathbb{R}^2_+$, determine and draw the indifference sets $I(\bar{x})$ and the upper contour set $U(\bar{x})$ for the following three cases: i) $a = b$; ii) $a > b$; and iii) $b > a$.
b) Determine the following properties of $u(x_1, x_2)$: continuity, differentiability, strictly increasing, strictly quasiconcavity.
c) Consider the following real-valued function $f(z) = (z)^{\frac{1}{a+b}}$. Show that the utility function defined by $v(x_1, x_2) = f(u(x_1, x_2))$ represents the same Cobb-Douglas preferences for all $x = (x_1, x_2) \in \mathbb{R}^2_+$;
d) Consider now the real-valued function $g(z) = \ln(z)$. Show that the utility function defined by $\tilde{u}(x_1, x_2) = g(u(x_1, x_2))$ also represents the same Cobb-Douglas preferences for all $x = (x_1, x_2) \in \mathbb{R}^2_{++}$.

e) Let $p = (p_1, p_2) \gg 0$ be a price system and $w > 0$ be the wealth of the consumer with the above Cobb-Douglas preferences. Determine the ordinary demand of this consumer.

f) Provide a graphical representation of the solution of the previous point in the (x_1, x_2)-space; assume in the diagram that the price of good x_1 changes. Show diagrammatically the offer curve. Assume that wealth changes and show the wealth-consumption path.

C.14* As usual, let $x(p_1, p_2, w) = (x_1(p_1, p_2, w), x_2(p_1, p_2, w))$ denote the demand of the consumer. For every commodity $l = 1, 2$, the demand of commodity l is given by

$$x_l(p_1, p_2, w) = \frac{w}{p_1 + p_2}$$

a) Prove that this demand is homogeneous of degree zero.

b) Prove that this demand satisfies Walras' Law.

c) State the Weak Axiom of Revealed Preferences (WARP) in the framework of the demand.

d) Prove that this demand satisfies WARP.

C.15* Let L be the number of commodities. Let $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ be the consumption set. The consumer has strictly convex preferences which are represented by a utility function $u(x) = x_1 + \phi(x_2, x_3, ..., x_L)$. We assume $p \gg 0$, and we normalize $p_1 = 1$.

a) Show that the demand for commodities $\{2, 3, ..., L\}$ must be independent of wealth. How does demand for commodity 1 react to changes in wealth w?

b) Using the previous result, define the indirect utility function as usual, i.e. $v(p, w) := u(x^*)$, where x^* belongs to the demand, given p and w. Show that $v(p, w)$ is linear in wealth: $v(p, w) = w + \psi(p)$ for some function $\psi : \mathbb{R}_{++}^L \to \mathbb{R}$.

c) Now let $L = 2$ and $\phi(x_2) = \alpha \ln(x_2)$. Solve the UMP as a function of (p, w) (Recall that we allow demand for commodity 1 to be negative).

C.16 (Ex 3.C.6 in MCWG). Suppose that in a two-commodity world, the consumer's utility function takes the form $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$. This utility function is known as *constant elasticity of substitution* (CES) utility function.

a) Show that when $\rho = 1$, indifference curves become linear.

b) Show that as $\rho \to 0$, this utility function comes to represent the same preferences as the (generalized) Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

c) Show that as $\rho \to -\infty$, indifference curves become right angles; that is, this utility function has in the limit the indifference map of the perfect complement - Leontief utility function $u(x) = \min\{x_1, x_2\}$.

C.17 (Ex 2.E.1 in MCWG). Suppose $L = 3$ and consider the demand function $x(p, w)$ defined by:

$$\begin{aligned} x_1(p,w) &= \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1}, \\ x_2(p,w) &= \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2}, \\ x_3(p,w) &= \frac{\beta p_1}{p_1+p_2+p_3}\frac{w}{p_3}, \end{aligned}$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0, 1)$?

C.18* (Ex 2.E.4 in MCWG): Show that if $x(p, w)$ is homogeneous of degree one with respect to w and satisfies Walras' law, then the elasticity of a generic commodity l to wealth is $\varepsilon_{lw}(p, w) = 1$. Interpret. Can you say something about the form of the Engel functions and curves in this case?

C.19 (Ex 2.E.7 in MCWG): A consumer in a two-good economy has a demand function $x(p, w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p, w) = \alpha w/p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree 0?

C.20 (Ex 2.D.1 in MCWG). A consumer living for two periods consumes a single consumption good denoted c_1 and c_2 in period 1 and period 2, respectively. His wealth in period 1 is $w_1 > 0$ and in period 2 is $w_2 > 0$. There are perfect capital markets so that wealth can be transferred between the two periods at a constant interest rate r. Prices in the two periods are $p_1 = p_2 = 1$.

a) What is the consumer's Walrasian (lifetime) budget set?
b) Provide a graphical representation of the budget set.

C.21*. A consumer consumes one consumption good x and hours of leisure R. The time endowment is T while the consumer has no exogenous wealth endowment. The price of the consumption good is p and the consumer can work at a hourly wage rate of s.

a) What is the consumer's Walrasian budget set?
b) Assuming Cobb-Douglas preferences $u(x, R) = x^\alpha R^{1-\alpha}$, with $\alpha > 0$, compute the consumer's labour supply which is given by the time endowment minus the optimal hours of leisure R (solution of the UMP).
c) Provide a graphical representation of the previous solution.
d) Suppose now that the salary s increases. Compute analytically and show diagrammatically the change in the labour supply. What can you say about the substitution effect and the income effect generated by this salary change?
e) Suppose now that, in addition to the time endowment T, the consumer has also an exogenous wealth endowment E. Repeat all steps a)-d) above.

C.22** (Ex 2.E.2 in MCWG): Show that the equations expressing the *Cournot aggregation* and the *Engel aggregation* lead to the following two elasticity formulas:

$$\sum_{l=1}^{L} b_l(p, w)\varepsilon_{lk}(p, w) + b_k(p, w) = 0, \quad \text{all } k = 1, ..., L$$

and

$$\sum_{l=1}^{L} b_l(p, w)\varepsilon_{lw}(p, w) = 1,$$

where $b_l(p, w) = p_l x_l(p, w)/w$ is the budget share of the consumer's expenditure on good l given price p and wealth w.

C.23**. Consider a UMP in $L = 3$ with prices $p \in \mathbb{R}^3_{++}$ and wealth $w > 0$. The agent's preferences $\succeq$ over bundles (x_1, x_2, x_3) in $\mathbb{R}^3_+$ can be represented by the utility function:

$$u(x_1, x_2, x_3) = x_1 + x_2 \cdot x_3$$

a) Given the definition of convex, strictly convex, and homothetic preferences,. Verify whether or not preferences in equation (1) are: (a.1) homothetic; (a.2) convex; (a.3) strictly convex.
b) Let the price of commodity x_1 be $p_1 = 1$. Compute the optimal consumption bundle which maximizes the consumer's utility.
c) Compute now the ordinary demands of the three goods. Do the ordinary demand functions satisfy: (c.i) the Walras' law? (c.ii) the so called property of *no-money illusion*?
d) Compute the Slutsky substitution matrix and verify that is symmetric.

C.24**. In a two-commodity world and for utility functions: A) $u(x) = ax_1 + bx_2$, B) $u(x) = \mathsf{Min}\{ax_1, bx_2\}$, C) $u(x) = x_1^a x_2^b$ (with $a > 0$ and $b > 0$ in all the three cases). Sovle for the following:

a) the compensated demand functions;
b) the expenditure function and verify that the derivative of the expenditure function with respect to price of good l delivers the compensated demand of good l;
c) using duality, compute the ordinary demand function from the compensated demand functions and the indirect utility function from the expenditure function.

C.25** (Ex 3.G.6 in MWG). A consumer in a three-goods economy (goods denoted x_1, x_2, x_3 and prices p_1, p_2, p_3) with wealth $w > 0$ has demand functions for commodities 1 and 2 given by:

$$\begin{aligned} x_1(p,w) &= 100 - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \delta\frac{w}{p_3}, \\ x_2(p,w) &= \alpha + \beta\frac{p_1}{p_3} + \gamma\frac{p_2}{p_3} + \delta\frac{w}{p_3}, \end{aligned}$$

where Greek letters are nonzero constant.

a) Indicate how to calculate the demand for good 3 (but do not actually do it).
b) Are the demand functions for x_1 and x_2 appropriately homogeneous?
c) Calculate the restrictions on the numerical values α, β, γ and δ implied by utility maximization.
d) Given your results in part (c), for a fixed level of x_3 draw the consumer's indifference curves in the (x_1,x_2) plane.
e) What does your answer to (d) imply about the form of the consumer's utility function $u(x_1, x_2, x_3)$?

Exercises on 'Production'

P.1* The basic properties of the production set Y are the following ones:

- Possibility of inaction
- Closedness
- Impossibility of free production ("no free lunch")
- Free-disposal
- Irreversibility
- Convexity
- Increasing/decreasing/constant returns to scale.

Let $L = 2$ be the number of commodities. A firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha z$ with $\alpha > 0$ and $z \geq 0$.

a) Determine, both formally and graphically, the production set Y that corresponds to the production function f.
b) Determine if the production Y verifies the basic properties.

Now answer questions a) and b) for the two alternative production functions below:

- $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.
- $f(z) = \alpha z^2 + \beta z$ with $\alpha > 0$, $\beta > 0$ and $z \geq 0$.

P.2* For a general single output technology, show that the production set Y is convex if and only if the production function $q = f(z)$ is concave.

P.3* Let L be the finite number of commodities. A firm produces commodity L using the other $L - 1$ commodities as inputs. $z = (z_1, ..., z_L, ..., z_{L-1}) \in \mathbb{R}_+^{L-1}$ denotes a generic bundle of inputs. Show that if the production function $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is concave, then the transformation function defined by

$$F(y) := y_L - f(z)$$

is quasi-convex on the convex set $A = \{y = (-z, y_L) \in \mathbb{R}_+^L : z \geq 0 \text{ and } y_L \geq 0\}$.

P.4* Let $L = 3$ be the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by $f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta$ with $\alpha > 0$, $\beta > 0$, $z_1 \geq 0$ and $z_2 \geq 0$.

a) Write the production set Y determined by the production function f.

b) Determine if the production Y verifies the basic properties (same as in EX1).
c) Discuss under which conditions of the scalars α and β the technology exhibits decreasing, increasing or constant return to scale.

P.5* (Ex 5.C.1 in MCWG). Show that, in general, if the production set Y exhibits nondecreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$.

P.6* Let $L = 2$ be the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha z$ with $\alpha > 0$ and $z \geq 0$.

a) Write the profit maximization problem of this firm.
b) Consider the production set Y determined by the production function f. Using the shape of Y and the iso-profit lines, determine graphically the supply of this firm.
c) Determine the profit function of this firm.

P.7* Let L be the finite number of commodities. Assume that the production set Y of the firm is represented by a transformation function F such that $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.

a) State the profit maximization problem (PMP) of the firm.
b) Let F be continuous and strictly quasi-convex. Show that if PMP has a solution for $p \gg 0$, then it must be unique.

P.8*. Let $L = 2$ be the number of commodities. The firm produces commodity 2 using commodity z as an input. The production function is given by $f(z) = \alpha\sqrt{x}$ with $\alpha > 0$ and $z \geq 0$.

a) Write the transformation function and the profit maximization problem (PMP) of this firm.
b) Show that if $\bar{y} = (\bar{y}_1, \bar{y}_2)$ belongs to the supply of the firm, then $\bar{y}_1 < 0$ and $\bar{y}_2 > 0$.
c) Consider the open and convex set $A = \{y = (-z, y_2) \in \mathbb{R}^2 : z > 0 \text{ and } y_2 > 0\}$. Write the first order conditions associated with (PMP) on the set A.
d) Compute the supply and the profit function of this firm.

P.9* (Ex 5.C.9 in MCWG). Derive the profit function $\pi(p)$ and supply function (or correspondence) $y(p)$ for the single-output technologies with production functions given by: **(a)** $f(z) = \sqrt{z_1 + z_2}$; **(b)** $f(z) = \sqrt{Min\{z_1, z_2\}}$; **(c)** $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$.

P.10* Let L be the number of commodities. A firm uses a single-output technology to produce output q using $(L-1)$ commodities as inputs. Let $z := (z_1, ..., z_l, ..., z_{L-1}) \in \mathbb{R}^{L-1}$ denote a generic bundle of inputs.

b) Define the cost function $c(w, q)$ of the firm.

b) Show that the cost function $c(w, q)$ is a concave function of the input price vector $w := (w_1, ..., w_l, ..., w_{L-1})$.

c) Show that if the production function $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is concave, then the cost function $c(w, q)$ is a convex function of the output level q.

P.11 A firm has the single-output technology, $f(z) = z_1^a + z_2$, with $0 < a < 1$, and where p is the price of output and w_1 and w_2 are the prices of inputs.

a) Determine the conditional input demands, the profit and the supply function of the firm;

b) How does you answer to point a) change when $a > 1$?

P.12*. Let $L = 2$ be the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha(1 - \exp(-kz))$ with $k > 0$, $\alpha > 0$ and $z \geq 0$.

a) Write and draw the production set Y determined by the production function f.

b) For every level of output $y_2 \geq 0$, determine and draw the following set $Y(y_2) := \{z \in \mathbb{R}^2 : z \geq 0 \text{ and } f(z) \geq y_2\}$.

c) Write the cost minimization problem of this firm.

d) Determine the conditional demand of inputs and the cost function of the firm.

Now answer questions a) to d) for the two alternative production functions below:

- $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.
- $f(z) = \alpha z^2 + \beta z$ with $\alpha > 0$, $\beta > 0$ and $z \geq 0$.

P.13* A firm uses a single-output technology to produce output q using two commodities as inputs with prices $w = (w_1, w_2)$. The cost function is given by $c(w, q) = 2(q)^2(w_1)^{\frac{2}{3}}(w_2)^{\frac{1}{3}}$.

a) Show that the cost function is homogeneous of degree one in the inputs prices w.

b) Verify that this cost function is a convex function of the output level q.

c) Compute the supply and the profit function of the firm.

P.14* Let $L = 3$ be the number of commodities. The firm produces commodity q using commodities 1 and 2 as inputs. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ z_1 \geq 0 \text{ and } z_2 \geq 0$$

a) Determine the conditional input demands.

b) Determine the cost function of the firm and verify that the derivatives of the cost function with respect to price of input $l = 1, 2$ deliver the conditional input demands.

c) Discuss the conditions under which the technology admits a strictly positive, unique and finite level of output maximizing the firm's profit. For the case in which it is admitted, determine the supply and the profit function of the firm.

P.15* (Ex 5.D.1 in MCWG). Let $AC(\cdot)$ and $C'(\cdot)$ denote a firm's average cost function and marginal cost function, respectively. Show that $AC(\overline{q}) = C'(\overline{q})$ at any $\overline{q}$ satisfying $AC(\overline{q}) \leq AC(q)$ for all q. Does this result depend on differentiability of $C(\dot{})$ everywhere?

P.16 For the single-output technology, with two inputs, $q = (Min\{z_1; 2z_2\})^{\lambda}$, with $\lambda > 0$:

a) Discuss under which conditions of λ the technology exhibits decreasing, increasing or constant return to scale.

b) Compute the cost function.

c) Derive the profit maximizing output for the firm for the different values of λ.

d) Suppose now that factor z_2 is fixed in the short run. Derive the cost function in the short run.

e) Provide a diagrammatic representation of the cost curves in the short run and use the diagram to discuss the profit maximizing output of the firm in the short run, taking care to consider the conditions under which the firm decides to operate in the market.

Micro IB - Exercises with (partial) solutions[1]

QEM Master Program - Ca' Foscari University of Venice

Lecturer: Pietro Dindo
Version: 200821

Note: I expect all students to be confident with the solution of unmarked questions and to be able to solve the questions marked with * with a bit more effort. Questions marked with ** are more involved. I encourage to answer these ** questions after you have covered all the others.

1 Edgeworth Box

Exercise 1

Consider an economy with two consumption goods, $L = \{1, 2\}$, and two consumers $I = \{a, b\}$. Consumer a has endowment $\omega_a = (3, 0)$ while consumer b has endowment $\omega_2 = (1, 3)$. Both consumers have utility $U(x) = (x_1)^2(x_2)^3$.

1. Find the set of Pareto optimal allocations.
2. Determine the subset of Pareto optimal allocations where both consumers are at least as well-off as with their endowment (the contract curve).
3. Find the competitive equilibria. Are they Pareto optimal?

Exercise 2

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. Consumer 1 has utility $U_1(x_1) = x_{11} + \sqrt{x_{21}}$ and endowment $\omega_1 = (2, 0)$. Consumer 2 values only good 2, $U_2(x_2) = x_{22}$, and has endowment $\omega_2 = (0, 2)$.

1.* Find the set of Pareto optimal allocations (a graphical solution is enough).

2.* Find the competitive equilibria (a graphical solution is enough).

3.** Repeat the same analysis (both Pareto set and competitive equilibria) with the endowments $\omega_1 = (2, 1)$ and $\omega_1 = (0, 1)$.

[1] Exercises have been gathered from various sources: exercise sessions of the QEM Micro1B course, Ca' Foscari University of Venice; the textbook "Microeconomic Theory" by Mas-Colell, Whinston, and Green [MWG]; past Microeconomics I exams of the QEM Joint Degree Program; Class Notes written by Jean-Marc Bonisseau and Elena del Mercato for the QEM1 Microeconomics I course, Université Paris 1.

Note: This exercise is inspired by Figure 15.B.10(a) of the MWG textbook, you are encouraged to have a look at it.

Exercise 3

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. Consumer 1 has utility $U_1(x_1) = \sqrt{x_{11}} + x_{21}$. Consumer 2 values only good 1, $U_2(x_2) = x_{12}$. The total endowment is $\bar{\omega}$.

1.⋆ Find the set of Pareto optimal allocations (a graphical solution is enough).

2.⋆ Find the competitive equilibria when $\omega_{21} = 0$ (a graphical solution is enough).

3.⋆ Find the competitive equilibria when $\omega_{12} = 0$ (a graphical solution is enough).

Note: This exercise is inspired by Figure 15.B.10(a) of the MWG textbook, you are encouraged to have a look at it.

Exercise 4

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. The endowments are ω_1 and ω_2. Assume that both consumers have locally non satiated preferences. Prove that if the market for good l clears and $p \gg 0$, then also the market for good $l' \neq l$ clears.

Exercise 5

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. The total endowment is $\bar{\omega} = (4, 3)$. Find the set of Pareto optimal allocations in the following cases:

1. $U_1(x_1) = \sqrt{x_{11}}\sqrt{x_{21}}$ and $U_2(x_2) = (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}}$;
2. $U_1(x_1) = x_{11} + x_{21}$ and $U_2(x_2) = 2x_{12} + x_{22}$ (a graphical solution is enough);

3.⋆ $U_1(x_1) = \sqrt{x_{11}}\sqrt{x_{21}}$ and $U_2(x_2) = x_{12} + 2x_{22}$.

Exercise 6

Consider an economy with two consumption goods, $L = \{1, 2\}$, and two consumers $I = \{1, 2\}$. Consumer 1 has endowment $\omega_1 = (1, 2)$ while consumer 2 has endowment $\omega_2 = (2, 1)$. Consumer 1 has utility $U_1(x_1) = x_{11}(x_{21})^2$ while 2 has utility $U_2(x_2) = x_{12}x_{22}$.

1. Find the set of Pareto optimal allocations.
2. Determine the subset of Pareto optimal allocations where both consumers are at least as well-off as with their endowment (the contract curve).

3. Find whether the allocation on the Pareto set with $x_{11} = 2$ can be supported as a competitive equilibrium with transfers. If so, for which transfer using only good 1 ? Is there a transfer of only good 2 that achieves the same allocation?

Exercise 7 (*JD-QEM '20-'21*)

We consider an exchange economy with two consumers and two goods.

- Consumer 1 has consumption set $\mathbb{R}^2_+$, initial endowment $e_1 = (1, 1)$ and utility

$$u_1(x_{11}, x_{12}) = x_{11}^{1/3}(x_{12})^{2/3}$$

- Consumer 1 has consumption set $\mathbb{R}^2_+$, initial endowment $e_2 = (1, 3)$ and utility

$$u_2(x_{21}, x_{22}) = x_{21}^{1/2} x_{22}^{1/2}$$

1. Represent in the Edgeworth box the initial endowment and the indifference curves going through the initial endowment for both agents.
2. Represent in the Edgeworth box the set B of allocations that are better, in the sense of Pareto, than the initial endowments.
3. Give the definition of a Pareto Optimum for this economy.
4. Determine the set C of Pareto Optima of the economy.
5. Represent (approximately) the set C in the Edgeworth box. Do B and C coincide ? Explain why/not ?
6. Give the definition of a general equilibrium for this economy.
7. Determine the general equilibrium of this economy (normalize to 1 the price of commodity 2).

Exercise 8

Solve Example 15.B.2 from the MWG textbook.

Exercise 9

Solve Exercise 15.B.2 from the MWG textbook.

Exercise 10

Solve Exercise 15.B.9 from the MWG textbook.

2 Robinson Crusoe Economy

Exercise 11

Solve Exercise 15.C.2 from the MWG textbook.

Exercise 12

Consider an economy with two periods, $t = 0, 1$, and a consumption good per period. The consumer has utility $U(x_0, x_1) = u(x_0) + \beta u(x_1)$ with $\beta \in (0, 1)$ and endowment k_0 of consumption good in $t = 0$. The economy has one firm, $J = 1$, which transforms the good in $t = 0$ (input, k_1), into an output (y) in $t = 1$. The firm production function is $f(k) = Ak^\alpha$ with $A > 0$ and $\alpha \in (0, 1]$. Assume that $u(x) = \log x$ and $\alpha \in (0, 1)$.

1. Find the competitive equilibrium.
2. Find the firm profit in the competitive equilibrium and discuss its dependence on α.
3. Find the Pareto optimal allocation of this economy and compare it with the allocation found in 1.

Exercise 13

Solve Exercise 12 when

1.* $u(x) = x$ and $\alpha \in (0, 1)$;

2. $u(x) = \log x$ and $\alpha = 1$;

3.** $u(x) = x$ and $\alpha = 1$.

Exercise 14

Consider an economy with two consumption goods, $L = 2$, one consumer, $I = 1$, and one firm, $J = 1$. The consumer has endowment $\omega = (4, 1)$ and utility $U_1(x) = x_1 x_2$. The firm has production set $Y_1 = \{y_2 \leq -ay_1,\ y_1 \leq 0\}$ with $a > 0$. The consumer is the owner of the firm.

1. Write down the definition of a competitive equilibrium for this economy.
2. Fix $a = 1$, find the set of competitive equilibria and the corresponding firm profits.
3. Are the the competitive equilibria found above Pareto optimal?
4. Find the Pareto optimal allocations for $a = 1$.

5.* Answer to 2. and 3. for each given $a > 0$.

6.* Find the Pareto optimal allocations for each given $a > 0$.

3 Competitive equilibrium and welfare theorems

Exercise 15

Consider an economy with L commodities, I consumers, and J firms. The endowments and ownership shares are $\left(\omega_i, (\theta_{ji})_{j=1}^J\right)_{i=1}^I$. Assume that all consumers have locally non satiated preferences. Prove that, for all choices of a good l', if the markets for all goods $l \neq l'$ clears and $p \gg 0$, then also the market for good l' clears.

Exercise 16

Consider an exchange economy with two consumption goods, $L = 2$, and two consumers, $I = 2$. Consumer 1 has endowment $\omega_1 = (2, 0)$ and utility $U_1(x_1) = (x_{11})^\alpha + (x_{21})^\alpha$. Consumer 1, has endowment $\omega_2 = (1, 3)$ and utility $U_2(x_2) = x_{12}x_{22}$. Consider $\alpha = \frac{1}{2}$ first.

1. Find the competitive equilibria and the set of Pareto optimal allocations.
2. Discuss the validity of both Welfare Theorems.
3. The social planner has the possibility to implement some transfer between the initial endowments of commodity 2. Is there a transfer such that, in the competitive equilibrium achieved after trading, both agents consume the same amount of good 1. If so, find it. Can the same competitive equilibrium arise after a transfer of good 1? If so, find it.
4. Repeat the exercise taking $\alpha = 1$.

Exercise 17

Consider an economy with two consumption goods, $L = 2$, two consumers, $I = 2$, and one firm, $J = 1$. Consumer 1 has endowment $\omega_1 = (4, 1)$ and utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2 has endowment $\omega_2 = (1, 1)$ and utility $U(x_2) = x_{12} + x_{22}$. The firm has production set $Y = \{y_2 \leq \sqrt{-y_1},\ y_1 \leq 0\}$. Consumer two is the owner of the firm.

1. Write down the definition of a competitive equilibrium in this economy.
2. Find the set of competitive equilibria and the corresponding firm profits.
3. Are the competitive equilibria allocation found above Pareto optimal?

4.** Find the set of Pareto optimal allocation of this economy.

Exercise 18

Consider three commodities, time for leisure/labor (commodity 1), durable good (commodity 2) and consumption good (commodity 3). There is a single consumer with utility function $U(x_2, x_3) = x_2 + x_3$, so that the consumption of leisure does not have any impact on his utility. The consumer has an endowment of leisure equal to 1 and no endowment of durable and consumption good.

There are two firms. Firm 1 produces only durable goods using labor as input. Firm 1 production set is $Y_1 = \{-\sqrt{-y_{11}} + y_{21} \leq 0,\ y_{11} \leq 0\}$. Firm 2 produces only consumption good using labor as input. Firm 2 production set is $Y_2 = \{-2\sqrt{-y_{12}} + y_{32} \leq 0,\ y_{12} \leq 0\}$. The consumer is the owner of both firms.

1. Write down the definition of a competitive equilibrium in this economy.
2. Check that
$$\left\{\frac{p_1}{p_2} = \frac{\sqrt{5}}{2}, \frac{p_3}{p_2} = 1, x = \left(0, \frac{1}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), y_1 = \left(-\frac{1}{5}, \frac{1}{\sqrt{5}}\right), y_2 = \left(-\frac{4}{5}, \frac{4}{\sqrt{5}}\right)\right\}$$
is the only competitive equilibrium of this economy.
3. Is the competitive equilibria allocation found above Pareto optimal?

4.* Find the set of Pareto optimal allocation of this economy.

Exercise 19

Find the competitive equilibrium and the Pareto set in an economy as above with $U(x_2, x_3) = x_2 x_3$ and $Y_2 = \{y_{12} + y_{32} \leq 0,\ y_{12} \leq 0\}$ (firm 1 keeps the same technology).

Exercise 20

Consider an economy with two consumption goods, $L = 2$, two consumers, $I = 2$, and one firm, $J = 1$. Consumer 1, the worker, has endowment $\omega_1 = (1, 0)$ and utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2, the capitalist, is the owner of the firm, $\theta_{21} = 1$, and has utility $U_2(x_1) = x_{12}x_{22}$. The firm has production set $Y = \{y_1 \leq 2\sqrt{-y_2} \leq 0, y_1 < 0\}$.

1. Provide the definition of a competitive equilibrium for this economy.
2. Find the set of competitive equilibria and the set of Pareto optimal allocations.
3. Discuss the validity of both Welfare Theorems.
4. Find the competitive equilibrium with transfer such that both consumers consume the same amount of output. How can you implement such a transfer starting from the given endowment/ownership?

Exercise 21

Consider an exchange economy with two consumption goods, $L = 2$, and two consumers, $I = 2$. Consumer 1 has endowment $\omega_1 = (2, 2 - \delta)$ with $\delta \in (0, 2)$ and utility $U(x_1) = \epsilon x_{11} + x_{21}$ for $\epsilon > 0$. Consumer 2 has endowment $\omega_2 = (2, \delta)$ and utility $U(x_2) = x_{12} + x_{22}$.

1. Consider $\epsilon = \delta = 0$. Find the set of competitive equilibria and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.

2. Consider $\epsilon = \delta = 0$. State whether $p = (0, 1)$, $p = (1, 1)$, $p = (1, 2)$, $p(2, 1)$ support $x_1^* = (0, 2), x_2^* = (4, 0)$ as a quasi-equilibrium transfer. If so, is each $\{p, x^*\}$ also a competitive equilibrium?

3. Consider $\epsilon > 0$ and $\delta = 0$. Find and plot the excess demand $z_1(p) = x_{11}(p) + x_{12}(p) - \bar{\omega}_1$ of good 1, the set of competitive equilibria, and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.

4. Consider $\epsilon = 0$ and $\delta \in (0, 2)$. Find and plot the excess demand function of good 1, the set of competitive equilibria, and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.

Exercise 22

Consider an economy with two consumption goods, $L = 2$, one consumer, $I = 1$, and one firm, $J = 1$. Consumer 1 has endowment $\omega_1 = (2, 1)$ and utility $U(x_1) = x_{11}x_{21}$. The firm has production set $Y = \{y_1 + y_2 \leq 0 \text{ if } y_1 \leq -a,\ y_2 \leq 0 \text{ if } y_1 \in (-a, 0]\}$ with $a \geq 0$. Consumer one is the owner of the firm. Fix $a = 1$.

1. Provide the definition of a competitive equilibrium for this economy.

2. Find the set of competitive equilibria.

3. Find the set of Pareto optimal allocations.

4. Discuss the validity of both Welfare Theorems.

Exercise 23

Solve Ex. 23 for each given $a \in [0, 1]$.

Exercise 24

Consider an exchange economy with $I = \{1, 2\}$, $L = \{1, 2\}$, $U_1(x_1) = x_{21}$, $U_2(x_2) = x_{12} + x_{22}$, and $\omega_1 = (1, 0)$, $\omega_2 = (0, 1)$.

1.* Find the utility possibility set

2.* Find the Pareto frontier

3.** Show that for some levels of $\bar{U}_2$ the solutions x^* of

$$\begin{array}{l} \text{Max } U_1(x_1) \\ x \in \mathbf{R}^4_+ \\ x_1 + x_2 \leq \bar{\omega} \\ U_2(x_2) \geq \bar{U}_2 \end{array}$$

are not Pareto optimal (and thus $(U_1(x_1^*), U(x_2^*))$ does not belong to the Pareto frontier).

Exercise 25 (*Economics-QEM exam '20-'21 I*)

Consider an economy with two commodities, two consumers, and one firm. Consumer 1 has endowment $\omega_1 = (6, 0)$, owns half of the firm, and has utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2 has endowment $\omega_2 = (4, 2)$, owns the other half of the firm, and has utility $U_2(x_2) = x_{12}x_{22}$. The firm uses commodity 1 to produce commodity 2 and has production function $f(z) = z$ with $z \geq 0$ (the amount of commodity 1).

1. Derive supply and profit of the firm.
2. Derive the demand of both consumers.
3. Provide the definition of a competitive equilibrium for this economy.
4. Compute the unique competitive equilibrium of this economy.
5. Is the competitive equilibrium found above Pareto optimal? Reply without computing the Pareto set.
6. Provide a definition of Pareto optimal allocation.
7. Explain how to find all the Pareto optimal allocation of this economy (write down the related maximization problem and the first order conditions for an interior solution).

4 Expected Utility

Exercise 26

Solve Exercise 6.B.2 in the MWG textbook.

Exercise 27

Show that a preference relation on lotteries that can be represented in an Expected Utility form satisfies transitivity.

Exercise 28

Consider the space of simple lotteries $L = \{C; (p_1, p_2, p_3)\}$. Show graphically that if a preference relation on lotteries satisfies the independence axiom, then indifference curves are parallel. (hint: show that if two indifference curves are not parallel, then the independence axiom is violated)

Exercise 29

Solve Exercise 6.B.4 in the MWG textbook.

Exercise 30

Solve Exercise 6.B.7 in the MWG textbook. (Note: a preference over lotteries is monotone if, given two real outcomes $C_1 > C_2$, then the lottery with sure outcome C_1 is strictly preferred to the lottery with sure outcome C_2)

Exercise 31

When faced with the choice between the lottery $A = \{(3500, 2800, 0); p = (0.3, 0.66, 0.04)\}$ and $B = \{3500; p = 1\}$, a decision maker chooses lottery B. When, instead, he is asked to choose between the lottery $A' = \{(3500, 2800, 0); p = (0.3, 0, 0.7)\}$ and lottery $B' = \{(3500, 2800, 0); p = (0, 0.34, 0.66)\}$, he chooses A'. Say whether the decision maker's preferences are consistent with the expected utility form, explaining how you have reached your conclusion. What if lottery B becomes $B'' = \{2800; 1\}$?

Exercise 32 (*JD-QEM exam '19-'20*)

We consider an expected-utility decision-maker facing the following possible professional occupations: working in the financial industry (A), working in the movie industry (B), working in the car industry (C). We further assume that the decision-maker can apply to three different schools.

- After School 1, he is certain to find a job in the car industry.
- After School 2, he is certain to find a job in the financial industry.

- After School 3, he would find a job in the movie industry with probability 0.1 and in the financial industry with probability 0.9.

We assume that School 2 is the least preferred option of the decision-maker and that he is indifferent between School 1 and School 3.

1. Give a representation of the utility function of this decision-maker.
2. We assume that a new school (School 4) opens. After school 4, a student would find a job in each industry with probability $\frac{1}{3}$. How would the new school rank compared to Schools 1, 2, 3?
3. There are too many application in School 4. The ministry of education decides that rather than applying to School 4 directly, the decision-maker must apply to a lottery that leads to admission in School 4 with probability $\alpha \in [0,1]$ and to admission in School 2 with probability $1-\alpha$. For which value of α does the decision-maker prefer to apply to School 1 rather than to the lottery that may lead to admission to School 4?

Exercise 33 (*JD-QEM exam '20-'21*)

We consider an expected-utility decision-maker with utility of the form $u(x) = x^a$ with $a < 1$.

1. Let X be a lottery whose outcome is uniformly distributed over $[0,1]$. Determine $\mathbb{E}(u(X))$.
2. Let Y be a lottery whose outcome is 0 with probability $1/3$ and 1 with probability $2/3$. Determine $\mathbb{E}(u(Y))$.
3. Determine the value a^* (of a) for which the decision-maker is indifferent between X and Y.
4. If $a > a^*$, which lottery, X or Y, is preferred by the decision-maker?
5. Determine, as a function of a, the coefficient of absolute risk-aversion of the decision-maker.

Exercise 34

Solve Exercise 6.C.1 in MWG assuming $u(x) = \log(x)$ and find* for which value of q the agent does not insure.

Exercise 35

Consider an investor who has to choose between two assets. Asset A has payoff $x_A = (12, 6, 9)$, all with equal probability. Asset B is risk free with payoff $x_B = (\bar{x}, \bar{x}, \bar{x})$.

1. Assume $u(x) = \log(x)$. Find the certainty equivalent of x_B.
2. Find the level of $\bar{x}$ such that the investor is indifferent between asset A and asset B.
3. Assume now that $\bar{x} = E[x_A]$, find how many units h_A of asset A should be given to the agent to make him indifferent between h_A and one unit of the risk free asset. Provide an intuition.
4. Compute the coefficient and absolute risk aversion.

Exercise 36 (*Economics-QEM exam '20'21 I*)

A decision maker has preferences over lotteries represented by an expected utility with Bernoulli utility $u(x)$. Consider the lottery $L = \{(0, 4, 9); (1/6, 1/2, 1/3)\}$

1. Provide the definition of certainty equivalent of L given u.
2. Compute the certainty equivalent for L given $u(x) = \sqrt{x}$.
3. Compute the coefficient of absolute risk aversion given u and x.
4. Assume that the agent starts with $w = 12$ and owns the lottery. Find the minimum price he is willing to accept to sell the lottery.
5. Compare the minimum price found above and the certainty equivalent and provide an intuition of their difference based on the coefficient of absolute risk aversion.

Exercise 37

Solve Exercise 6.C.18 in MWG.

Exercise 38

Solve Exercise 6.C.16 in MWG assuming from the beginning the parametrization of point (d) with $p = 0.5$. Other than the proposed $u(x) = \sqrt{x}$, consider also $u(x) = x$ and $u(x) = \log x$. Give and interpretation in terms of the certainty equivalent and the change of the risk aversion coefficient with wealth.

5 Solutions

Solution of Exercise 1

Consider an economy with two consumption goods, $L = \{1, 2\}$*, and two consumers* $I = \{a, b\}$*. Consumer a has endowment* $\omega_a = (3, 0)$ *while consumer b has endowment* $\omega_2 = (1, 3)$*. Both consumers have utility* $U(x) = (x_1)^2(x_2)^3$*.*

1. *Find the set of Pareto optimal allocations.*
2. *Determine the subset of Pareto optimal allocations where both consumers are at least as well-off as with their endowment (the contract curve).*
3. *Find the competitive equilibria. Are they Pareto optimal?*

1. In order to find the Pareto set one should maximize the utility of one consumer by choosing non-wasteful feasible allocation that do not decrease the utility of the other consumer. The first order conditions (which are both necessary and sufficient given strict concavity of U) for such a problem, considering all possible levels of utility, lead to

$$\begin{cases} \frac{2}{3}\frac{x_{2a}}{x_{1a}} = MRS^a_{12} = MRS^b_{12} = \frac{2}{3}\frac{x_{2b}}{x_{1b}} \\ x_{1a} + x_{1b} = 4 \\ x_{2a} + x_{2b} = 3 \end{cases}$$

whose solution is

$$PS = \left\{ x_1 = \left(t, \frac{3}{4}t\right), x_2 = \left(4 - t, 3 - \frac{3}{4}t\right), t \in [0, 4] \right\},$$

the Pareto Set. Note that for consumer b on the Pareto set $x_{2b} = \frac{3}{4}x_{1b}$.

2. To find the contract curve, first we compute both agents' level of utility in the endowment. It holds

$$U_a(\omega_a) = (3)^2(0)^3 = 0 \quad \text{and} \quad U_b(\omega_b) = (1)^2(3)^3 = 27\,.$$

Then, we find the allocation on the Pareto set where both agents are at least as well of as in their endowment by solving

$$\begin{cases} x_{2a} = \frac{3}{4}x_{1a} \\ x_{1b} = 4 - x_{1a} \\ x_{2b} = 3 - x_{2a} \\ x_{1a} \in [0, 4] \\ x_{1a}^2 x_{2a}^3 \geq 0 \\ x_{1b}^2 x_{2b}^3 \geq 27 \end{cases}$$

Solving the above, we find the contract curve as the subset of the Pareto set PS where $x_{1a} \in \left[0, 4 - 4^{\frac{3}{5}}\right]$.

3. To find the competitive equilibrium, we first find both agents offer curves (demand as a function of the price ratio), getting

$$\begin{aligned} OC_a &= \left\{ \left(\tfrac{2}{5}\tfrac{3p_1}{p_1}, \tfrac{3}{5}\tfrac{3p_1}{p_2} \right) \; p_1 > 0, p_2 > 0 \right\}, \\ OC_b &= \left\{ \left(\tfrac{2}{5}\tfrac{p_1+3p_2}{p_1}, \tfrac{3}{5}\tfrac{p_1+3p_2}{p_2} \right) \; p_1 > 0, p_2 > 0 \right\}. \end{aligned}$$

(we can exclude the case of $p_l = 0$ because it leads to an infinite demand of good l for both agents and thus not to an equilibrium). Having the offer curves, we can impose the market clearing conditions (by Ex. 4 imposing one condition is sufficient for solving the problem). Equilibrium in the market for the first good leads to

$$\frac{2}{5}\frac{3p_1}{p_1} + \frac{2}{5}\frac{p_1 + 3p_2}{p_1} = 4 \Rightarrow \frac{p_1}{p_2} = \frac{1}{2}.$$

The competitive equilibrium is the price found above and the corresponding optimal allocation:

$$CE = \left\{ \frac{p_1^*}{p_2^*} = \frac{1}{2}, \, x_a^* = \left(\frac{6}{5}, \frac{9}{10} \right), \, x_b^* = \left(\frac{14}{5}, \frac{21}{10} \right) \right\}$$

Note that the allocation $(x_a^*, x_b^*) \in PS$.

Solution of Exercise 2

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. Consumer 1 has utility $U_1(x_1) = x_{11} + \sqrt{x_{21}}$ and endowment $\omega_1 = (2, 0)$. Consumer 2 values only good 2, $U_2(x_2) = x_{22}$, and has endowment $\omega_2 = (0, 2)$.

1.⋆ *Find the set of Pareto optimal allocations (a graphical solution is enough).*

2.⋆ *Find the competitive equilibria (a graphical solution is enough).*

3.⋆⋆ *Repeat the same analysis (both Pareto set and competitive equilibria) with the endowments $\omega_1 = (2, 1)$ and $\omega_1 = (0, 1)$.*

<u>Note</u>: *This exercise is inspired by Figure 15.B.10(a) of the MWG textbook, you are encouraged to have a look at it.*

1. The Pareto set of this economy is

$$PS = \{(2, t), (0, 2 - t), t \in [0, 2]\}$$

All other allocations of the Edgeworth box are not in the Pareto set.
At interior allocations agents have different MRS and first order conditions are not satisfied (note that both can improve their welfare by letting agent 1 increase good 1 and decrease good 2 while agent 2 increases good 2. For the same reason also allocations with $x_{22} = 0$ or $x_{11} = 0$ can be improved upon. Allocations with $x_{21} = 0$ (but $x_{11} < 2$ are not in the Pareto set because in all these allocations agent 2 has the same utility and agent 1

can be made better off by having $x_{11} = 2$.
NOTE: Agent 2 preferences not being strictly monotone implies that $O_1 = (0, 0) \notin PS$.

2. In this economy there are no prices that support a competitive equilibrium with the given endowment. See also the text commenting Example 15.B.10(a) in the MWG.

3.a Consider $\omega_1 = (2, 1)$ and $\omega_2 = (0, 1)$. The endowment is in the Pareto Set and it is interior for agent 1. Thus, provided

$$\frac{p_1}{p_2} = MRS^1_{12}(\omega_1) = 2\sqrt{(\omega_{21})} = 2$$

we have a competitive equilibrium. The competitive equilibrium is thus

$$CE = \left\{ \frac{p_1^*}{p_2^*} = 2,\ x_1^* = (2, 1)\,,\ x_2^* = (0, 1) \right\}$$

3.b Consider $\omega_1 = (0, 1)$ and $\omega_2 = (2, 1)$. The endowment is not in the Pareto Set. We expect the C.E. to be in the Pareto Set, thus we look for an allocation in the Pareto Set that can be reached trading from ω. Assume fist that the C.E. allocation is interior for agent 1, we look for a solution of

$$\begin{cases} x_{11} = 2 \\ x_{21} = t > 0 \\ MRS^1_{12} = 2\sqrt{t} = \frac{p_1}{p_2} \geq MRS^2_{12} = 0 \\ x_{11} + x_{12} = 2 \\ x_{21} + x_{22} = 2 \\ p_2 = p_1 2 + t p_2 \\ p_1 2 + p_2 = (2 - t) p_2 \end{cases}$$

Solving the above gives

$$CE = \left\{ \frac{p_1^*}{p_2^*} = p^* = \sqrt{20} - 4,\ x_1^* = \left(2, \frac{(p^*)^2}{4}\right),\ x_2^* = \left(0, 2 - \frac{(p^*)^2}{4}\right) \right\}$$

Finally note that the allocation $\{(2, 0), (0, 2)\}$ in the Pareto Set, not interior for agent 1, cannot be a C.E. (see point 2. above).

Solution of Exercise 3

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. Consumer 1 has utility $U_1(x_1) = \sqrt{x_{11}} + x_{21}$. Consumer 2 values only good 1, $U_2(x_2) = x_{12}$. The total endowment is $\bar{\omega}$.

1.* *Find the set of Pareto optimal allocations (a graphical solution is enough).*

2.* *Find the competitive equilibria when $\omega_{21} = 0$ (a graphical solution is enough).*

3.* *Find the competitive equilibria when* $\omega_{12} = 0$ *(a graphical solution is enough).*

Note: This exercise is inspired by Figure 15.B.10(a) of the MWG textbook, you are encouraged to have a look at it.

1. The Pareto set of this economy is

$$PS = \{(t, \bar{\omega_2}), (\bar{\omega_1} - t, 0), t \in [0, \bar{\omega_1}]\}$$

All other allocations of the Edgeworth box are not in the Pareto set for similar reasons as given in the solution of point 1. *Ex.* 3.
NOTE: Also here the fact that agent 2 preferences are not strictly monotone implies that $O_1 = (0, 0) \notin PS$.

2. and 3. To be proposed in class (see the notes at the end of Lecture IV, '20-21).

Solution of Exercise 4

Consider an economy with two consumers, $I = \{1, 2\}$, *and two commodities,* $L = \{1, 2\}$. *The endowments are* ω_1 *and* ω_2. *Assume that both consumers have locally non satiated preferences. Prove that if the market for good* l *clears and* $p \gg 0$, *then also the market for good* $l' \neq l$ *clears.*

First note that if preferences are LNS, then the optimal consumption is chosen on the budget line:

$$px_i = p\omega_i \quad i = 1, 2.$$

Adding up the latter leads to

$$\sum_{i=1,2} \sum_{l=1,2} p_l \left(x_{li} - \omega_{li}\right) = 0$$

or, equivalently,

$$\sum_{l=1,2} p_l \sum_{i=1,2} \left(x_{li} - \omega_{li}\right) = 0.$$

The latter together with market clearing in good l, $\sum_i (x_{li} - \omega_{li}) = 0$, leads to

$$\sum_{i=1,2} p_{l'} (x_{l'i} - \omega_{l'i}) = 0.$$

If $p_{l'} > 0$, then it must be $\sum_i \left(x_{l'i} - \omega_{l'i}\right) = 0$ so that also the market for good l' clears.

Solution of Exercise 5

Consider an economy with two consumers, $I = \{1, 2\}$, *and two commodities,* $L = \{1, 2\}$. *The total endowment is* $\bar{\omega} = (4, 3)$. *Find the set of Pareto optimal allocations in the following cases:*

1. $U_1(x_1) = \sqrt{x_{11}}\sqrt{x_{21}}$ *and* $U_2(x_2) = (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}}$*;*

2. $U_1(x_1) = x_{11} + x_{21}$ *and* $U_2(x_2) = 2x_{12} + x_{22}$ *(a graphical solution is enough);*

3.* $U_1(x_1) = \sqrt{x_{11}}\sqrt{x_{21}}$ *and* $U_2(x_2) = x_{12} + 2x_{22}$.

1. We proceed as in point 1. of Ex. 1. Both agents preferences are strictly convex and strongly monotone, thus the FOC are necessary and sufficient. Computing MRS for both agents gives

$$\begin{cases} \frac{x_{21}}{x_{11}} = MRS^1_{12} = MRS^2_{12} = \frac{1}{2}\frac{x_{22}}{x_{12}} \\ x_{11} + x_{12} = 4 \\ x_{21} + x_{22} = 3 \end{cases},$$

whose solution is

$$PS = \left\{ x_1 = \left(t, \frac{3t}{8-t}\right), x_2 = \left(4-t, 3 - \frac{3t}{8-t}\right), t \in [0,4] \right\}.$$

2. Given that both agents' preferences are represented by a linear utility, and that $MRS^1_{12} = 1 \neq MRS^2_{12} = 2$, there is not interior solution that satisfies the FOC conditions for Pareto optimality. Evaluating the frontier of the Edgeworth box, we notice that both when $x_{12} = 0$ and when $x_{21} = 0$ agent 1 (2) can decrease (increase) consumption of good 1 and increase (decrease) consumption of good 2 to achieve a higher utility. Thus these allocations are Pareto optimal. Instead, when $x_{11} = 0$ agent 1 cannot decrease further consumption of good 1 even if she values, relative to agent 2, good 1 less than good 2, $MRS^1_{12} = 1 < MRS^2_{12} = 2$. In the same way, when $x_{22} = 0$ agent 2 cannot decrease further the consumption of good 2 even if she values, relative to agent 1, good 2 less than good 1, $MRS^2_{21} = 1/2 < MRS^1_{21} = 1$. Thus

$$PS = \{x_1 = (0,t), x_2 = (4, 3-t), t \in [0,3]\} \cup \{x_1 = (t,3), x_2 = (t,0), t \in [0,4]\}.$$

3. To find the Pareto set we can proceed as in point 1., i.e. by maximizing agent 1 utility by choosing allocations in the Edgeworth both where the utility of agent 2 does not fall below a given level. Linearity of agent 2 utility implies that the Pareto Set might be on the frontier of the Edgeworth box. In particular, to maximize agent 2 utility we might end up giving to her only one good. The FOC for an internal solutions are

$$\begin{cases} \frac{x_{21}}{x_{11}} = MRS^1_{12} = MRS^2_{12} = \frac{1}{2} \\ x_{11} + x_{11} = 4 \\ x_{21} + x_{21} = 3 \\ x_{li} > 0 \ \forall l, i \end{cases}.$$

The latter is solved by $x_{21} = x_{11}/2$ provided $x_{11} \in (0,4)$ (the bundle of agent 2 follows by imposing the F.C.). Another set of FOC considers the possibility that agent 2 has no consumption of either good. Let us start with good 1, we obtain:

$$\begin{cases} \frac{x_{21}}{x_{11}} = MRS^1_{12} > MRS^2_{12} = \frac{1}{2} \\ x_{11} + x_{11} = 4 \\ x_{21} + x_{21} = 3 \\ x_{12} = 0, x_{11} = 4, x_{21} > 0, x_{22} > 0 \end{cases},$$

with solution $\{x_{12} = 0, x_{11} = 4, x_{21} = t \in [2, 3), x_{22} = 3 - t\}$. When, instead, agent 2 does not consume good 2 we obtain:

$$\begin{cases} \frac{x_{21}}{x_{11}} = MRS^1_{12} < MRS^2_{12} = \frac{1}{2} \\ x_{11} + x_{11} = 4 \\ x_{21} + x_{21} = 3 \\ x_{12} > 0, x_{11} > 0, x_{21} = 3, x_{22} = 0 \end{cases}.$$

The latter implies, at the same time, $x_{11} > 6$ and $x_{11} \leq 4$, thus leading to no solutions. The Pareto set is thus

$$PS = \left\{x_1 = \left(t, \frac{t}{2}\right), x_2 = \left(4 - t, 3 - \frac{t}{2}\right), t \in [0, 4)\right\} \cup \{x_1 = (4, t), x_2 = (0, 3 - t), t \in [2, 3]\}.$$

(note that we have added the allocations when either agent has all the endowment)

Solution of Exercise 6

Consider an economy with two consumption goods, $L = \{1, 2\}$, and two consumers $I = \{1, 2\}$. Consumer 1 has endowment $\omega_1 = (1, 2)$ while consumer 2 has endowment $\omega_2 = (2, 1)$. Consumer 1 has utility $U_1(x_1) = x_{11}(x_{21})^2$ while 2 has utility $U_2(x_2) = x_{12}x_{22}$.

1. *Find the set of Pareto optimal allocations*
2. *Determine the subset of Pareto optimal allocations where both consumers are at least as well-off as with their endowment (the contract curve)*
3. *Find whether the allocation on the Pareto set with $x_{11} = 2$ can be supported as a competitive equilibrium with transfers. If so, for which transfer using only good 1 ? Is there a transfer of only good 2 that achieves the same allocation?*

1. In order to find the Pareto set one should maximize the utility of one consumer by choosing non-wasteful feasible allocation that do not decrease the utility of the other consumer. The first order conditions (which are both necessary and sufficient given strict concavity of U) for such a problem and considering all possible levels of utility and interior solutions are

$$\begin{cases} \frac{1}{2}\frac{x_{21}}{x_{11}} = MRS^1_{12} = MRS^2_{12} = \frac{x_{22}}{x_{12}} \\ x_{11} + x_{12} = 3 \\ x_{21} + x_{22} = 3 \\ x_{li} > 0 \quad \forall l, i \end{cases}$$

whose solution is (including the two "origins" where either consumer has the total endowment)

$$PS = \left\{x_1 = \left(t, \frac{6t}{3 + t}\right), x_2 = \left(3 - t, 3 - \frac{6t}{3 + t}\right), t \in [0, 3]\right\},$$

the Pareto Set.

2. To find the contract curve, first we compute both agents' level of utility in the endowment. It holds

$$U_1(\omega_1) = (1)^1(2)^2 = 4 \quad \text{and} \quad U_2(\omega_2) = (2)(1) = 2\,.$$

Then, we find the allocation on the Pareto set where both agents are at least as well of as in their endowment by solving

$$\begin{cases} x_{21} = \frac{6x_{11}}{3+x_{11}} \\ x_{12} = 3 - x_{11} \\ x_{22} = 3 - x_{21} \\ x_{11} \in [0,3] \\ x_{11}x_{21}^2 \geq 4 \\ x_{12}^2 x_{22}^3 \geq 2 \end{cases}$$

Graphically it can be shown that there exist two levels of x_{11}, x_{11}^d and x_{11}^u with $x_{11}^u > x_{11}^d$ such that the solution of the above (the contract curve) is the subset of the Pareto set PS with $x_{11} \in \left[x_{11}^d, x_{11}^u\right]$.

4. The allocation of the Pareto Set with $x_{11} = 2$ has

$$x_{21} = \frac{6x_{11}}{3+x_{11}}|_{x_{11}=2} = \frac{12}{5}, \quad x_{12} = x_{11} - 3|_{x_{11}=2} = 1, \quad x_{22} = 3 - \frac{6x_{11}}{3+x_{11}}|_{x_{11}=2} = \frac{3}{15}\,.$$

At this allocation we have

$$MRS_{12}^1(x_1) = \frac{3}{5} = MRS_{12}^2(x_2),$$

thus the allocation can not be supported as a competitive equilibrium with transfer with supporting price ration

$$\frac{p_1}{p_2} = \frac{3}{5}.$$

To find the transfer, we solve for T_1 such that

$$px_1 = w_1 = p\omega_1 + T_1.$$

Equivalently

$$px_2 = w_2 = p\omega_2 + T_2.$$

with $T_2 = -T_1$. Expressing everything in terms of good 1, i.e. dividing left and write by p_1, we find the transfer T_1 in good 1

$$\frac{T_1}{p_1} = x_{11} + \frac{p_2}{p_1}x_{21} - \omega_{11} - \frac{p_2}{p_1}\omega_{21}.$$

Using prices and allocation found above we obtain

$$\frac{T_1}{p_1} = 2 + \frac{5}{3}\frac{12}{5} - 1 - \frac{5}{3}2 = \frac{5}{3}.$$

The latter is the amount of good 1 that need to be given to consumer 1 to support $x_{11} = 2$ as a competitive equilibrium. Note that the transfer can be implemented because $\omega_{11} + \frac{T_1}{p_1} = \frac{8}{3} < 3 = \bar{\omega}_1$ and gives thus a feasible new endowment. Note also that working with T_2 we would have found the same result:

$$\frac{T_2}{p_1} = -\frac{5}{3} \Rightarrow \frac{T_1}{p_1} = \frac{5}{3}.$$

The same transfer can be expressed in good 2 by using

$$\frac{T_1}{p_2} = \frac{T_1}{p_1}\frac{p_1}{p_2} = 1$$

Also this transfer can be implemented leading to the new endowment $(3, 2)$ for agent 1 and $(0, 1)$ for agent 2.

Solution of Exercise 7

We consider an exchange economy with two consumers and two goods.

- *Consumer* 1 *has consumption set* $\mathbb{R}^2_+$, *initial endowment* $e_1 = (1, 1)$ *and utility*

$$u_1(x_{11}, x_{12}) = x_{11}^{1/3}(x_{12})^{2/3}$$

- *Consumer* 1 *has consumption set* $\mathbb{R}^2_+$, *initial endowment* $e_2 = (1, 3)$ *and utility*

$$u_2(x_{21}, x_{22}) = x_{21}^{1/2}x_{22}^{1/2}$$

1. *Represent in the Edgeworth box the initial endowment and the indifference curves going through the initial endowment for both agents.*
2. *Represent in the Edgeworth box the set* B *of allocations that are better, in the sense of Pareto, than the initial endowments.*
3. *Give the definition of a Pareto Optimum for this economy.*
4. *Determine the set* C *of Pareto Optima of the economy.*
5. *Represent (approximately) the set* C *in the Edgeworth box. Do* B *and* C *coincide ? Explain why/not ?*
6. *Give the definition of a general equilibrium for this economy.*
7. *Determine the general equilibrium of this economy (normalize to* 1 *the price of commodity* 2*).*

TO BE PREPARED

Solution of Exercise 8

Solve Example 15.B.2 from the MWG textbook.

The solution is provided in the MWG textbook.

Solution of Exercise 9

Solve Exercise 15.B.2 from the MWG textbook.

The exercise generalizes Example 15.B.1 in the MWG textbook.

Solution of Exercise 10

Solve Exercise 15.B.9 from the MWG textbook.

Both consumers have Leontief preferences. Deriving the offer curves for both consumers and crossing them leads to

$$CE = \left\{p_1 = 0, p_2 = p, x_\alpha = (x, 0), x_\beta = (30 - x, 20), p > 0, x \in [0, 30 - \sqrt{20}]\right\}$$

when $\omega_1 = (30, 0)$ and $\omega_2 = (0, 20)$. The latter is a set of competitive equilibria also when $\omega_1 = (5, 0)$ and $\omega_2 = (0, 20)$, provided 30 is replaced by 5. When $\omega_1 = (5, 0)$ and $\omega_2 = (0, 20)$, in addition, we have

$$CE^1 = \left\{\frac{p_1}{p_2} = \frac{9 - \sqrt{61}}{1 + \sqrt{61}}, x_\alpha = \left(\frac{9 - \sqrt{61}}{2}, \frac{9 - \sqrt{61}}{2}\right), x_\beta = \left(\frac{1 + \sqrt{61}}{2}, \frac{31 + \sqrt{61}}{2}\right)\right\}$$

and

$$CE^2 = \{p_1 = p, p_2 = 0, x_\alpha = (5, 20 - x), x_\beta = (0, x), p > 0, x \in [0, 15]\}.$$

Solution of Exercise 11

Solve Exercise 15.C.2 from the MWG textbook.

The exercise is a special case of Ex. 12 with $A = 1$, $\alpha = \frac{1}{2}$, $\beta = 1$. Please have a look at the solution of Ex. 12.

Solution of Exercise 12

Consider an economy with two periods, $t = 0, 1$, and a consumption good per period. The consumer has utility $U(x_0, x_1) = u(x_0) + \beta u(x_1)$ with $\beta \in (0, 1)$ and endowment k_0 of consumption good in $t = 0$. The economy has one firm, $J = 1$, which transforms the good in $t = 0$ (input, k_1), into an output (y) in $t = 1$. The firm production function is $f(k) = Ak^\alpha$ with $A > 0$ and $\alpha \in (0, 1]$. Assume that $u(x) = \log x$ and $\alpha \in (0, 1)$.

1. *Find the competitive equilibrium*
2. *Find the firm profit in the competitive equilibrium and discuss its dependence on α*
3. *Find the Pareto optimal allocation of this economy and compare it with the allocation found in* 1

1. To find the competitive equilibrium we first solve for the the firms optimal decision as a function of the price ratio. This gives an optimal demand $k_1(p)$, optimal supply $y(p)$, and an equilibrium profit $\pi(p)$. Setting $p_0 = p$ and $p_1 = 1$ and solving for

$$\begin{aligned}&\text{Max } y - pk_1, \text{ such that}\\&0 \leq y \leq Ak_1^{\alpha}\\&k_1 \geq 0\end{aligned}$$

leads to

$$k_1(p) = \left(\frac{\alpha A}{p}\right)^{\frac{1}{1-\alpha}}, \; y(p) = A\left(\frac{\alpha A}{p}\right)^{\frac{\alpha}{1-\alpha}}, \; \pi(p) = A(1-\alpha)\left(\frac{\alpha A}{p}\right)^{\frac{\alpha}{1-\alpha}}.$$

Having the profit of the firm, we can compute agent optimal demand for each good. Agent 1 has Cobb Douglas utility with coefficient $\frac{1}{1+\beta}$ for good 1 and $\frac{\beta}{1+\beta}$ for good 2, her demand is thus

$$\begin{aligned}x_0(p) &= \tfrac{1}{1+\beta}\tfrac{pk_0+\pi(p)}{p},\\x_1(p) &= \tfrac{\beta}{1+\beta}\left(pk_0 + \pi(p)\right).\end{aligned}$$

Imposing market clearing, e.g. in good 1, leads to

$$x_0(p) + k_1(p) = k_0 \Rightarrow p^* = \alpha A\left(\frac{1+\alpha\beta}{\alpha\beta k_0}\right)^{1-\alpha}$$

Note that market clearing for good 2 is implied by the latter together with the agent BC (which includes the firm profit) and the definition of firm profit. The CE is thus

$$CE = \left\{p^*, x = \left(\frac{k_0}{1+\alpha\beta}, A\left(\frac{k_0\alpha\beta}{1+\alpha\beta}\right)^{\alpha}\right), k_1 = \frac{k_0\alpha\beta}{1+\alpha\beta}, y_1 = A\left(\frac{k_0\alpha\beta}{1+\alpha\beta}\right)^{\alpha}\right\}$$

2. Substituting the market clearing price p^* in the optimal profit function $\pi(p)$ leads to

$$\pi(p^*) = A(1-\alpha)\left(\frac{\alpha\beta k_0}{1+\alpha\beta}\right)^{\alpha}$$

The latter goes to 0 when $\alpha \to 1$. Note that this is consistent with a linear technology (and thus zero profit in equilibrium) when $\alpha = 1$. When $\alpha \to 0$, the profits goes to A (taking the log and using de L'Hopital rule to derive that $\lim_{x\to 0} x \log x = 0$). This is because, on the cost side, the use of input goes to zero but its price does not diverge, so that the cost goes to zero; on the revenue side, the price of output is normalized to one and the output goes to A, so that the revenue goes to A.

3. To find the Pareto Set, we maximize the agent utility given the feasibility constraint:

$$\begin{aligned}&\text{Max } \log x_0 + \beta \log x_1, \text{ such that}\\&x_0 + k_1 \leq k_0\\&0 \leq y \leq Ak_1^{\alpha}\\&x_1 \leq y\end{aligned}$$

Imposing all constraints with equality the problem becomes

$$\begin{array}{l}\text{Max } \log(k_0 - k_1) + \beta\alpha \log Ak_1, \text{ such that}\\ k_1 \in [0, k_0]\end{array}$$

whose unique (internal) solution is $k_1 = \frac{k_0\alpha\beta}{1+\alpha\beta}$. Imposing the constraints leads to

$$PS = \left\{x = \left(\frac{k_0}{1+\alpha\beta}, A\left(\frac{k_0\alpha\beta}{1+\alpha\beta}\right)^{\alpha}\right), k_1 = \frac{k_0\alpha\beta}{1+\alpha\beta}, y_1 = A\left(\frac{k_0\alpha\beta}{1+\alpha\beta}\right)^{\alpha}\right\}.$$

Note that the marginal productivity of capital in the Pareto optimal allocation is equal to the supporting price in the CE:

$$f'(k_1) = \alpha A k_1^{\alpha-1} = \alpha A \left(\frac{1+\alpha\beta}{k_0\alpha\beta}\right)^{1-\alpha} = p^*.$$

Solution of Exercise 13

Solve Exercise 12 when

1.* $u(x) = x$ *and* $\alpha \in (0,1)$*;*

2. $u(x) = \log x$ *and* $\alpha = 1$*;*

3.** $u(x) = x$ *and* $\alpha = 1$.

1. Normalize $p_1 = 1$ and name $p_0 = p$. The firm supply is the same as computed in point 1. of Ex. 12. Given linear utility, consumer demand is

$$(x_0(p), x_1(p)) = \begin{cases} \text{any bundle on the budget line if } p = \frac{1}{\beta} \\ \text{only good } x_0 \geq k_0 \text{ if } p < \frac{1}{\beta} \\ \text{only good } x_1 \text{ if } p > \frac{1}{\beta} \end{cases}$$

No equilibrium is possible if $p < \frac{1}{\beta}$. In fact, if $p < \frac{1}{\beta}$, then firm has positive demand of k_1 but the consumer consumes (at least) all her endowment k_0. Thus, in a competitive equilibrium it must be $p \geq \frac{1}{\beta}$.
Let us consider first the case $p = \frac{1}{\beta}$. Firm optimal demand is

$$k_1\left(p = \frac{1}{\beta}\right) = (\alpha\beta A)^{\frac{1}{1-\alpha}}.$$

Such demand is compatible with an equilibrium only if lower or equal than k_0, leading to

$$CE^1 = \left\{p = \frac{1}{\beta}, x = (k_0 - t, At^{\alpha}), (k_1, y) = (t, At^{\alpha}), 0 < t = (\alpha\beta A)^{\frac{1}{1-\alpha}} \leq k_0\right\}.$$

Note that when $(\alpha\beta A)^{\frac{1}{1-\alpha}} = k_0$, the marginal product of capital in k_0 is exactly $\frac{1}{\beta}$. In this case, the consumer is giving away all her endowment k_0 in exchange of consumption $f(k_0)$ in date $t = 1$. A similar situation occurs when $(\alpha\beta A)^{\frac{1}{1-\alpha}} > k_0$. In this case there is no equilibrium with $p = \frac{1}{\beta}$, in fact the demand of input at this price is larger than k_0 and cannot be satisfied. However there is another supporting price. Choosing $p = f'(k_0)$ leads to $p > \frac{1}{\beta}$ if (and only if) $(\alpha\beta A)^{\frac{1}{1-\alpha}} > k_0$. At this price the firm demand for input is k_0 and the consumer, facing a too high price for the consumption good in $t = 0$ chooses $x_0 = 0$. This competitive equilibrium is as follows

$$CE^2 = \left\{p = \alpha A k_0^{\alpha-1}, x = (0, Ak_0^\alpha), (k_1, y) = (k_0, Ak_0^\alpha), (\alpha\beta A)^{\frac{1}{1-\alpha}} > k_0\right\}.$$

The set of equilibria is thus

$$CE = CE^1 \cup CE^2$$

By maximizing consumer welfare on the feasible allocations, it can be shown that $PS = CE$.

2. Keeping the same price normalization as above, let us start to consider the firm optimal choice of input and output. Note that the firm has a linear, constant return to scale, technology. The firm optimal choice is

$$(k_1(p), y(p)) = \begin{cases} k_1 \in [0, +\infty), y = Ak_1 \text{ when } p = A, \\ (+\infty, +\infty) \text{ when } p < A, \\ (0, 0) \text{ when } p > A. \end{cases}$$

Note that there can not be an equilibrium with $p < A$ -unbounded demand (supply) of input (output)- and that when $p \geq A$ the firm profits are zero. Turing to the consumer, and imposing zero profits for the firm, we have (as in Ex. 12)

$$\begin{aligned} x_0(p) &= \tfrac{1}{1+\beta} k_0, \\ x_1(p) &= \tfrac{\beta}{1+\beta} p k_0. \end{aligned}$$

Imposing the market clearing conditions we note that when $p \geq A$, the consumer has positive demand also for good x_1 and thus needs the firm to produce. Thus, there cannot be an equilibrium with $p > A$. When $p = A$, the consumer has a demand

$$x_0(A) = \frac{1}{1+\beta} k_0 \quad x_1(A) = \frac{\beta}{1+\beta} A k_0$$

and the firm, being indifferent on all bundles on the production set frontier, is willing to provide it. Note that indeed when

$$k_1 = k_0 - x_0(A) = \frac{\beta}{1+\beta} k_0$$

the firm has output

$$f(k_1) = \frac{\beta}{1+\beta} A k_0 = x_1(A),$$

so that both markets clear. The competitive equilibrium is thus

$$CE = \left\{ p = A, x = \left(\frac{1}{1+\beta} k_0, \frac{\beta}{1+\beta} A k_0 \right), (k_1, y) = \left(\frac{\beta}{1+\beta} k_0, \frac{\beta}{1+\beta} A k_0 \right) \right\}.$$

Also in this case $CE = PS$.

3. We turn to the case when the consumer has a linear utility with $MRS^1_{12} = \frac{1}{\beta} > 1$ and the firm has a linear production function with $f'(k) = A$. Optimal behaviors are extreme, as derived in point 1. above for the consumer and in point 2. above for the firm. For all possible values of A, we can rule out $p < A$ because it would lead to an unbounded demand of the firm. Possible supporting prices are thus $p \geq A$, and for all these prices the firm has zero profits.

Let us first consider the case $A = \frac{1}{\beta} > 1$. The only possible supporting price is $p = A$, with zero profits for the firm and all allocations on the budget line (which coincides with part of the production frontier) as possible equilibria. We have

$$CE^1 = \left\{ p = A, x = (k_0 - t, At), (k_1, y) = (t, At), t \in [0, k_0], A = \frac{1}{\beta} \right\}.$$

Let us turn to $A > \frac{1}{\beta} > 1$. Then, when $p \geq A$ it also holds $p > \frac{1}{\beta}$ so that the consumer wants to sell k_0 and buy only $x_1 = pk_0$ (remember that firm profits are zero). If $p = A$ the firm is willing to absorb all the supply of k_0 and produce Ak_0 to satisfy the consumer. If $p > A$, the firm chooses not to produce and the goods market do not clear. Thus

$$CE^2 = \left\{ p = A, x = 0, Ak_0), (k_1, y) = (k_0, Ak_0), A > \frac{1}{\beta} \right\}.$$

The last case to consider has $A < \frac{1}{\beta}$. When $p = A$ the firm is indifferent and the consumer, facing $p < \frac{1}{\beta}$, chooses $x = (k_0, 0)$. We have

$$CE^3 = \left\{ p = A, x = k_0, 0), (k_1, y) = (0, 0), A < \frac{1}{\beta} \right\}.$$

When $\frac{1}{\beta} \geq p > A$, the situation is as above, with the difference that the firm only choice is not to produce (and still consistent with the consumer choice). We have

$$CE^4 = \left\{ p \in \left(A, \frac{1}{\beta} \right], x = k_0, 0), (k_1, y) = (0, 0), A < \frac{1}{\beta} \right\}.$$

Finally when $p > \frac{1}{\beta} > A$, the firm does not produce but the consumer wants to consume only the firm output, no equilibrium in this case. Summarizing we have

$$CE = CE^1 \cup CE^2 \cup CE^3 \cup CE^4.$$

and it can be shown that $CE = PS$.

Solution of Exercise 14

Consider an economy with two consumption goods, $L = 2$, one consumer, $I = 1$, and one firm, $J = 1$. The consumer has endowment $\omega = (4, 1)$ and utility $U_1(x) = x_1 x_2$. The firm has production set $Y_1 = \{y_2 \leq -ay_1,\ y_1 \leq 0\}$ with $a > 0$. The consumer is the owner of the firm.

1. *Write down the definition of a competitive equilibrium for this economy.*
2. *Fix $a = 1$, find the set of competitive equilibria and the corresponding firm profits.*
3. *Are the the competitive equilibria found above Pareto optimal?*
4. *Find the Pareto optimal allocations for $a = 1$.*

5.* *Answer to 2. and 3. for each given $a > 0$.*

6.* *Find the Pareto optimal allocations for each given $a > 0$.*

1. A competitive equilibrium for this economy is a supporting price vector $p^* = (p_1^*, p_2^*)$ and an allocation (x^*, y^*) with $x^* \in \mathbf{R}_+^2$ and $\{y_2^* + ay_1^* \leq 0, y_1 \leq 0\}$ such that

- The firm maximizes profits given p, that is, $p^*y^* \geq p^*y$ for all y with $y_2 + ay_1 \leq 0$ and $y_1 \leq 0$;
- The consumer maximizes his utility given p^* and income $p^*\omega + p^*y^*$, that is, $U(x^*) \geq U(x)$ for all $x \in \mathbf{R}_+^2$ with $p^*x \leq p^*\omega + p^*y^*$;
- The two goods market clear, that is, $x_l^* = \omega_l + y_l^*$ for $l = 1, 2$.

2. Take $a = 1$. To find an equilibrium we start from the firm (we need the firm profit into the consumer demand). The firm has a linear technology and its optimal choice as a function of the price vector p is

$$(y_1(p), y_2(p)) = \begin{cases} -y_1 \in [0, +\infty), y_2 = -y_1 \text{ when } \frac{p_1}{p_2} = 1, \\ (-\infty, +\infty) \text{ when } \frac{p_1}{p_2} < 1, \\ (0, 0) \text{ when } \frac{p_1}{p_2} > 1. \end{cases}$$

Note that there can not be an equilibrium with $\frac{p_1}{p_2} < 1$ -due to unbounded demand (supply) of input (output)- and that when $\frac{p_1}{p_2} \geq 1$ the firm profits are zero. Turing to the consumer, and imposing zero profits for the firm, we have

$$x_1(p_1/p_2) = \tfrac{1}{2}\tfrac{4p_1+p_2}{p_1},$$
$$x_2(p_1/p_2) = \tfrac{1}{2}\tfrac{4p_1+p_2}{p_2}.$$

Imposing the market clearing conditions we note that when $\frac{p_1}{p_2} \geq 1$, the consumer has positive demand also for good x_1 and thus needs the firm to produce. Thus, there cannot be an equilibrium with $\frac{p_1}{p_2} > 1$. When $\frac{p_1}{p_2} = 1$, the consumer has demand

$$x_1(1) = \frac{5}{2} \quad x_2(1) = \frac{5}{2}$$

and the firm, being indifferent on all bundles on the production set frontier, is willing to provide it. Note that indeed when

$$y_1 = x_1(1) - 4 = -\frac{3}{2}$$

the firm has output

$$y_2 = -y_1 = \frac{3}{2} = x_2(1) - 1,$$

so that both markets clear. The competitive equilibrium is thus

$$CE = \left\{ \frac{p_1^*}{p_2^*} = 1, x^* = \left(\frac{5}{2}, \frac{5}{2}\right), y^* = \left(-\frac{3}{2}, \frac{3}{2}\right) \right\}.$$

<u>Note</u> Exploiting the I welfare theorem (see below), we could have derived the CE also by finding the Pareto Set first (here a unique allocation, see point 4.) and a suitable supporting price vector.
3. Yes, the CE found above is Pareto optimal due to the I welfare theorem, which holds due to strong monotonicity, and thus LNS, of the consumer preferences as represented by $U(x) = x_1 x_2$.
4. Fix $a = 1$. To find the Pareto set we must maximize the consumer utility in the feasible set. We can solve

$$\begin{aligned}
&\text{Max } x_1 x_2, \ \text{ such that} \\
&x \in \mathbf{R}_+^2 \\
&y_1 \leq 0 \\
&y_2 \leq -y_1 \\
&x_1 = 4 + y_1 \\
&x_1 = 1 + y_2
\end{aligned}$$

Imposing all constraints with equality the problem becomes

$$\begin{aligned}
&\text{Max } (4 + y_1)(1 - y_1), \ \text{ such that} \\
&y_1 \in [-4, 0]
\end{aligned}$$

whose unique (internal) solution is $y_1 = -\frac{3}{2}$. Imposing the constraints leads to

$$PS = \left\{ x = \left(\frac{5}{2}, \frac{5}{2}\right), y_1 = \left(-\frac{3}{2}, \frac{3}{2}\right) \right\}.$$

5. We proceed similarly to point 2.. First note that with $a = 1$, the firm provides the transformation of inputs into output that pleases the consumer with a unitary price vector. As a changes, we expect the price vector to respond (so that the linear firm is still happy to supply what the consumer asks) and change as well as $\frac{p_1}{p_2} = a$. However there will be low values of a such that the consumer would like to increase his consumption of good 1 beyond $\omega_1 = 4$ and decrease the consumption of good 2 below $\omega_2 = 1$. Such configuration cannot be an equilibrium as the firm is bound to use good 1 as input, and cannot supply

it to the consumer.
Having this is mind, the competitive equilibrium is the same "interior" allocation as found in point 2., that is, with $\frac{p_1}{p_2} = a$, as long as $x_1(a) \leq 4$. Imposing the condition we find

$$x_1(a) = \frac{4a+1}{2a} \leq 4 \;\Rightarrow\; a \geq \frac{1}{4}$$

Using the demand of good 2 and the market clearing condition(s) to find the firm choice in equilibrium we obtain that if $a \geq \frac{1}{4}$, then

$$CE^a = \left\{ \frac{p_1^*}{p_2^*} = a, x^* = \left(\frac{4a+1}{2a}, \frac{4a+1}{2} \right), y^* = \left(\frac{1-4a}{2a}, \frac{4a-1}{2} \right) \right\}$$

Let's turn to the case $a < \frac{1}{4}$. Now the price ratio cannot be equal to a, otherwise the consumer would demand good 1 in exchange of good 2. A possibility is that $\frac{p_1}{p_2} = \frac{1}{4}$ and the consumer consumes her endowment $(4, 1)$. The latter can be part of an equilibrium only if the firm is willing not to operate at this price ratio. Having $\frac{p_1}{p_2} = \frac{1}{4} > a = MRT$ implies that this is the case. We obtain that if $a < \frac{1}{4}$, then

$$CE^a = \left\{ \frac{p_1^*}{p_2^*} = \frac{1}{4}, x^* = (4, 1), y^* = (0, 0) \right\}.$$

As the preferences of the consumer have not changed, for each $a > 0$ the CE is in the PS.

6. To find the Pareto set we must maximize the consumer utility in the feasible set. We can solve

$$\begin{aligned}
&\text{Max } x_1 x_2, \text{ such that} \\
&x \in \mathbf{R}^2_+ \\
&y_1 \leq 0 \\
&y_2 \leq -a y_1 \\
&x_1 = 4 + y_1 \\
&x_1 = 1 + y_2
\end{aligned}$$

Imposing all constraints with equality the problem becomes

$$\begin{aligned}
&\text{Max } (4 + y_1)(1 - a y_1), \text{ such that} \\
&y_1 \in [-4, 0]
\end{aligned}$$

The solution of the latter is internal and equal to

$$y_1 = \frac{1-4a}{2a}$$

when $a > \frac{1}{4}$. Otherwise, when $ain \left(0, \frac{1}{4}\right]$ the solution is at the right-end border, $y_1 = 0$. Using the feasibility constraints lead to the Pareto Set. If $a > \frac{1}{4}$, then

$$PS^a = \left\{ x = \left(\frac{1+4a}{2a}, \frac{1+4a}{2} \right), y_1 = \left(\frac{1-4a}{2a}, \frac{4a-1}{2} \right) \right\}.$$

If $a \leq \frac{1}{4}$, then

$$PS^a = \{x = (4,1), y_1 = (0,0)\}.$$

Solution of Exercise 15

Consider an economy with L commodities, I consumers, and J firms. The endowments and ownership shares are $\left(\omega_i, (\theta_{ji})_{j=1}^J\right)_{i=1}^I$. Assume that all consumers have locally non satiated preferences. Prove that, for all choices of a good l', if the markets for all goods $l \neq l'$ clears and $p \gg 0$, then also the market for good l' clears.

First note that if preferences are LNS, then the optimal consumption is chosen on the budget line

$$px_i = p\omega_i + \sum_{j\in J} \theta_{ji} p y_j \quad i = 1, \dots, L.$$

Adding up the latter leads to

$$\sum_{i\in I}\left(\sum_{l\in L} p_l\left(x_{li} - \omega_{li}\right) - \sum_{j\in J}\sum_{l\in L}\theta_{ij}p_l y_{lj}\right) = 0$$

or, equivalently,

$$\sum_{l\in L} p_l \left(\sum_{i\in I}\left(x_{li} - \omega_{li}\right) - \sum_{j\in J} y_{lj}\right) = 0.$$

The latter together with market clearing in all goods $l \neq l'$, $\sum_{i\in I}(x_{li} - \omega_{li}) = \sum_{j\in J} y_{lj}$, leads to

$$p_{l'}\left(\sum_{i\in I}(x_{l'i} - \omega_{l'i}) - \sum_{j\in J} y_{l'j}\right) = 0.$$

If $p_{l'} > 0$, then it must be $\sum_{i\in I}\left(x_{l'i} - \omega_{l'i}\right) = \sum_{j\in J} y_{l'j}$, so that also the market for good l' clears.

Solution of Exercise 16

Consider an exchange economy with two consumption goods, $L = 2$, and two consumers, $I = 2$. Consumer 1 has endowment $\omega_1 = (2,0)$ and utility $U_1(x_1) = (x_{11})^\alpha + (x_{21})^\alpha$. Consumer 1, has endowment $\omega_2 = (1,3)$ and utility $U_2(x_2) = x_{12}x_{22}$. Consider $\alpha = \frac{1}{2}$ first.

1. *Find the competitive equilibria and the set of Pareto optimal allocations.*
2. *Discuss the validity of both Welfare Theorems.*
3. *The social planner has the possibility to implement some transfer between the initial endowments of commodity 2. Is there a transfer such that, in the competitive equilibrium achieved after trading, both agents consume the same amount of good 1. If so, find it. Can the same competitive equilibrium arise after a transfer of good 1? If so, find it.*

4. Repeat the exercise taking $\alpha = 1$.

1. To find the competitive equilibrium of this economy, we first derive the demand of both agents by solving their utility maximization given their budget constraint. For agent 1 we obtain:

$$x_{11}(p) = \frac{2}{1+\frac{p_1}{p_2}} \text{ and } x_{21}(p) = \frac{2p_1}{p_2\left(1+\frac{p_2}{p_1}\right)},$$

while for agent 2 we have

$$x_{12}(p) = \frac{1}{2}\frac{p_1+3p_2}{p_1} \text{ and } x_{22}(p) = \frac{1}{2}\frac{p_1+3p_2}{p_2}.$$

Market clearing in the first good gives the equilibrium price ratio

$$\frac{2}{1+\frac{p_1}{p_2}} + \frac{1}{2}\frac{p_1+3p_2}{p_1} = 3 \quad \Rightarrow \frac{p_1}{p_2} = 1.$$

The resulting equilibrium allocation is thus

$$x_1^* = (1,1) \quad x_2^* = (2,2)$$

(note that also the market for good 2 clears). As for Pareto optimality, strong monotonicity of both consumers preferences, allows us to pick either agent as the one whose utility is to be maximized, given different utility levels of the other agent. Choosing agent 1, we have to solve

$$\begin{array}{l} Max\ U_1(x_1) \\ x_1 \in \mathbf{R}_2^+, x_2 \in \mathbf{R}_2^+ \\ x_{11}+x_{12} \le 3 \\ x_{21}+x_{22} \le 3 \\ U_2(x_2) \ge \bar{U} \end{array}$$

for given levels of $\bar{U}$. The first order conditions at an interior and non wasteful allocation are

$$\begin{array}{l} MRS_{1,2}^1 = \frac{\sqrt{x_{21}}}{\sqrt{x_{11}}} = \frac{x_{22}}{x_{12}} = MRS_{1,2}^2 \\ x_{11}+x_{12} = 3 \\ x_{21}+x_{22} = 3. \end{array}$$

Solving the above gives

$$PS = \{x_1 = (t,t), x_2 = (3-t, 3-t), t \in [0,3]\},$$

where we have extended the solution to the border cases having $x_1 = \bar{\omega}$ or $x_2 = \bar{\omega}$.

2. Both consumers have local non satiated preferences (implied by monotonicity), thus the First Welfare Theorem holds. In fact, the competitive equilibrium allocation found above belongs to the Pareto Set. Both consumers have also convex and continuous preferences.

Thus, for all allocations x in the Pareto Set, there exists a price vector $p = (p_1, p_2) \neq 0$ such that $\{p, x\}$ is a quasi-equilibrium with transfer. In all such allocations, computing the MRS, $p = (p, p) \gg 0$. Thus all quasi-equilibrium with transfer are also equilibria: if the allocation is interior, each agent has positive wealth; if the allocation is at the border, agent i with $x_i = 0$ has only 0 in the budget set.

3. A social planner can transfer amounts of good 2 to achieve an allocation where both consumers consume the same amount of the first good. Such allocation, to be "stable", must also be in the Pareto set (otherwise agents would want to trade away from it). The allocation we are looking for solves

$$\begin{cases} x_{11} = x_{12} \\ x_{11} = x_{21} \\ x_{12} = 3 - x_{11} \\ x_{22} = 3 - x_{21} \end{cases},$$

leading to

$$x'_1 = x'_2 = \left(\frac{3}{2}, \frac{3}{2}\right).$$

Such allocation is a competitive equilibrium together with $p = (p, p)$. The transfer that support such an equilibrium is

$$\begin{cases} px'_1 = p\omega_1 + T_1 & \Rightarrow \quad 3p = 2p + T_1 \\ px'_2 = p\omega_2 + T_2 & \Rightarrow \quad 3p = 4p + T_2 \\ T_1 + T_2 = 0 \end{cases}$$

giving $T_1 = -T_2 = p$. For such a transfer to be implemented using good 2 the social planner needs to transfer

$$t = \frac{T_1}{p_2} = \frac{T_1}{p} = 1$$

from consumer 2 to consumer 1. Given that consumer 2 is endowed with 3 units of the second good, such transfer is possible and leads to

$$\omega'_1 = \omega_1 + (0, 1) = (2, 1) \quad \text{and} \quad \omega'_2 = \omega_2 - (0, 1) = (1, 2).$$

In terms of good 1 the transfer should be the same (the two goods have the same price in equilibrium) and it is also implementable, giving

$$\omega'_1 = \omega_1 + (1, 0) = (3, 0) \quad \text{and} \quad \omega'_2 = \omega_2 - (1, 0) = (0, 3).$$

4. When $\alpha = 1$ the first consumer has linear preferences with $MRS^1_{12} = 1$. It can be checked that the Pareto set and the competitive equilibrium are the same (the first agent is indifferent with $p_1 = p_2 = p$. The discussion on the Welfare theorems and the transfer are also identical.

Solution of Exercise 17

Consider an economy with two consumption goods, $L = 2$, two consumers, $I = 2$, and one firm, $J = 1$. Consumer 1 has endowment $\omega_1 = (4, 1)$ and utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2 has endowment $\omega_2 = (1, 1)$ and utility $U(x_2) = x_{12} + x_{22}$. The firm has production set $Y = \{y_2 \leq \sqrt{-y_1},\ y_1 \leq 0\}$. Consumer two is the owner of the firm.

1. *Write down the definition of a competitive equilibrium in this economy.*

2. *Find the set of competitive equilibria and the corresponding firm profits.*

3. *Are the competitive equilibria allocation found above Pareto optimal?*

4.** *Find the set of Pareto optimal allocation of this economy.*

1. A competitive equilibrium for this economy is a supporting price vector $p^* = (p_1^*, p_2^*)$ and an allocation (x_1^*, x_2^*, y^*) with $x^* \in \mathbf{R}_+^4$ and $\{y_2^* - \sqrt{-y_1^*} \leq 0, y_1 \leq 0\}$ such that

- The firm maximizes profits given p, that is, $p^*y^* \geq p^*y$ for all y with $y_2 - \sqrt{-y_1} \leq 0$ and $y_1 \leq 0$;

- Consumer 1 maximizes her utility given p^* and income $p^*\omega_1$, that is, $U_1(x_1^*) \geq U_1(x_)$ for all $x_1 \in \mathbf{R}_+^2$ with $p^*x_1 \leq p^*\omega_1$;

- Consumer 2 maximizes her utility given p^* and income $p^*\omega_2 + p^*y^*$, that is, $U_2(x_2^*) \geq U_2(x_2)$ for all $x_2 \in \mathbf{R}_+^2$ with $p^*x_2 \leq p^*\omega_2 + p^*y^*$;

- The two goods market clear, that is, $x_{l1}^* + x_{l2}^* = \omega_{l1} + \omega_{l2} + y_l^*$ for $l = 1, 2$.

2. To find an equilibrium we start from the firm (we need the firm profit into consumer 2 demand). The firm has a concave production function, its optimal choice as a function of the price vector p is a solution of

$$\begin{aligned}&\text{Max } p_2y_2 + p_1y_1, \ \text{ such that}\\&y_1 \leq 0\\&y_2 \leq \sqrt{-y_1}\end{aligned}$$

Using FOC we derive

$$y_1(p) = -\frac{p_2^2}{4p_1^2},\ y_2(p) = \frac{p_2}{2p_1} \quad \text{and thus} \quad \pi(p) = \frac{p_2^2}{4p_1}$$

Turing to the consumers, for consumer one (no profits in her income and Cobb Douglas utility) we have

$$\begin{aligned}x_{11}(p) &= \tfrac{1}{2}\tfrac{4p_1+p_2}{p_1},\\x_{21}(p) &= \tfrac{1}{2}\tfrac{4p_1+p_2}{p_2}.\end{aligned}$$

Consumer 2 has linear utility and thus her demand is given by

$$(x_{12}(p), x_{22}(p)) = \begin{cases} \left(x \in \left[0, \frac{9}{4}\right], \frac{9}{4} - x\right) & \text{when } \frac{p_1}{p_2} = 1, \\ \left(\frac{p_1+p_2+\pi(p)}{p_1}, 0\right) & \text{when } \frac{p_1}{p_2} < 1, \\ \left(0, \frac{p_1+p_2+\pi(p)}{p_2}\right) & \text{when } \frac{p_1}{p_2} > 1, \end{cases}$$

where we have used that $\pi(p)/p_1 = \pi(p)/p_2 = 1/4$ when $p_1/p_2 = 1$.
Imposing the market clearing conditions gives the supporting price ratio. Let us first check if $\frac{p_1}{p_2} = 1$ can be part of a competitive equilibrium. Optimal behavior of the firm and of consumer 1 gives

$$x_{11}(1) = \frac{5}{2},\ x_{21}(1) = \frac{5}{2},\ y_1(1) = -\frac{1}{4},\ y_2(1) = \frac{1}{2}.$$

Using market 1 clearing condition with $x_{12} = x \in \left[0, \frac{9}{4}\right]$ (which is optimal for this price ratio) gives

$$\frac{5}{2} + x = 4 + 1 - \frac{1}{4} \Rightarrow x = \frac{9}{4}.$$

As a double check, in the market for good 2 we have

$$\frac{5}{2} + \frac{9}{4} - x = 1 + 1 + \frac{1}{2} \Rightarrow x = \frac{9}{4}.$$

In principle we could have competitive equilibria with other price ratios. Assuming $\frac{p_1}{p_2} < 1$ implies $x_{22} = 0$ and leads to the following market clearing condition for good 2

$$\frac{1}{2}\frac{4p_1 + p_2}{p_2} = 2 + \frac{1}{2}\frac{p_2}{p_1}.$$

Solving the II order equation in $\frac{p_1}{p_2}$ and considering only the positive solution leads to

$$\frac{p_1}{p_2} = 1 \geq 1$$

The solution cannot be accepted because not smaller than 1. Assuming now $\frac{p_1}{p_2} > 1$ implies $x_{12} = 0$ and leads to the following market clearing condition for good 1

$$\frac{1}{2}\frac{4p_1 + p_2}{p_1} = 5 - \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2.$$

Solving the II order equation in $\frac{p_2}{p_1}$ and considering only the positive solution leads to

$$\frac{p_1}{p_2} = \frac{1}{-1 + \sqrt{13}} < 1$$

The solution cannot be accepted because not greater than 1. To summarize the unique competitive equilibrium is

$$CE = \left\{ \frac{p_1^*}{p_2^*} = 1, x_1^* = \left(\frac{5}{2}, \frac{5}{2} \right), x_2^* = \left(\frac{9}{4}, 0 \right), y = \left(-\frac{1}{4}, \frac{1}{2} \right) \right\}$$

3. Both consumers preferences are LNS (due to strong monotonicity) and thus the I first welfare theorem implies that the allocation of the CE found above is Pareto optimal.
4. To find the Pareto set, we can solve, for a given value of $\bar{U}_2$

$$\begin{array}{l} \text{Max } U_1(x_1) \text{ such that} \\ x \in \mathbf{R}_+^4 \\ U_2(x_2) \geq \bar{U}_2 \\ y_1 \leq 0 \\ 0 \leq y_2 \leq \sqrt{-y_1} \\ x_{11} + x_{12} = 5 + y_1 \\ x_{21} + x_{22} = 2 + y_2 \end{array}$$

(all these solutions are in the Pareto set due to strong monotonicity of both agents preferences). Equivalently, due to the concavity of utility functions we could solve

$$\begin{array}{l} \text{Max } \lambda_1 U_1(x_1) + \lambda_2 U_1(x_1) \text{ such that} \\ x \in \mathbf{R}_+^4 \\ y_1 \leq 0 \\ 0 \leq y_2 \leq \sqrt{-y_1} \\ x_{11} + x_{12} = 5 + y_1 \\ x_{21} + x_{22} = 2 + y_2 \end{array}$$

for all $\lambda \geq 0$, $\lambda \neq 0$. we consider first the case of $\lambda \gg 0$. The set of FOC of the problems depends on whether positivity constraints on x are binding. An interior solution, $x_{li} > 0$ for all l, i, solves

$$\begin{cases} MRS_{12}^1 = \frac{x_{21}}{x_{11}} = MRS_{12}^2 = 1 \\ MRS_{12}^2 = 1 = MRT_{12} = \frac{1}{2\sqrt{-y_1}} \\ x_{11} + x_{12} = 5 + y_1 \\ x_{21} + x_{22} = 2 + y_2 \\ y_2 = \sqrt{-y_1} \\ x_{li} > 0, y_1 < 0, y_2 > 0 \end{cases}$$

Solving the system gives

$$PS^1 = \left\{ x_1 = (t, t), x_2 = \left(\frac{19}{4} - t, \frac{5}{2} - t \right), y = \left(-\frac{1}{4}, \frac{1}{2} \right), t \in \left(0, \frac{5}{2} \right) \right\}$$

Next we relax the positivity constraint, one at a time. We do so only for consumer 2 because consumer 1, as long as $\lambda_1 > 1$, gets the lowest level of utility and she is thus never satisfied when $x_{11} = 0$ or $x_{21} = 0$. Let us start with $x_{22} = 0$. The set of FOCs gives

$$\begin{cases} MRS^1_{12} = \frac{x_{21}}{x_{11}} \leq MRS^2_{12} = 1 \\ MRS^1_{12} = \frac{x_{21}}{x_{11}} = MRT_{12} = \frac{1}{2\sqrt{-y_1}} \\ x_{11} + x_{12} = 5 + y_1 \\ x_{21} = 2 + y_2 \\ y_2 = \sqrt{-y_1} \\ x_{11} > 0, x_{12} > 0, x_{21} > 0, y_1 < 0, y_2 > 0 \end{cases}$$

Solving the system gives

$$\begin{aligned} PS^2 = \quad & \Big\{ x_1 = \Big(2x_{21}(t)(x_{21} - 2), x_{21}(t) = \tfrac{4}{3} + \sqrt{\tfrac{64+(1-t)12}{36}} \Big), \\ & x_2 = (t, 0), y = (-(x_{21}(t) - 2)^2, (x_{21}(t) - 2)), t \in \big(0, \tfrac{9}{4}\big) \Big\} \end{aligned}$$

Note that when $t = \frac{9}{4}$ it holds $x_{11} = x_{22} = \frac{5}{2}$ and $MRS^1_{12} = MRS^2_{12}$, the same allocation in PS^1 when $t = \frac{5}{2}$. We turn to $x_{12} = 0$, implying

$$\begin{cases} MRS^1_{12} = \frac{x_{21}}{x_{11}} \geq MRS^2_{12} = 1 \\ MRS^1_{12} = \frac{x_{21}}{x_{11}} = MRT_{12} = \frac{1}{2\sqrt{-y_1}} \\ x_{11} = 5 + y_1 \\ x_{21} + x_{22} = 2 + y_2 \\ y_2 = \sqrt{-y_1} \\ x_{11} > 0, x_{12} > 0, x_{22} > 0, y_1 < 0, y_2 > 0 \end{cases}$$

It can be checked that the system has no solutions, in particular the feasibility conditions and positivity constraint lead to

$$x_{11} \geq \frac{19}{4} \quad \text{and} \quad x_{21} \geq \frac{10}{4}$$

whereas $MRS^1_{12} \geq MRS^2_{12} = 1$ implies $x_{21} \geq x_{11}$. Two more (sets of) Pareto optimal allocations are found when $\lambda_1 = 0$ or when $\lambda_2 = 0$. Setting $t = 0$ in PS^1 and in PS^2 gives, respectively, these allocations.

Solution of Exercise 18

Consider three commodities, time for leisure/labor (commodity 1), durable good (commodity 2) and consumption good (commodity 3). There is a single consumer with utility function $U(x_2, x_3) = x_2 + x_3$, so that the consumption of leisure does not have any impact on his utility. The consumer has an endowment of leisure equal to 1 and no endowment of durable and consumption good.

There are two firms. Firm 1 produces only durable goods using labor as input. Firm 1 production set is $Y_1 = \{-\sqrt{-y_{11}} + y_{21} \leq 0,\ y_{11} \leq 0\}$. Firm 2 produces only consumption good using labor as input. Firm 2 production set is $Y_2 = \{-2\sqrt{-y_{12}} + y_{32} \leq 0,\ y_{12} \leq 0\}$. The consumer is the owner of both firms.

1. *Write down the definition of a competitive equilibrium in this economy.*

2. *Check that*

$$\left\{\frac{p_1}{p_2} = \frac{\sqrt{5}}{2}, \frac{p_3}{p_2} = 1, x = \left(0, \frac{1}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), y_1 = \left(-\frac{1}{5}, \frac{1}{\sqrt{5}}\right), y_2 = \left(-\frac{4}{5}, \frac{4}{\sqrt{5}}\right)\right\}$$

is the only competitive equilibrium of this economy.

3. *Is the competitive equilibria allocation found above Pareto optimal?*

4.* *Find the set of Pareto optimal allocation of this economy.*

1. A competitive equilibrium for this economy is a supporting price vector $p^* = (p_1^*, p_2^*, p_3^*)$ and an allocation (x^*, y_1^*, y_2^*) with $x^* \in \mathbf{R}^3_+$, $\{y_{21}^* - \sqrt{-y_{11}^*} \leq 0, y_{11} \leq 0\} = Y_1$, and $\{y_{22}^* - 2\sqrt{-y_{12}^*} \leq 0, y_{12} \leq 0\} = Y_2$ such that

- Both firms $j = 1, 2$ maximize profits given p, that is, $p^* y_j^* \geq p^* y_j$ for all $y_j \in Y_j$;
- the consumer maximizes her utility given p^* and income $p^*\omega$, that is, $U(x^*) \geq U(x)$ for all $x_1 \in \mathbf{R}^2_+$ with $p^* x_1 \leq p^*\omega + p^* y_1^* + p^* y_2$;
- The goods market clear, that is, $x_l^* = \omega_l + y_{l1}^* + y_{l2}^*$ for $l = 1, 2, 3$.

2. To check that an allocation is a competitive equilibrium, we check that it satisfies the market clearing conditions as well as firms and consumer optimal behavior. The latter is equivalent to first order conditions, which, given concavity of utility and convexity of the production sets are both necessary and sufficient. The first order conditions of firm $j = 1$ are

$$\begin{aligned} \tfrac{1}{2\sqrt{-y_{11}}} &= \tfrac{p_1}{p_2} \\ y_{21} &= -\sqrt{-y_{11}} \end{aligned}$$

giving

$$y_{11}(p) = -\left(\frac{p_2}{2p_1}\right)^2 \text{ and } y_{21}(p) = \frac{p_2}{2p_1}.$$

It can be easily checked that $(y_{11}, y_{21}) = \left(-\frac{1}{5}, \frac{1}{\sqrt{5}}\right)$ is optimal when $\frac{p_1}{p_2} = \frac{\sqrt{5}}{2}$. Note that with this allocation and price ratio, firm 1 profits are

$$\pi_1(p) = p_2 y_2 + p_1 y_1 = p_2\left(y_2 + \frac{p_1}{p_2} y_1\right) = p_2 \frac{1}{2\sqrt{5}} > 0.$$

Turning to firm $j = 2$, the focs are

$$\begin{aligned} \tfrac{1}{\sqrt{-y_{12}}} &= \tfrac{p_1}{p_3} \\ y_{32} &= -2\sqrt{-y_{12}} \end{aligned}$$

giving

$$y_{12}(p) = -\left(\frac{p_3}{p_1}\right)^2 \quad \text{and} \quad y_{32}(p) = 2\frac{p_3}{p_1}$$

It can be easily checked that $(y_{12}, y_{32}) = \left(-\frac{4}{5}, \frac{4}{\sqrt{5}}\right)$ is optimal when $\frac{p_1}{p_3} = \frac{p_1}{p_2}\frac{p_2}{p_3} = \frac{\sqrt{5}}{2}$. Note that with this allocation and price ratio, firm 3 profits are

$$\pi_2(p) = p_3 y_3 + p_1 y_1 = p_3\left(y_3 + \frac{p_1}{p_3} y_1\right) = p_3 \frac{2}{\sqrt{5}} > 0.$$

We turn to the consumer. Having linear utility in x_2 and x_3 and being $\frac{p_3}{p_2} = 1$ the consumer is indifferent between the two goods. We should check that she is choosing on her budget line. The latter takes into account the income coming from the endowment and the income coming from both firm profits, leading to

$$p_2 x_2 + p_3 x_3 = p_1 + \pi_1(p) + \pi_2(p)$$

or, equivalently,

$$x_2 + \frac{p_3}{p_2} x_3 = \frac{p_1}{p_2} + \frac{\pi_1(p)}{p_2} + \frac{p_3}{p_2}\frac{\pi_1(p)}{p_3}.$$

Plugging in the values of profits found above, it can be easily checked that the given consumption bundle is on the budget line for the given price ratios.
Have checked that we have a competitive equilibrium we should still prove that this is the unique one. First, we exploit consumer preferences and note

$$\frac{p_2}{p_3} > 1 \Rightarrow x_2 = 0, x_3 > 0 \Rightarrow y_2 = 0, y_3 > 0 \Rightarrow \frac{p_2}{p_1} = 0, \frac{p_3}{p_1} > 0 \Rightarrow \frac{p_2}{p_3} = 0$$

thus leading to a contradiction. No equilibrium exists with such a price ratio. Having $\frac{p_2}{p_3} < 1$ leads to a similar contradiction. Thus it must be $\frac{p_2}{p_3} = 1$, and thus also $\frac{p_3}{p_1} = \frac{p_2}{p_1}$. The value of $\frac{p_2}{p_1}$ can be found by imposing market clearing in the first market. Using the optimal demand of inputs by both firms gives

$$1 + y_{11}(p) + y_{12}(p) = 0 \Rightarrow \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2 + \left(\frac{p_3}{p_1}\right)^2 = 1$$

giving only $\frac{p_1}{p_2} = \frac{\sqrt{5}}{2}$ when $p_2 = p_3$.

3. The consumer preferences are LNS (due to monotonicity) and thus the I first welfare theorem implies that the allocation of the CE found above is Pareto optimal.

4. To find the Pareto set, we can solve

$$\begin{aligned}
&\text{Max } x_2 + x_3 \text{ such that} \\
&x \in \mathbf{R}^3_+ \\
&y_{11} \leq 0,\ 0 \leq y_{21} \leq \sqrt{-y_{11}} \\
&y_{12} \leq 0,\ 0 \leq y_{32} \leq 2\sqrt{-y_{12}} \\
&x_1 = 1 + y_{11} + y_{12} \\
&x_2 = y_{21} \\
&x_3 = y_{32}
\end{aligned}$$

Given that the consumer does not value good 1 we look for a solution with $x_1 = 0$, $x_2 > 0$ and $x_3 > 0$. The latter together with feasibility, imply also $y_{21} > 0$ and $y_{32} > 0$, leading to $y_{11} < 0$ and $y_{12} < 0$. Efficiency of the production process leads to $y_{21} = \sqrt{-y_{11}}$ and $y_{32} = 2\sqrt{-y_{12}}$. Deriving the related first order conditions, we obtain

$$\begin{cases}
MU_1 = 0 \leq \mu_1, MU_2 = 1 = \mu_2, MU_3 = 1 = \mu_3 \\
-\gamma_1 \frac{1}{2\sqrt{-y_{11}}} + \mu_1 = 0, -\gamma_1 + \mu_2 = 0, \\
-\gamma_2 \frac{1}{\sqrt{-y_{12}}} + \mu_1 = 0, -\gamma_2 + \mu_3 = 0 \\
0 = 1 + y_{11} + y_{12} \\
x_2 = y_{21} \\
x_3 = y_{32}
\end{cases}$$

where μ_l is the multiplier associated with the feasibility of good l and γ_j is the multiplier associated with the production frontier of firm j. Solving the above gives

$$\begin{cases}
\mu_1 \geq 0, \mu_3 = 1, \mu_3 = 1 \\
\frac{1}{2\sqrt{-y_{11}}} = \frac{\mu_1}{\mu_2} = \mu_1 = \frac{\mu_1}{\mu_3} = \frac{1}{\sqrt{-y_{12}}} \\
0 = 1 + y_{11} + y_{12} \\
x_2 = y_{21} \\
x_3 = y_{32},
\end{cases}$$

and, using the second and third equation to solve for y_{11} (or for y_{12}) we find

$$\begin{cases}
\mu_1 \geq 0, \mu_3 = 1, \mu_3 = 1 \\
x = \left(0, \frac{1}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) \\
y_1 = \left(-\frac{1}{5}, \frac{1}{\sqrt{5}}\right) \\
y_2 = \left(-\frac{4}{5}, \frac{4}{\sqrt{5}}\right).
\end{cases}$$

We have found the Pareto optimal allocation that coincides with the competitive equilibrium allocation. In principle, there could be other Pareto optimal allocation where the consumer consumes only one good, for example only good 3. Feasibility of this allocation would correspond to firm $j = 1$ not producing, $y_{21} = 0$, leading to $y_{11} = 0$ and thus to

$$\begin{cases} MU_1 = 0 \le \mu_1, MU_2 = 1 \le \mu_2, MU_3 = 1 = \mu_3 \\ -\gamma_1 \frac{1}{\sqrt{-y_{11}=0}} + \mu_1 \ge 0, -\gamma_1 + \mu_2 \le 0 \Rightarrow \mu_1 \ge +\infty \\ -\gamma_2 \frac{1}{\sqrt{-y_{12}=1}} + \mu_1 = 0, -\gamma_2 + \mu_3 = 0 \Rightarrow \mu_1 = 1 \\ y_{21} = -1 \\ x_2 = y_{21} = y_{11} = 0 \\ x_3 = y_{32} \end{cases}$$

Similarly, we have no solutions also when the consumer consumes only good 2. We can conclude the the Pareto set contains only the allocation found above with $x_1 = 0, x_2 > 0, x_3 > 0$.

Solution of Exercise 19

Find the competitive equilibrium and the Pareto set in an economy as above with $U(x_2, x_3) = x_2 x_3$ *and* $Y_2 = \{y_{12} + y_{32} \le 0,\ \ y_{12} \le 0\}$ *(firm 1 keeps the same technology).*

First note that given monotonicity of consumer preferences (and thus LNS) the I welfare theorem applies and we can search for the competitive equilibrium in the Pareto Set. Having a unique consumer, there is no issue of distributing resources among consumer and we expect all the allocations of the Pareto Set to be competitive equilibria (together with supporting prices). To find the Pareto Set we solve

$$\begin{array}{l} \text{Max } x_2 x_3 \ \text{ such that} \\ x \in \mathbf{R}^3_+ \\ y_{11} \le 0,\ \ 0 \le y_{21} \le \sqrt{-y_{11}} \\ y_{12} \le 0,\ \ 0 \le y_{32} \le -y_{12} \\ x_1 = 1 + y_{11} + y_{12} \\ x_2 = y_{21} \\ x_3 = y_{32} \end{array}$$

Given that the consumer does not value good 1 we look for a solution with $x_1 = 0$, $x_2 > 0$ and $x_3 > 0$. The latter together with feasibility, imply also $y_{21} > 0$ and $y_{32} > 0$, leading to $y_{11} < 0$ and $y_{12} < 0$. Efficiency of the production process leads to $y_{21} = \sqrt{-y_{11}}$ and $y_{32} = -y_{12}$. Deriving the related first order conditions, we obtain

$$\begin{cases} MU_1 = 0 \le \mu_1, MU_2 = x_3 = \mu_2, MU_3 = x_2 = \mu_3 \\ -\gamma_1 \frac{1}{2\sqrt{-y_{11}}} + \mu_1 = 0, -\gamma_1 + \mu_2 = 0, \\ -\gamma_2 + \mu_1 = 0, -\gamma_2 + \mu_3 = 0 \\ 0 = 1 + y_{11} + y_{12} \\ x_2 = y_{21} \\ x_3 = y_{32} \end{cases}$$

where μ_l is the multiplier associated with the feasibility of good l and γ_j is the multiplier associated with the production frontier of firm j. Solving the above gives the following

Pareto optimal allocation

$$\begin{cases} x_1 = 0, x_2 = \frac{1}{\sqrt{3}}, x_3 = \frac{2}{3} \\ y_{11} = -\frac{1}{3}, y_{21} = \frac{1}{\sqrt{3}} \\ y_{12} = -\frac{2}{3}, y_{32} = \frac{2}{3} \end{cases}$$

Note that having $x_2 = 0$ (or $x_3 = 0$) gives the consumer the lowest level of utility and thus cannot be maximal. Having have the Pareto optimal allocation of this economy, we have a candidate for the allocation of a competitive equilibrium (given for granted that a competitive equilibrium exists) if we find supporting prices. The latter are given my the marginal ratio of substitution and by the marginal rate of transformation computed at the Pareto optimal allocation (x^*, y^*). We find

$$\frac{p_2}{p_3} = MRS_{23} = \frac{x_3^*}{x_2^*} = \frac{2}{\sqrt{3}} \text{ and } \frac{p_1}{p_3} = MRT_{13}^2 = 1 .$$

Thus, the competitive equilibrium is

$$CE = \left\{ \frac{p_1^*}{p_3^*} = 1, \frac{p_2^*}{p_3^*} = \frac{2}{\sqrt{3}}, x^* = \left(0, \frac{1}{\sqrt{3}}, \frac{2}{3}\right), y_1^* = \left(-\frac{1}{3}, \frac{1}{\sqrt{3}}\right), y_2^* = \left(-\frac{2}{3}, \frac{2}{3}\right) \right\} .$$

Solution of Exercise 20

Consider an economy with two consumption goods, $L = 2$, two consumers, $I = 2$, and one firm, $J = 1$. Consumer 1, the worker, has endowment $\omega_1 = (1, 0)$ and utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2, the capitalist, is the owner of the firm, $\theta_{21} = 1$, and has utility $U_2(x_1) = x_{12}x_{22}$. The firm has production set $Y = \{y_1 \leq 2\sqrt{-y_2} \leq 0, y_1 < 0\}$

1. *Provide the definition of a competitive equilibrium for this economy.*
2. *Find the set of competitive equilibria and the set of Pareto optimal allocations.*
3. *Discuss the validity of both Welfare Theorems.*
4. *Find the competitive equilibrium with transfer such that both consumers consume the same amount of output. How can you implement such a transfer starting from the given endowment/ownership?*

1. This is standard.
2. To find the set of competitive equilibria we first derive firm optimal supply and demand of input, as well as consumers optimal demands. Starting from the firm we have

$$\begin{aligned} &Max\ p_2 y_2 + p_1 y_1 \\ &y_1 \leq 0 \\ &y_2 \leq 2\sqrt{-y_1} \end{aligned}$$

leading to

$$y_1(p) = -\left(\frac{p_2}{p_1}\right)^2, \quad y_2(p) = 2\frac{p_2}{p_1}, \quad \frac{\pi(p)}{p_2} = \frac{p_2}{p_1}.$$

Consumers optimal demand is

$$x_1 = \left(\frac{1}{2}, \frac{1}{2}\frac{p_1}{p_2}\right) \quad \text{and} \quad x_2 = \left(\frac{1}{2}\frac{\pi(p)}{p_1}, \frac{1}{2}\frac{\pi(p)}{p_2}\right).$$

Market clearing of good 1 leads to

$$\frac{1}{2} + \frac{1}{2}\left(\frac{p_2}{p_1}\right)^2 = 1 - \left(\frac{p_2}{p_1}\right)^2 \Rightarrow \frac{p_1}{p_2} = \sqrt{3}.$$

The corresponding equilibrium allocation is

$$x_1^* = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \ x_2^* = \left(\frac{1}{6}, \frac{1}{2\sqrt{3}}\right) \ y^* = \left(-\frac{1}{3}, \frac{2}{\sqrt{3}}\right)$$

(note that also the market for good 2 clears). As for Pareto optimality, strong monotonicity of both consumers preferences, allows us to pick either agent as the one whose utility is to be maximized, given different utility levels of the other agent. Choosing agent 1, we have to solve

$$\begin{array}{l} Max \ U_1(x_1) \\ x_1 \in \mathbf{R}_2^+, x_2 \in \mathbf{R}_2^+ \\ x_{11} + x_{12} = 1 + y_1 \\ x_{21} + x_{22} = y_2 \\ y_1 \leq 0 \\ y_2 \leq \sqrt{-y1} \\ U_2(x_2) \geq \bar{U} \end{array}$$

for given levels of $\bar{U}$. The first order conditions at an interior and non wasteful allocation give

$$\begin{cases} MRS_{1,2}^1 = \frac{x_{21}}{x_{11}} = \frac{x_{22}}{x_{12}} = MRS_{1,2}^2 \\ MRS_{1,2}^1 = \frac{x_{21}}{x_{11}} = \frac{1}{\sqrt{-y_1}} = MRT_{1,2} \\ x_{11} + x_{12} = 1 + y_1 \\ x_{21} + x_{22} = y_2 \\ y_2 = \sqrt{-y_1} \end{cases}$$

Solving the above gives

$$PS = \left\{ x_1 = (t, \sqrt{3}t), x_2 = \left(\frac{2}{3} - t, \frac{2}{\sqrt{3}} - \sqrt{3}t\right), y = \left(-\frac{1}{3}, \frac{2}{\sqrt{3}}\right), t \in \left[0, \frac{2}{3}\right] \right\}$$

where we have extended the solution to the border cases where either consumer consumes all the output of firm 2.

3. See the answer to point 2 in the exercise above. Note that in this economy the production function is concave (and thus the production set convex)

4. The allocation we are looking for solves

$$\begin{cases} x_{21} = x_{22} \\ x_{21} = \sqrt{3}x_{11} \\ y_1 = -\frac{1}{3} \\ y_2 = \frac{2}{\sqrt{3}} \\ x_{11} + x_{11} = 1 + y_1 \\ x_{21} + x_{22} = y_2 \end{cases}$$

leading to

$$x_1' = \left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad x_2' = \left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$$

Such allocation is a competitive equilibrium together with

$$\frac{p_1}{p_2} = MRS^1_{1,2} = MRS^2_{1,2} = MRT_{1,2} = \sqrt{3}$$

or $p = (\sqrt{3}p, p)$. The transfer that support such an equilibrium is thus

$$\begin{cases} px_1' = p\omega_1 + T_1 \;\; \Rightarrow \;\; \frac{2}{\sqrt{3}}p = \sqrt{3}p + T_1 \\ px_2' = \pi(p) + T_2 \;\; \Rightarrow \;\; \frac{2}{\sqrt{3}}p = \frac{1}{\sqrt{3}}p + T_2 \\ T_1 + T_2 = 0 \end{cases}$$

giving $T_1 = -T_2 = -\frac{1}{\sqrt{3}}p$. Such a transfer can be implemented using good 1, in this case the social planner needs to transfer

$$t = \frac{T_1}{p_1} = \frac{T_1}{\sqrt{3}p} = -\frac{1}{3}.$$

from consumer 2 to consumer 1. Given that consumer 1 is endowed with 1 unit of the first good, such transfer is possible and leads to

$$\omega_1' = \omega_1 + (-1/3, 0) = (2/3, 0) \quad \text{and} \quad \omega_2' = \omega_2 + (1/3, 0) = (1/3, 0).$$

Alternatively, the transfer could be implemented by transferring ownership of the firm:

$$\theta' = \frac{T_1}{\pi(p)} = -\frac{1}{\sqrt{3}}.$$

However, consumer one is not endowed with any share of the firm and thus such transfer can not be implemented in this economy.

Solution of Exercise 21

Consider an exchange economy with two consumption goods, $L = 2$, and two consumers, $I = 2$. Consumer 1 has endowment $\omega_1 = (2, 2 - \delta)$ with $\delta \in (0, 2)$ and utility $U(x_1) = \epsilon x_{11} + x_{21}$ for $\epsilon > 0$. Consumer 2 has endowment $\omega_2 = (2, \delta)$ and utility $U(x_2) = x_{12} + x_{22}$.

1. *Consider $\epsilon = \delta = 0$. Find the set of competitive equilibria and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.*

2. *Consider $\epsilon = \delta = 0$. State whether $p = (0,1)$, $p = (1,1)$, $p = (1,2)$, $p(2,1)$ support $x_1^* = (0,2), x_2^* = (4,0)$ as a quasi-equilibrium transfer. If so, is each $\{p, x^*\}$ also a competitive equilibrium?*

3. *Consider $\epsilon > 0$ and $\delta = 0$. Find and plot the excess demand $z_1(p) = x_{11}(p) + x_{12}(p) - \bar{\omega}_1$ of good 1, the set of competitive equilibria, and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.*

4. *Consider $\epsilon = 0$ and $\delta \in (0,2)$. Find and plot the excess demand function of good 1, the set of competitive equilibria, and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.*

1. The Pareto set is the set of allocations where consumer 1 consumes only good 2. Both See also the lecture notes of Tuesday November 26.
2. The given allocation x^* belongs to the Pareto Set, and due to convexity of both agents preferences the II welfare theorem apply. $p = (0,1)$ supports a quasi-equilibrium but not an equilibrium at x^*. Both $p = (1,1)$ and $p = (1,2)$ support an equilibrium (and thus also a quasi-equilibrium). $p = (2,1)$ does not support x^* as quasi-equilibrium. 3. The excess demand is

$$z_1(p) = \begin{cases} \frac{2}{p_1} & \text{when } \frac{p_1}{p_2} < \epsilon < 1 \\ \left[-2, \frac{2}{\epsilon}\right] \text{ when } \frac{p_1}{p_2} = \epsilon \\ -2 & \text{when } \frac{p_1}{p_2} \in (\epsilon, 1) \\ [-4, -2] & \text{when } \frac{p_1}{p_2} = 1 \\ -4 & \text{when } \frac{p_1}{p_2} > 1 \end{cases}$$

The equilibrium is achieved for p such that $z_1(p) = 0$, giving

$$CE = \{p_1/p_2 = \epsilon, x_1^* = (2,2), x_2^* = (0,2)\}.$$

The Pareto set is the set of allocations where agent 1 has only good 2 or where agent 2 has only good 1. Both welfare theorems apply.
4. The excess demand is

$$z_1(p) = \begin{cases} \frac{\delta}{p_1} - 2 & \text{when } \frac{p_1}{p_2} \in (0,1) \\ [-2-\delta, -2] & \text{when } \frac{p_1}{p_2} = 1 \\ -4 & \text{when } \frac{p_1}{p_2} > 1 \end{cases}$$

The equilibrium is achieved for p such that $z_1(p) = 0$, giving

$$CE = \{p_1/p_2 = \delta/2, x_1^* = (0,2), x_2^* = (4,0)\}.$$

The Pareto set is the set of allocations found in 1.

Solution of Exercise 22

Consider an economy with two consumption goods, $L = 2$, one consumer, $I = 1$, and one firm, $J = 1$. Consumer 1 has endowment $\omega_1 = (2, 1)$ and utility $U(x_1) = x_{11}x_{21}$. The firm has production set $Y = \{y_1 + y_2 \leq 0 \text{ if } y_1 \leq -a,\ y_2 \leq 0 \text{ if } y_1 \in (-a, 0]\}$ with $a \geq 0$. Consumer one is the owner of the firm. Fix $a = 1$.

1. *Provide the definition of a competitive equilibrium for this economy.*
2. *Find the set of competitive equilibria.*
3. *Find the set of Pareto optimal allocations.*
4. *Discuss the validity of both Welfare Theorems.*

The production technology is linear but there are positive set-up costs when $a > 0$. Positive set-up costs make the production set non-convex and the II welfare Theorem cannot be applied. The numerical solution is provided in the solution of Exercise 5.

Solution of Exercise 23

Solve Ex. 23 for each given $a \in [0, 1]$.

When $a = 0$ the firm has a standard linear production function. The firm is active, and indifferent on the efficient frontier, when $\frac{p_1}{p_2} = 1$. For such prices, given the endowment and zero profit, the consumer wants to consume $\left(\frac{3}{2}, \frac{3}{2}\right)$ leading to an equilibrium where the firm choose $y = \left(-\frac{1}{2}, \frac{1}{2}\right)$.
The same equilibrium is present as long as $a \leq \frac{1}{2}$. At $a = \frac{1}{2}$ the equilibrium coincides with the minimum output of the firm.
As long as a increases, there is no equilibrium. In fact, at the allocation $(2 - a, 1 + a)$ the consumer would have be happy only with a price ratio larger than one, but for such price ratio the firm would not want to produce. The consumer would be happy to consume her endowment for a price ratio lower than one, but for such price ratio the firm would want to have an infinite output. The allocation is however Pareto optimal. There is no other feasible allocation where the consumer is better off. For $a = 1$ there is another Pareto optimal allocation, the endowment in fact

$$U(2 - a, a) = U(\omega)$$

is solved by $a = 1$. The first welfare theorem always applies. The second welfare theorem applies only when $a = 0$. Note that the II welfare theorem provides only sufficient conditions for a Pareto optimal allocation to be a (quasi)-equilibrium with transfer. As a result, it is well possible that a Pareto optimal allocation is supported as a competitive equilibrium even if the production set is not convex, as when $a \in \left(0, \frac{1}{2}\right)$.

Solution of Exercise 24

Consider an exchange economy with $I = \{1, 2\}$, $L = \{1, 2\}$, $U_1(x_1) = x_{21}$, $U_2(x_2) = x_{12} + x_{22}$, and $\omega_1 = (1, 0)$, $\omega_2 = (0, 1)$.

1.* *Find the utility possibility set*

2.* *Find the Pareto frontier*

3.** *Show that for some levels of* $\bar{U}_2$ *the solutions* x^* *of*

$$\begin{array}{l} Max\ U_1(x_1) \\ x \in \mathbf{R}^4_+ \\ x_1 + x_2 \leq \bar{\omega} \\ U_2(x_2) \geq \bar{U}_2 \end{array}$$

are not Pareto optimal (and thus $(U_1(x_1^*), U(x_2^*))$ *does not belong to the Pareto frontier).*

1. To find the UPS find (U_1, U_2) such that

$$\begin{cases} x_{li} \geq 0 \ \ \forall l, i \\ x_{11} + x_{12} \leq 1 \\ x_{21} + x_{22} \leq 1 \\ U_1(x_1) = x_{21} \geq U_1 \\ U_2(x_2) = x_{12} + x_{22} \geq U_2 \end{cases}$$

The set can be written as

$$\begin{cases} U_1 \leq 1 \\ U_2 \leq 2 \\ U_1 + U_2 \leq 2 \end{cases}$$

2. The Pareto frontier is the subset of the Pareto set where U_i cannot be increased without increasing U_j, $j \neq i$. This is the segment $\{U_2 = 2 - U_1, U_1 \in (0, 1)\}$.
3. See the lecture notes of Tuesday November 26.

Solution of Exercise 25

Consider an economy with two commodities, two consumers, and one firm. Consumer 1 has endowment $\omega_1 = (6, 0)$*, owns half of the firm, and has utility* $U_1(x_1) = x_{11}x_{21}$*. Consumer 2 has endowment* $\omega_2 = (4, 2)$*, owns the other half of the firm, and has utility* $U_2(x_2) = x_{12}x_{22}$*. The firm uses commodity 1 to produce commodity 2 and has production function* $f(z) = z$ *with* $z \geq 0$ *(the amount of commodity 1).*

1. *Derive supply and profit of the firm.*
2. *Derive the demand of both consumers.*
3. *Provide the definition of a competitive equilibrium for this economy.*
4. *Compute the unique competitive equilibrium of this economy.*
5. *Is the competitive equilibrium found above Pareto optimal? Reply without computing the Pareto set.*

6. *Provide a definition of Pareto optimal allocation.*
7. *Explain how to find all the Pareto optimal allocation of this economy (write down the related maximization problem and the first order conditions for an interior solution).*

TO BE PREPARED
Solution of Exercise 26
Solve Exercise 6.B.2 in the MWG textbook.

We have to show that if a preference relation over lotteries that can be represented by an utility function that has an expected utility form, then its satisfies the independence axiom. We shall exploit the linearity of the EU form.
Consider two lotteries $L = \{C; p\}$ and $L' = \{C; p'\}$ with

$$L \succsim L'.$$

The EU form implies that the latter is equivalent to

$$\sum_n p_n u(C_n) \geq \sum_n p'_n u(C_n).$$

Take now any other lottery $L'' = \{C; p''\}$ and any $\alpha \in (0,1)$. The former inequality implies also

$$\alpha \sum_n p_n u(C_n) + (1-\alpha) \sum_n p''_n u(C_n) \geq \alpha \sum_n p'_n u(C_n) + (1-\alpha) \sum_n p''_n u(C_n).$$

By linearity of the EU form, the latter can be re-written as

$$\sum_n [\alpha p_n + (1-\alpha) p''_n] u(C_n) \geq \sum_n [\alpha p'_n + (1-\alpha) p''_n] u(C_n),$$

which, by definition of compound lottery, is equivalent to

$$\alpha L + (1-\alpha) L'' \succsim \alpha L + (1-\alpha) L''.$$

Solution of Exercise 27
Show that a preference relation on lotteries that can be represented in an Expected Utility form satisfies transitivity.

Consider three lotteries $L = \{C; p\}$, $L' = \{C; p'\}$, and $L'' = \{C; p''\}$ with

$$L \succsim L' \quad \text{and} \quad L' \succsim L''.$$

The latter, together with the fact that the preference relation can be represented in an EU form, imply, respectively,

$$\sum_n p_n u(C_n) \geq \sum_n p'_n u(C_n) \quad \text{and} \quad \sum_n p'_n u(C_n) \geq \sum_n p''_n u(C_n).$$

Thus

$$\sum_n p_n u(C_n) \geq \sum_n p'_n u(C_n) \geq \sum_n p''_n u(C_n),$$

so that

$$\sum_n p_n u(C_n) \geq \sum_n p''_n u(C_n).$$

Given the EU form, the latter implies

$$L \succsim L''.$$

Solution of Exercise 28

Consider the space of simple lotteries $L = \{C; (p_1, p_2, p_3)\}$. Show graphically that if a preference relation on lotteries satisfies the independence axiom, then indifference curves are parallel. (hint: show that if two indifference curves are not parallel, then the independence axiom is violated)

The solution is at page $175 - 176$, figure $6.B.5.$ panels $a)$ and $c)$ of the MWG textbook.

Solution of Exercise 29

Solve Exercise 6.B.4 in the MWG textbook.

First not that given the set of outcomes $C = \{A, B, C, D\}$ with

A No flood and no evacuation

B No flood and evacuation

C Flood and evacuation

D Flood and no evacuation

the expected utility of a generic lottery $L = \{C, ; p\}$ is

$$U(L) = \sum_{i=A,B,C,D} p_i u_i$$

with

p_A Probability of no flood and no evacuation

p_B Probability of no flood and evacuation

p_C Probability of flood and evacuation

p_D Probability of flood and no evacuation

(a) To find the EU form we need to assign values to u_i, $i = A, B, C, D$. We shall use, the revealed preferences of the decision maker. In particular we have, for $p \in (0,1)$ and $q \in (0,1)$,

$$u_B = pu_A + (1-p)u_D \quad \text{and} \quad u_C = qu_B + (1-q)u_D \quad \text{and} \quad u_A > u_D.$$

It follows that $u_A > u_B > u_C > u_D$. Without loss of generality, we can normalize $u_D = 0$ and $u_A = 1$ and get

$$u_B = p \quad \text{and} \quad u_C = pq.$$

(b) To evaluate each criterion we have to compute the corresponding probabilities. The probability of having a flood is given and equal to $\pi = 0.01$, so that the probability of not having a flood is $1 - \pi = 0.99$. Name p_N the probability of a necessary evacuation and p_U the probability on an unnecessary evacuation as implied by a criterion. The resulting probabilities over outcomes are

$$(p_A, p_B, p_C, p_D) = ((1-p_U)(1-\pi), p_U(1-\pi), p_N\pi, (1-p_N)\pi)$$

Applying the above to the criterion 1 and 2 we have, respectively

$$p^1 = \left(\frac{90}{100}\frac{99}{100}, \frac{10}{100}\frac{99}{100}, \frac{90}{100}\frac{1}{100}, \frac{10}{100}\frac{1}{100}\right)$$

and

$$p^2 = \left(\frac{95}{100}\frac{99}{100}, \frac{5}{100}\frac{99}{100}, \frac{95}{100}\frac{1}{100}, \frac{5}{100}\frac{1}{100}\right).$$

Note that the second criterion implies a higher probability of necessary evacuation and also a lower probability of unnecessary evacuation, we thus expected that the second is always better than the first. Using the EU representation found in (a) and computing the difference in utility between the two criteria, we find

$$U(L^2) - U(L^1) = \frac{5}{100}\frac{99}{100}(1-p) + \frac{5}{100}\frac{1}{100}pq.$$

The latter, being strictly positive for all p and q in $(0,1)$ confirms our intuition: criterion 2 is always preferred to criterion 1.

Solution of Exercise 30

Solve Exercise 6.B.7 in the MWG textbook. (Note: a preference over lotteries is monotone if, given two real outcomes $C_1 > C_2$, then the lottery with sure outcome C_1 is strictly preferred to the lottery with sure outcome C_2)

The decision maker is indifferent between L and the sure amounts x_L, so that

$$L \sim x_L \quad \Rightarrow \quad x_L \succcurlyeq L,$$

and between L' and the sure amounts $x_{L'}$, so that

$$L \sim x_L \quad \Rightarrow \quad L' \succcurlyeq x_{L'}.$$

(note that above with x_L or $x_{L'}$ we denote the lottery that give the sure amount for sure). Assume now that the decision maker strictly prefers L to L', then by transitivity

$$x_L \succ x_{L'}.$$

Given monotonicity, the latter is implied by

$$x_L > x_{L'}.$$

To show the "only if" case we also need that when two real outcomes are equal, then the decision maker is indifferent between the two lotteries having that amount for sure. The latter is always the case as the two lotteries coincide.

Solution of Exercise 31

When faced with the choice between the lottery $A = \{(3500, 2800, 0); p = (0.3, 0.66, 0.04)\}$ *and* $B = \{3500; p = 1\}$*, a decision maker chooses lottery* B*. When, instead, he is asked to choose between the lottery* $A' = \{(3500, 2800, 0); p = (0.3, 0, 0.7)\}$ *and lottery* $B' = \{(3500, 2800, 0); p = (0, 0.34, 0.66)\}$*, he chooses* A'*. Say whether the decision maker's preferences are consistent with the expected utility form, explaining how you have reached your conclusion. What if lottery* B *becomes* $B'' = \{2800; 1\}$*?*

Given the EU form of the utility that represents the decision maker preferences, we can write, for a generic lottery $L = \{(3500, 2800, 0); p\}$

$$U(L) = p_{3500}u_{3500} + p_{2800}u_{2800} + p_0u_0.$$

Without loss of generality we can normalize $u_0 = 0$ and $u_{3500} = 1$ and get

$$U(L) = p_{3500} + p_{2800}u_{2800}.$$

Using this representation of the utility, having $U(B) > U(A)$ implies the following restriction on u_{2800}

$$U(B) = 1 > U(A) = 0.3 + 0.66u_{2800} \quad \Rightarrow \quad u_{2800} < \frac{70}{66}.$$

Repeating the same reasoning for A' and B', we find

$$U(A') = 0.3 > U(B') = 0.34u_{2800} \quad \Rightarrow \quad u_{2800} < \frac{30}{34}.$$

Any value of u_{2800} that satisfies both inequalities, e.g. $u_{2800} = 0.5$, gives an expected utility representation of consumer's preferences.
If we replace lottery B with B'', the first inequality becomes

$$U(B'') = u_{2800} > U(A) = 0.3 + 0.66u_{2800} \quad \Rightarrow \quad u_{2800} > \frac{30}{34},$$

and no values of u_{2800} such that preferences are consistent with an EU form exist.

Solution of Exercise 32

We consider an expected-utility decision-maker facing the following possible professional occupations: working in the financial industry (A), working in the movie industry (B), working in the car industry (C). We further assume that the decision-maker can apply to three different schools.

- *After School* 1*, he is certain to find a job in the car industry.*
- *After School* 2*, he is certain to find a job in the financial industry.*
- *After School* 3*, he would find a job in the movie industry with probability* 0.1 *and in the financial industry with probability* 0.9*.*

We assume that School 2 *is the least preferred option of the decision-maker and that he is indifferent between School* 1 *and School* 3*.*

1. *Give a representation of the utility function of this decision-maker.*
2. *We assume that a new school (School* 4*) opens. After school* 4*, a student would find a job in each industry with probability* $\frac{1}{3}$*. How would the new school rank compared to Schools* 1*,* 2*,* 3*?*
3. *There are too many application in School* 4*. The ministry of education decides that rather than applying to School* 4 *directly, the decision-maker must apply to a lottery that leads to admission in School* 4 *with probability* $\alpha \in [0,1]$ *and to admission in School* 2 *with probability* $1-\alpha$*. For which value of* α *does the decision-maker prefer to apply to School* 1 *rather than to the lottery that may lead to admission to School* 4*?*

1. Given the three outcomes, $\{A, B, C\}$, to each school there corresponds a lottery that can be identified with the following probability vectors

$$L_1 = (0, 0, 1), \quad L_2 = (1, 0, 0), \quad L_3 = (0.9, 0.1, 0).$$

The decision maker evaluations are thus

$$U(L_1) = u_C, \quad U(L_2) = u_A, \quad U(L_3) = 0.9u_A + 0.1u_B.$$

Thus, having $U(L_1) = U(L_3) > U(L_2)$, implies

$$u_C = 0.9u_A + 0.1u_B > u_A.$$

Without loss of generality we can take $u_A = 0$, $u_B = 1$, $u_C = 0.1$.
2. The new school corresponds to the lottery $L_4 = (1/3, 1/3, 1/3)$. Using the values found above, we find

$$U(L_4) = \frac{1}{3} + \frac{1}{3}\frac{1}{10} = \frac{11}{30} > \frac{1}{3} = U(L_1) = U(L_3),$$

so that the new school is the most preferred.
3. The new procedure corresponds to the compound lottery

$$\alpha L_4 + (1-\alpha)L_2.$$

By linearity of the EU, the utility of the latter is easily found as

$$U(\alpha L_4 + (1-\alpha)L_2) = \alpha U(L_4) + (1-\alpha)U(L_2) = \alpha\frac{11}{30}.$$

The decision maker prefers school 1 rather than the compound lottery when

$$U(L_1) = \frac{1}{10} > \alpha\frac{11}{30} = U(\alpha L_4 + (1-\alpha)L_2) \quad \Leftrightarrow \quad \alpha < \frac{3}{11}.$$

Solution of Exercise 33

We consider an expected-utility decision-maker with utility of the form $u(x) = x^a$ with $a < 1$.

1. *Let X be a lottery whose outcome is uniformly distributed over $[0,1]$. Determine $\mathbb{E}(u(X))$.*
2. *Let Y be a lottery whose outcome is 0 with probability $1/3$ and 1 with probability $2/3$. Determine $\mathbb{E}(u(Y))$.*
3. *Determine the value a^* (of a) for which the decision-maker is indifferent between X and Y.*
4. *If $a > a*$, which lottery, X or Y, is preferred by the decision-maker?*
5. *Determine, as a function of a, the coefficient of absolute risk-aversion of the decision-maker.*

TO BE PREPARED

Solution of Exercise 34

Solve Exercise 6.C.1 in MWG assuming $u(x) = \log(x)$ and find* for which value of q the agent does not insure.

Let us consider $u(x) = \log x$ first. From the first order conditions one gets

$$\frac{\pi(1-q)}{w - \alpha q - D + \alpha} - \frac{q(1-\pi)}{w - \alpha q} \leq 0$$

with equality when $\alpha > 0$. Let us first check that $\alpha = D > 0$ is not a solution. Substituting $\alpha = D$ in the above, one gets

$$\pi = q$$

which contradicts that the insurance premium q is higher than π. To show that the solution must have $\alpha < D$ rewrite the FOC as

$$\frac{\pi}{q}\frac{1-q}{1-\pi}\frac{1}{w-\alpha q - D + \alpha} \leq \frac{1}{w-\alpha q}.$$

If the solution is interior, $\alpha > 0$, then equality of the latter and $q > \pi$ imply

$$\frac{1}{w-\alpha q - D + \alpha} > \frac{1}{w-\alpha q} \quad \Leftrightarrow \quad \alpha < D.$$

If the solution is not interior, $\alpha = 0$, then also $\alpha < D$.
To find the conditions on q such that the agent does not insure we must find bounds on the value of q such that the FOC with inequality is satisfies at $\alpha = 0$. Assuming positive final wealth, the FOC with inequality can be re-written as

$$\alpha \geq D\frac{1-\pi}{1-q} - w\frac{q-\pi}{q(1-q)}.$$

With $\alpha = 0$, and solving for q, we get

$$q \geq \pi\frac{w}{w - D(1-\pi)}.$$

Note that the higher the loss, the higher the insurance price beyond which the agent does not insure.

Solution of Exercise 35

Consider an investor who has to choose between two assets. Asset A has payoff $x_A = (12, 6, 9)$, all with equal probability. Asset B is risk free with payoff $x_B = (\bar{x}, \bar{x}, \bar{x})$.

1. *Assume $u(x) = \log(x)$. Find the certainty equivalent of x_B.*
2. *Find the level of $\bar{x}$ such that the investor is indifferent between asset A and asset B.*
3. *Assume now that $\bar{x} = E[x_A]$, find how many units h_A of asset A should be given to the agent to make him indifferent between h_A and one unit of the risk free asset. Provide an intuition.*
4. *Compute the coefficient and absolute risk aversion.*

The certainty equivalent is 8.653. The average is 9.

Solution of Exercise 36

A decision maker has preferences over lotteries represented by an expected utility with Bernoulli utility $u(x)$. Consider the lottery $L = \{(0, 4, 9); (1/6, 1/2, 1/3)\}$

1. *Provide the definition of certainty equivalent of L given u.*

2. *Compute the certainty equivalent for L given $u(x) = \sqrt{x}$.*
3. *Compute the coefficient of absolute risk aversion given u and x.*
4. *Assume that the agent starts with $w = 12$ and owns the lottery. Find the minimum price he is willing to accept to sell the lottery.*
5. *Compare the minimum price found above and the certainty equivalent and provide an intuition of their difference based on the coefficient of absolute risk aversion.*

TO BE PREPARED

Solution of Exercise 37

Solve Exercise 6.C.18 in MWG.

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Solution of Exercise 38

Solve Exercise 6.C.16 in MWG assuming from the beginning the parametrization of point (d) with $p = 0.5$. Other than the proposed $u(x) = \sqrt{x}$, consider also $u(x) = x$ and $u(x) = \log x$. Give and interpretation in terms of the certainty equivalent and the change of the risk aversion coefficient with wealth.

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