Ex 15 : $L = 3$

$$f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$$

$\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$

(CMP) : Min $p_1 z_1 + p_2 z_2$.

st : $z_1, z_2 \geq 0$

$z_1^{\alpha} \cdot z_2^{\beta} \geq q$

1) If $q = 0$

$\Rightarrow z_1 = z_2 = 0$ is the unique solution.

1) If $q > 0$

$\Rightarrow z_1, z_2 > 0$

• If not then $0 \geq q$ (Contradiction)

$$\mathcal{L} = p_1 z_1 + p_2 z_2 + \lambda (q - z_1^{\alpha} z_2^{\beta})$$

FOC :

$$\frac{\partial \mathcal{L}}{\partial z_1} = \boxed{p_1 - \alpha \lambda z_1^{\alpha - 1} z_2^{\beta} = 0 \quad (1)}$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = p_2 - \beta \lambda z_1^{\alpha} z_2^{\beta - 1} = 0 \quad (2)$$

$$\lambda \cdot \left(q - z_1^{\alpha} z_2^{\beta}\right) = 0 \quad (3)$$

$$(1),(2) \Leftrightarrow \begin{cases} p_1 = \alpha \lambda z_1^{\alpha-1} z_2^{\beta} \\ p_2 = \beta \lambda z_1^{\alpha} z_2^{\beta-1} \end{cases}$$

$$\Rightarrow \frac{p_1}{p_2} = \frac{\alpha \lambda z_1^{\alpha-1} z_2^{\beta}}{\beta \lambda z_1^{\alpha} z_2^{\beta-1}} = \frac{\alpha . z_2}{\beta z_1}$$

$$\Rightarrow z_2 = \frac{p_1}{\alpha p_2} \cdot \beta z_1$$

It is obvious that $\lambda > 0$

$$(3) \Leftrightarrow q = z_1^{\alpha} z_2^{\beta}.$$

$$\Leftrightarrow q = z_1^{\alpha} \left(\frac{p_1}{\alpha p_2} \beta z_1\right)^{\beta}$$

$$\Leftrightarrow q = z_1^{\alpha+\beta} \left(\frac{\beta p_1}{\alpha p_2}\right)^{\beta}$$

$$\Rightarrow z_1 = \left(q \cdot \left(\frac{\alpha p_2}{\beta p_1}\right)^{\beta} \right)^{\frac{1}{\alpha+\beta}}$$

$$\Rightarrow \quad z_2 = \frac{\beta p_1}{\alpha p_2} q^{1/\alpha+\beta} \left(\frac{\alpha p_2}{\beta p_1} \right)^{\beta/\alpha+\beta}$$

$$c(p, q) = p_1 \cdot q^{1/\alpha+\beta} \cdot \left(\frac{\alpha p_2}{\beta p_1} \right)^{\beta/\alpha+\beta}$$

$$+ \; p_2 \cdot \frac{\beta p_1}{\alpha p_2} q^{1/\alpha+\beta} \cdot \left(\frac{\alpha p_2}{\beta p_1} \right)^{\beta/\alpha+\beta}$$

simplify it !

Ex 16 :

(CMP_q) $$\begin{aligned} \text{Min } & p \cdot z \\ \text{st : } & z \geq 0 \\ & f(z) \geq q \end{aligned}$$

closed

We have :

$$C(q) = \begin{aligned} \text{Min } & p \cdot z \\ \text{st : } & z \geq 0 \\ & f(z) \geq q \end{aligned}$$

We have to prove that $C(\cdot)$ is convex

$$\Longleftrightarrow \boxed{\lambda C(q) + (1-\lambda)\, C(q') \geq C(\lambda q + (1-\lambda) q')} \quad 1$$

$$\forall \; q, q' \in \mathbb{R}$$
$$\lambda \in (0,1)$$

Let $K_q = \{ z \geq 0 \text{ st: } f(z) \geq q \}$ ✓

* Assume that:

$$K_q = \emptyset \text{ or } K_{q'} = \emptyset.$$

$$\boxed{\min_{z \in \emptyset} g(z) = +\infty}$$

$\Rightarrow C(q) = +\infty$ or $C(q') = +\infty$

$\Rightarrow$ 1 is true

* Assume that:

$$K_q \neq \emptyset \text{ and } K_{q'} \neq \emptyset$$

$\Rightarrow (CMP_q)$ and $(CMP_{q'})$ are well-defined

$\Downarrow$

they have solutions

Let $z(q)$ and $z(q')$ be the solutions of (CMP_q) and $(CMP_{q'})$ respectively.

By definition,

$$p \cdot z \geqslant p \cdot z(q)$$
$$\forall \ z \in K_q$$

$$p \cdot z \geqslant p \cdot z(q')$$
$$\forall \ z \in K_{q'}$$

and $f(z(q)) \geqslant q$; $f(z(q')) \geqslant q'$

① $(\Rightarrow) \ \lambda p \cdot z(q) + (1-\lambda) p \, z(q')$

$$\geqslant C(\lambda q + (1-\lambda) q')$$

$$C(\lambda q + (1-\lambda) q') = \min \ p \cdot z$$
$$\text{s.t. } z \geqslant 0$$
$$f(z) \geqslant \lambda q + (1-\lambda) q'.$$

We have:

$$f\left(\lambda z(q) + (1-\lambda) z(q')\right)$$

$$\geq \lambda f(z(q)) + (1-\lambda) f(z(q'))$$
(f is concave)

$$\geq \lambda q + (1-\lambda) q'$$
(By yellow)

$$\Rightarrow \lambda z(q) + (1-\lambda) z(q') \in K_{\lambda q + (1-\lambda) q'}$$

$$\Rightarrow \boxed{p \cdot \left(\underbrace{\lambda z(q) + (1-\lambda) z(q')}_{\in K_{\lambda q + (1-\lambda) q'}}\right) \geq \min_{z \in K_{\lambda q + (1-\lambda) q'}} p \cdot z}$$

$$= C(\lambda q + (1-\lambda) q')$$

$$\Leftrightarrow \lambda c(q) + (1-\lambda) c(q') \geq c(\lambda q + (1-\lambda) q')$$

$\Rightarrow$ $c(\cdot)$ is convex.