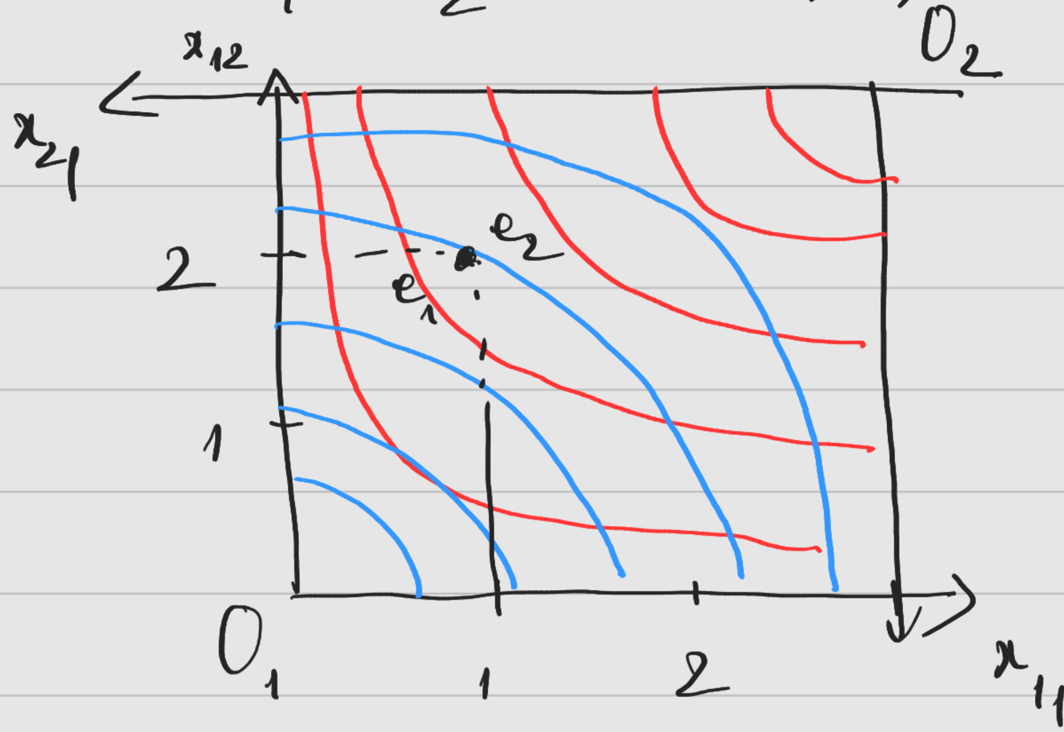
Ex 20

1) $e = e_1 + e_2 = (3, 3)$



2) (x_1^*, x_2^*, p^*) is a C.E of the economy iff

(i) $\forall\ i = 1, 2$

$x_i^* \in d_i(p^*, p^* \cdot e_i)$

(ii) Market clearing condition:

$x_1^* + x_2^* = e_1 + e_2$

3) For a given $(p^*, 1)$

– Consumer 1's problem

$$\text{Max } u_1(x_{11}, x_{12})$$

$$\text{st}: \quad x_{11}, x_{12} \geq 0$$

$$p^* . x_{11} + 1 . x_{12} \leq p^* e_{11} + 1 . e_{12}$$

$$\Leftrightarrow \quad \text{Max } x_{11}^{1/3} \, x_{12}^{2/3}$$

$$\text{st}: \quad x_{11}, x_{12} \geq 0 \qquad (1)$$

$$p^* x_{11} + x_{12} \leq p^* + 2$$

$$\mathcal{L} = x_{11}^{1/3} . x_{12}^{2/3} + \mu_1 . x_{11} + \mu_2 x_{12} + \lambda (p^* + 2 - p^* x_{11} - x_{12})$$

FOC :

$$\frac{\partial \mathcal{L}}{\partial x_{11}} = \frac{1}{3} x_{11}^{-2/3} . x_{12}^{2/3} + \mu_1 - \lambda p^* = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{2}{3} x_{11}^{1/3} . x_{12}^{-1/3} + \mu_2 - \lambda = 0 \qquad (2)$$

$$\mu_1 . x_{11} = 0 \quad ; \quad \mu_2 x_{12} = 0$$

$$\lambda . (p^* + 2 - p^* x_{11} - x_{12}) = 0 \qquad (3)$$

If $x_{11} = 0$

(1) $\Leftrightarrow \quad \infty = \lambda p^* - \mu_1$

(Contradictory)

$\Rightarrow x_{11} > 0$

Similarly, $x_{12} > 0$

$\Rightarrow \mu_1 = \mu_2 = 0$

(1), (2) $\Leftrightarrow \begin{cases} \frac{1}{3} x_{11}^{-2/3} x_{12}^{2/3} = \lambda p^* \\ \frac{2}{3} x_{11}^{1/3} x_{12}^{-1/3} = \lambda \end{cases}$

$\Rightarrow \lambda > 0$

$\Rightarrow \frac{1/3 \, x_{11}^{-2/3} x_{12}^{2/3}}{2/3 \, x_{11}^{1/3} x_{12}^{-1/3}} = p^*$

$\Leftrightarrow \frac{1}{2} \frac{x_{12}}{x_{11}} = p^*$

$$\Leftrightarrow \quad x_{12} = 2p^* x_{11}$$

(3) $\Leftrightarrow \quad p^* x_{11} + x_{12} = p^* + 2$

$$\Leftrightarrow \quad p^* x_{11} + 2p^* x_{11} = p^* + 2$$

$$\Rightarrow \quad x_{11} = \frac{p^* + 2}{3p^*}$$

$$x_{12} = 2p^* \cdot \frac{p^* + 2}{3p^*} = \frac{2}{3}(p^* + 2)$$

— Consumer 2's problem

$$\text{Max } x_{21}^{1/2} x_{22}^{1/2}$$

st: $x_{21}, x_{22} \geq 0$

$$p^* x_{21} + x_{22} \leq 2p^* + 1$$

$$\mathcal{L} = x_{21}^{1/2} x_{22}^{1/2} + \mu_1 x_{21} + \mu_2 x_{22} + \lambda (2p^* + 1 - p^* x_{21} - x_{22})$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x_{21}} = \frac{1}{2} x_{21}^{-1/2} x_{22}^{1/2} + \mu_1 - \lambda p^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_{22}} = \frac{1}{2} x_{21}^{1/2} x_{22}^{-1/2} + \mu_2 - \lambda = 0$$

$$\mu_1 x_{21} = \mu_2 x_{22} = 0$$

$$\lambda\,(2p^* + 1 - p^* x_{21} - x_{22}) = 0$$

Easily, $\mu_1 = \mu_2 = 0$, $\lambda > 0$

$$\Rightarrow \begin{cases} \dfrac{1}{2}\, x_{21}^{-1/2}\, x_{22}^{1/2} = \lambda p^* \\ \dfrac{1}{2}\, x_{21}^{1/2}\, x_{22}^{-1/2} = \lambda \end{cases}$$

$$\Rightarrow \quad \frac{x_{22}}{x_{21}} = p^*$$

$$\Leftrightarrow \quad x_{22} = p^* x_{21}$$

We also have:

$$p^* x_{21} + x_{22} = 2p^* + 1$$

$$\Leftrightarrow 2p^* x_{21} = 2p^* + 1$$

$$\Rightarrow \quad x_{21} = \frac{2p^* + 1}{2p^*}$$

$$x_{22} = \frac{2p^* + 1}{2}$$

— Using Market clearing condition to find p^*.

$$x_1 + x_2 = e_1 + e_2 = (3,3)$$

$$\Leftrightarrow \begin{cases} x_{11} + x_{21} = 3 \\ x_{12} + x_{22} = 3. \end{cases}$$

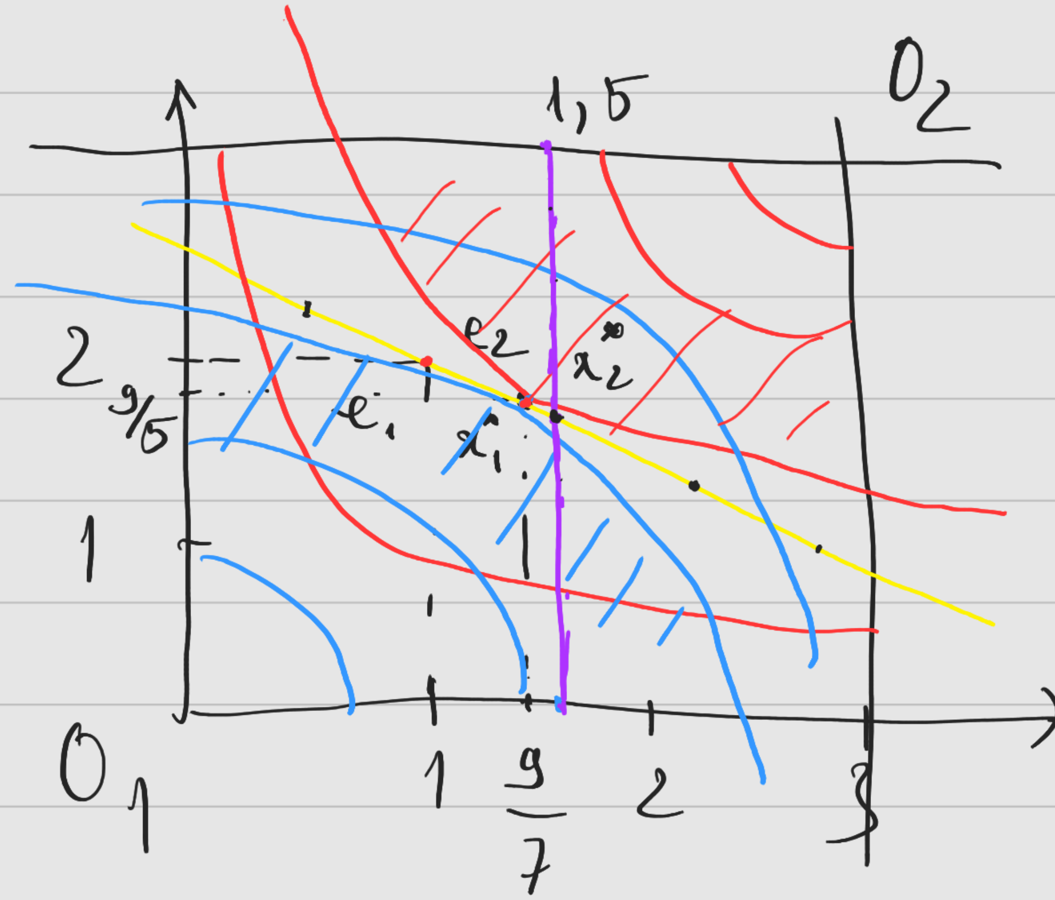
$$\Rightarrow \frac{p^*+2}{3p^*} + \frac{2p^*+1}{2p^*} = 3$$

$$\Leftrightarrow 2(p^*+2) + 3(2p^*+1) = 18p^*$$

$$\Leftrightarrow 8p^* + 7 = 18p^*$$

$$\Leftrightarrow p^* = \frac{7}{10}$$

$$\left\{ \begin{array}{l} x_1^* = \left(\frac{9}{7}, \frac{9}{5} \right) \\ x_2^* = \left(\frac{12}{7}, \frac{6}{5} \right) \end{array} \right\}, \left(\tfrac{7}{10}, 1\right) \text{ is C.E}$$



4) $\tilde{e}_{11} = \tilde{e}_{21}$ $\in$ purple line

$\tilde{e}_{11} + \tilde{e}_{21} = 3$

$\Rightarrow \boxed{\tilde{e}_{11} = \tilde{e}_{21} = 1{,}5}$

$\tilde{e}$ must be the intersection of the purple line and the budget line

Budget line:

$$\frac{7}{10} \cdot x_{11} + 1 \cdot x_{12} = \frac{7}{10} \cdot 1 + 2$$

$$\Rightarrow \frac{7}{10} \cdot 1.5 + 1 \cdot \tilde{e}_{12} = \frac{27}{10}$$

$$\Rightarrow \tilde{e}_{12} = \frac{27}{10} - \frac{21}{20} = \frac{33}{20}$$

$$\tilde{e}_{22} = 3 - \tilde{e}_{12} = 3 - \frac{33}{20} = \frac{27}{20}$$

$$\tilde{e}_1 = \left(\frac{3}{2}, \frac{33}{20} \right) \text{ and}$$

$$\tilde{e}_2 = \left(\frac{3}{2}, \frac{27}{20} \right)$$

Ex 21

1 For a given $p^* = (p_1^*, p_2^*)$

Consumer 1's problem :

$$\text{Max} \quad x_{11}^{\alpha_1} x_{12}^{1-\alpha_1}$$

$$\text{st} : \quad x_{11}, x_{12} \geqslant 0$$

$$p_1^* x_{11} + p_2^* x_{12} \leq p_1^* e_{11} + p_2^* e_{12}$$

$$\mathcal{L} = x_{11}^{\alpha_1} x_{12}^{1-\alpha_1} + \mu_1 x_{11} + \mu_2 x_{12} + \lambda \left(p_1^* e_{11} + p_2^* e_{12} - p_1^* x_{11} - p_2^* x_{12} \right)$$

FOC :

$$\frac{\partial \mathcal{L}}{\partial x_{11}} = \alpha_1 x_{11}^{\alpha_1 - 1} x_{12}^{1-\alpha_1} + \mu_1 - \lambda p_1^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_{12}} = (1-\alpha_1) x_{11}^{\alpha_1} x_{12}^{-\alpha_1} + \mu_2 - \lambda p_2^* = 0$$

$$\mu_1 x_{11} = \mu_2 x_{12} = 0$$

$$\lambda \cdot \left(p_1^* e_{11} + p_2^* e_{12} - p_1^* x_{11} - p_2^* x_{12} \right) = 0$$

It is immediate that $\mu_1 = \mu_2 = 0$ and $\lambda > 0$

$$\Rightarrow \begin{cases} \alpha_1 x_{11}^{\alpha_1 - 1} x_{12}^{1-\alpha_1} = \lambda p_1^* \\ (1-\alpha_1) x_{11}^{\alpha_1} x_{12}^{-\alpha_1} = \lambda p_2^* \end{cases}$$

$$\Rightarrow \frac{\alpha_1}{1-\alpha_1} \cdot \frac{x_{12}}{x_{11}} = \frac{p_1^*}{p_2^*}$$

$$\Leftrightarrow x_{12} = \frac{1-\alpha_1}{\alpha_1} \cdot \frac{p_1^*}{p_2^*} \cdot x_{11}$$

We have:

$$p_1^* e_{11} + p_2^* e_{12} = p_1^* x_{11} + p_2^* x_{12}$$

$$\Leftrightarrow p_1^* x_{11} + \cancel{p_2^*} \cdot \frac{(1-\alpha_1) p_1^*}{\alpha_1 \cancel{p_2^*}} \cdot x_{11} = p_1^* e_{11} + p_2^* e_{12}$$

$$\Rightarrow p_1^* x_{11} \left(1 + \frac{1-\alpha_1}{\alpha_1}\right) = p_1^* e_{11} + p_2^* e_{12}$$

$$\Leftrightarrow \quad x_{11} = \frac{(p_1^* e_{11} + p_2^* e_{12}) \cdot \alpha_1}{p_1^*}$$

$$x_{12} = \frac{(1-\alpha_1)(p_1^* e_{11} + p_2^* e_{12})}{p_2^*}$$

The excess demand function for consumer 1:

$$z_1 := x_1 - e_1$$

$$\alpha_1 p_2^* e_{12} - (1-\alpha_1) p_1^* e_{11}$$

$$\Leftrightarrow z_1 = \left(\frac{\alpha_1 p_2^* e_{12} - (1-\alpha_1) p_1^* e_{11}}{p_1^*} , \right.$$

$$\frac{(1-\alpha_1) \cdot p_1^* e_{11} - \alpha_1 p_2^* e_{12}}{p_2^*}$$

Similar,

$$z_2 = \left(\frac{\alpha_2 p_2^* e_{22} - (1-\alpha_2) p_1^* e_{21}}{p_1^*} , \right.$$

$$\frac{(1-\alpha_2)p_1^* e_{21} - \alpha_2 p_2^* e_{22}}{p_2^*} \Bigg)$$

The aggregate excess demand function

$$z := z_1 + z_2 = (z_1, z_2)$$

where:

$$z_1 = \frac{p_2^*(\alpha_1 e_{12} + \alpha_2 e_{22}) - p_1^*[(1-\alpha_1)e_{11} + (1-\alpha_2)e_{21}]}{p_1^*}$$

$$z_2 = \Big(\quad \ldots \quad \Big)$$

$$\left\{ \begin{array}{l} p_1^* \\ p_2^* \end{array} \right. \overset{\text{fixed}}{\uparrow} \Rightarrow z_1 \uparrow$$

(easy to check)