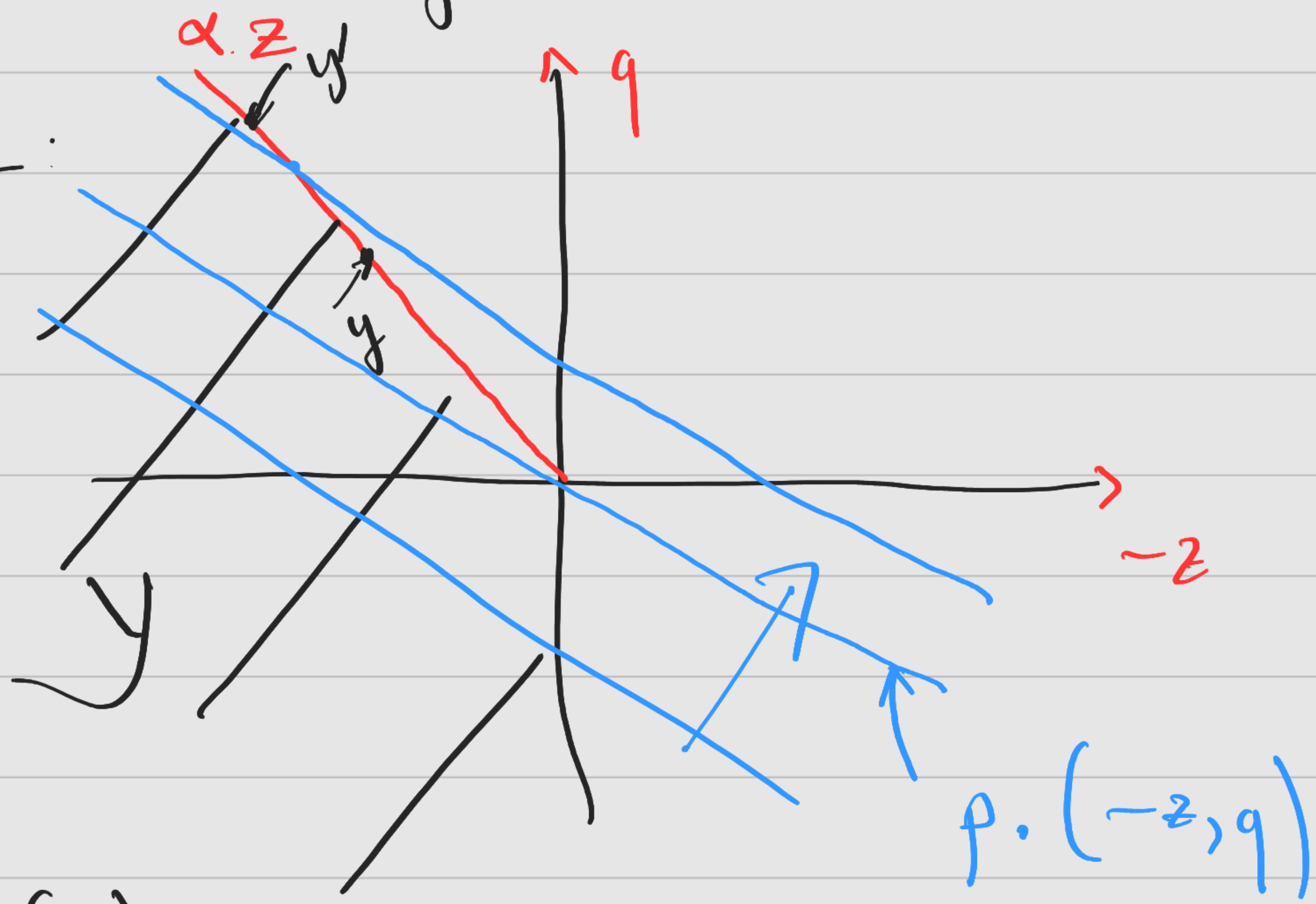
Ex 11 :

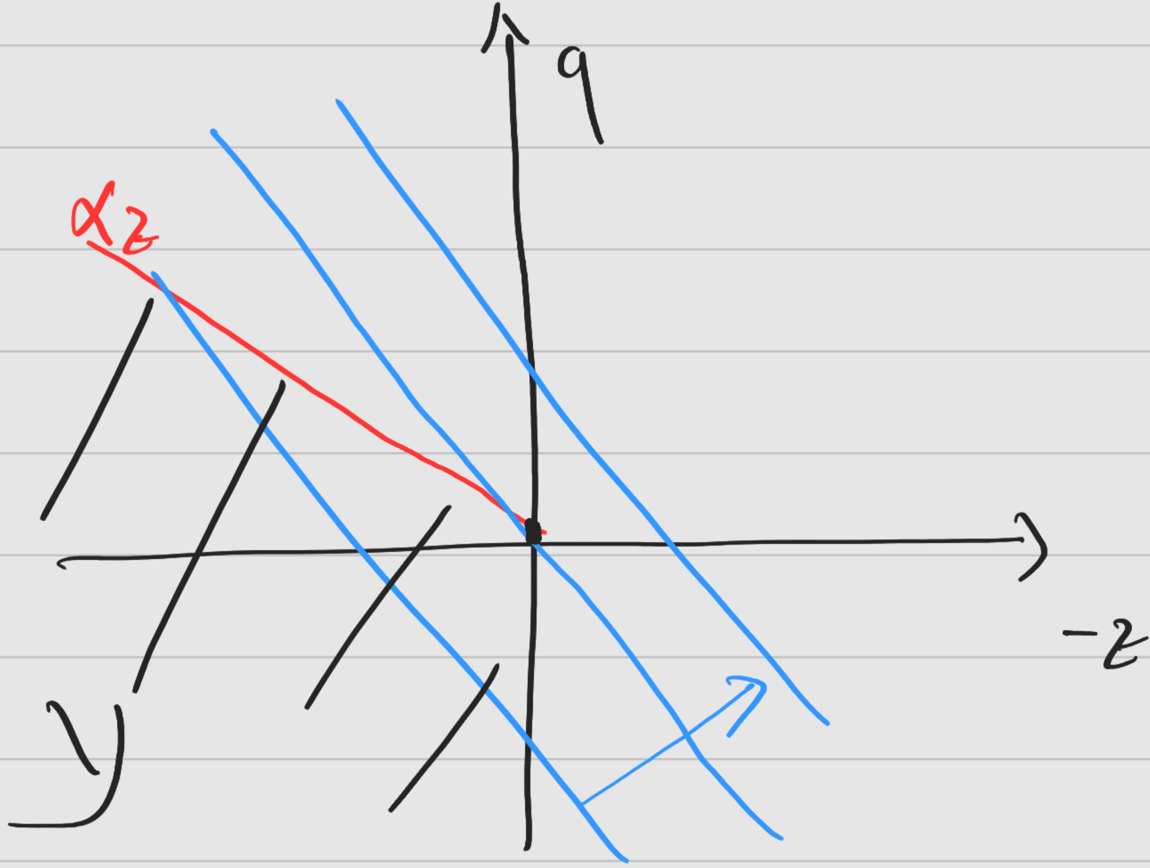
3. $\pi(p) = \max_{y \in \mathcal{Y}} p \cdot y$

Case 2 :



$\pi(p) = +\infty$

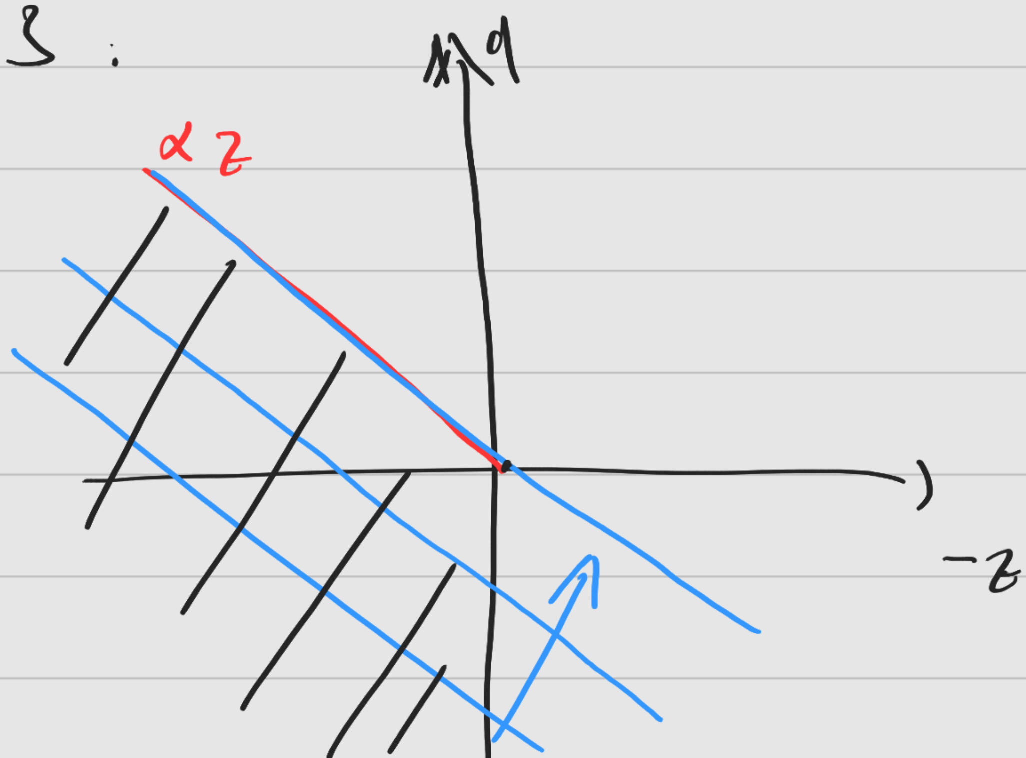
Case 1



$\pi(p) = 0$

$s(p) = \{(0,0)\}$

Case 3 :



$\pi(p) = 0$

$$s(p) = \{(z, q) \mid z \geq 0 \text{ and } q = \alpha z\}$$

<u>Ex 12</u> :

1) (PMP) : $\text{Max } p \cdot y$

st : $y \in \mathcal{Y}$

$\Leftrightarrow \quad \text{Max } p \cdot y$

st : $t(y) \leq 0$

2) $t(\cdot)$ is strictly quasi-convex iff $\forall\ y, y' \in \mathbb{R}^L$ and $\forall \lambda \in (0,1)$ with $y \neq y'$ then :

$$t(\lambda y + (1-\lambda) y') < \max\{t(y), t(y')\}$$

Assume that there exists $y, y' \in s(p)$ with $y \neq y'$.

$\Rightarrow$ $y, y' \in \tilde{Y}$, i.e, $\begin{cases} t(y) \leq 0 \\ t(y') \leq 0 \end{cases}$

Since t is strictly quasi-convex,

$$t(\lambda y + (1-\lambda) y') < \max\{t(y), t(y')\} \leq 0$$

for some $\lambda \in (0,1)$

Since $t(\cdot)$ is continuous, there must exist an closed ball $B(\lambda y + (1-\lambda) y', \varepsilon)$ such that:

$$t(y'') < 0 \quad \forall\, y'' \in B(\lambda y + (1-\lambda) y', \epsilon)$$

We have:

$$\lambda y + (1-\lambda) y' + \varepsilon . \mathbb{1}_L \in B(\lambda y + (1-\lambda) y', \varepsilon)$$

where: $\mathbb{1}_L = (\underbrace{1, 1, \ldots, 1}_{L})$.

$\Rightarrow t(\lambda y + (1-\lambda) y' + \epsilon \cdot \mathbb{1}_L) < 0$

$\Rightarrow \lambda y + (1-\lambda) y' + \epsilon \cdot \mathbb{1}_L \in \mathcal{Y}$

$+) \; p \cdot [\lambda y + (1-\lambda) y' + \epsilon \mathbb{1}_L]$

$$= \lambda \underbrace{p \cdot y}_{\pi(p)} + (1-\lambda) \underbrace{p \cdot y'}_{\pi(p)} + \underbrace{\epsilon \cdot (p_1 + \dots + p_L)}_{>0}$$

$$> \pi(p)$$

$\Rightarrow$ Contradiction.

$\Rightarrow$ The supply correspondence is single - valued.

Ex 13

1) The transformation function:

$$t : \mathbb{R}_- \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$t(y_1, y_2) = y_2 - f(-y_1)$$

$$\Leftrightarrow t(y_1, y_2) = y_2 - \alpha\sqrt{-y_1}$$

(PMP): $\operatorname{Max} \{ p_1 y_1 + p_2 y_2 \}$

st:

$$\begin{cases} y_2 - \alpha\sqrt{-y_1} \leq 0 \\ y_1 \leq 0. \end{cases}$$

2) Assume that $\bar{y} = (\bar{y}_1, \bar{y}_2) \in s(p)$

$p \gg 0$

$$\Rightarrow \begin{cases} \bar{y}_1 \leq 0 \\ \bar{y}_2 \leq \alpha\sqrt{-\bar{y}_1} \end{cases}$$

If $\bar{y}_1 = 0$

$$\Rightarrow \bar{y}_2 \leq 0$$

$$\pi(p) = p \cdot \bar{y} = p_2 \bar{y}_2 \leq 0$$

Choose : $y = \left(y_1 , \alpha \sqrt{-y_1} \right)$ with $-\left(\frac{p_2 \alpha}{p_1}\right)^2 < y_1 < 0$

It is immediate that $y \in \mathcal{Y}$

and $p \cdot y = p_1 y_1 + p_2 \alpha \sqrt{-y_1}$

$\Leftrightarrow \quad p \cdot y = \sqrt{-y_1} \left[p_2 \alpha - p_1 \sqrt{-y_1} \right] > 0$

$\Rightarrow \left(\bar{y}_1 , \bar{y}_2 \right)$ is not optimal.

$\Rightarrow \quad \boxed{\bar{y}_1 < 0.}$

We have :

$$\bar{y}_2 \leq \alpha \sqrt{-\bar{y}_1}$$

If $\bar{y}_2 = 0$

$p \cdot \bar{y} = p_1 \bar{y}_1$

Choose : $y = \left(\bar{y}_1 , \alpha \sqrt{-\bar{y}_1} \right) \in \mathcal{Y}$

$$p \cdot y = p_1 \bar{y}_1 + p_2 \alpha \sqrt{-\bar{y}_1}$$

$$> p_1 \bar{y}_1 = p \cdot \bar{y}$$

$\Rightarrow$ Contradictory,

$\Rightarrow$ $\bar{y}_2 > 0$.

3) $A = \{ y = (-z, y_2) \mid z > 0$ and $y_2 > 0 \}$.

(PMP) : Max $p \cdot (-z, y_2)$

s.t.:

$z > 0$, $y_2 > 0$

and

$y_2 \leq \alpha \sqrt{z}$

$$\mathcal{L} = -p_1 z + p_2 y_2 + \lambda [\alpha \sqrt{z} - y_2]$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial z} = -p_1 + \frac{\lambda \alpha}{2\sqrt{z}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = p_2 - \lambda = 0$$

$$\lambda . [\alpha \sqrt{z} - y_2) = 0$$

---

4) $\lambda = p_2 > 0$

$$\Rightarrow \frac{p_2 \alpha}{2\sqrt{z}} = p_1$$

$$\Rightarrow z = \left(\frac{p_2 \alpha}{2 p_1}\right)^2$$

$$\alpha \sqrt{z} - y_2 = 0$$

$$\Rightarrow y_2 = \alpha \sqrt{z} = \frac{p_2 \alpha^2}{2 p_1}$$

$$\Rightarrow \quad s(p) = \left\{ \left(-\left(\frac{p_2 \alpha}{2 p_1}\right)^2, \frac{p_2 \alpha^2}{2 p_1} \right) \right\}$$

$$\pi(p) = p \cdot s(p)$$

$$= -p_1 \cdot \left(\frac{p_2 \alpha}{2 p_1}\right)^2 + p_2 \cdot \frac{p_2 \alpha^2}{2 p_1}$$

$$= -\frac{p_2^2 \alpha^2}{4 p_1} + \frac{p_2^2 \alpha^2}{2 p_1}$$

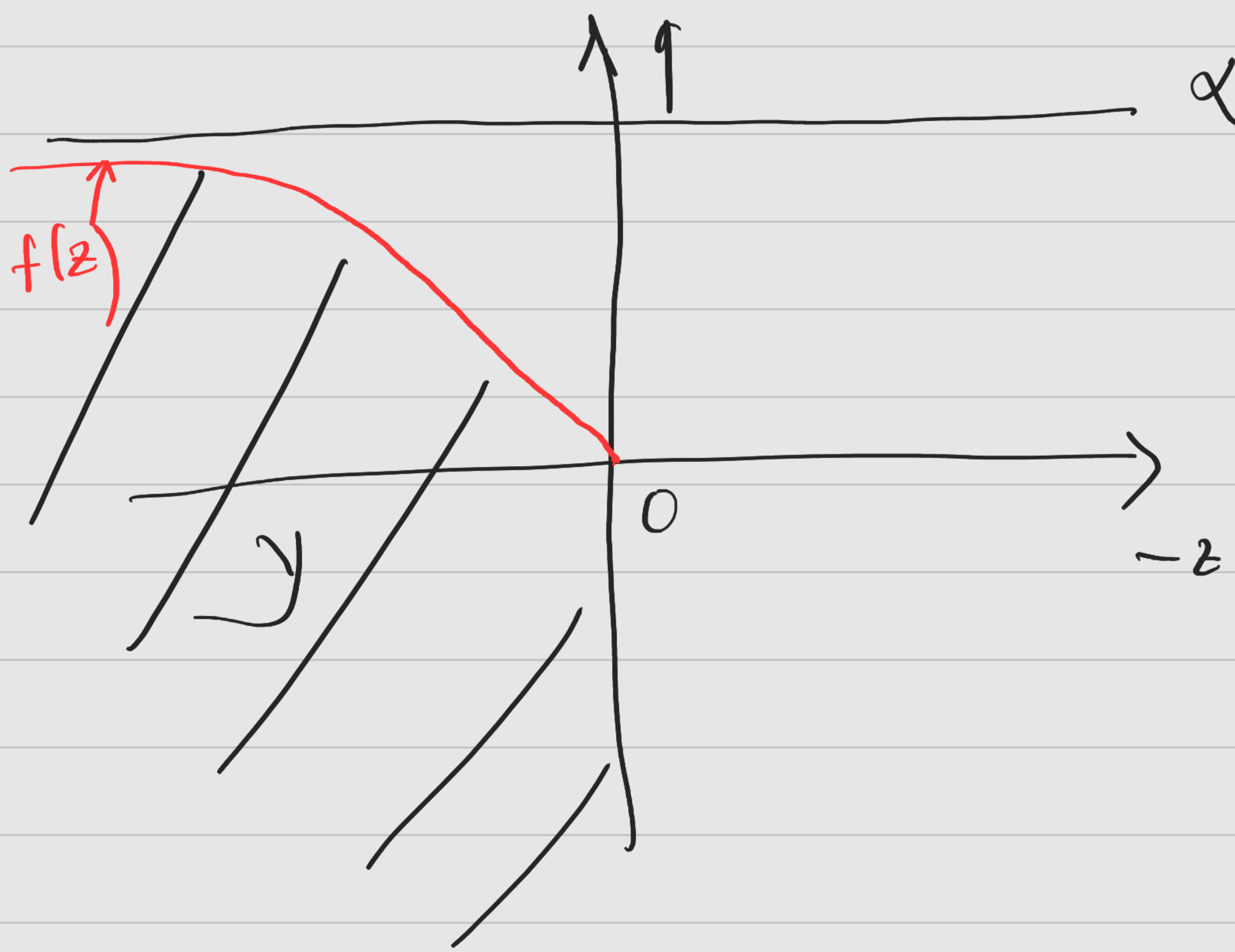
$$\boxed{\pi(p) = \frac{p_2^2 \alpha^2}{4 p_1}}$$

Ex 14 :

1) $$f(z) = \alpha\left(1 - e^{-kz}\right)$$

$$\mathcal{Y} = \{(-z, q) \mid z \geq 0 \text{ and } q \leq f(z)\}$$

$$\Leftrightarrow \mathcal{Y} = \{(-z, q) \mid z \geq 0 \text{ and } q \leq \alpha\left(1 - e^{-kz}\right)\}$$



Given $\bar{y}_2 \geqslant 0$.

$$\mathcal{Y}(\bar{y}_2) := \{ z \in \mathbb{R} : z \geqslant 0 \text{ and } f(z) \geqslant \bar{y}_2 \}$$

$$*) \quad f(z) \geqslant \bar{y}_2.$$

$$\Leftrightarrow \quad \alpha(1 - e^{-kz}) \geqslant \bar{y}_2.$$

$$\Leftrightarrow \quad \alpha - \bar{y}_2 \geqslant \alpha e^{-kz} \quad (1)$$

If: $\bar{y}_2 \geqslant \alpha$

then $\mathcal{Y}(\bar{y}_2) = \emptyset$

If: $\bar{y}_2 < \alpha$

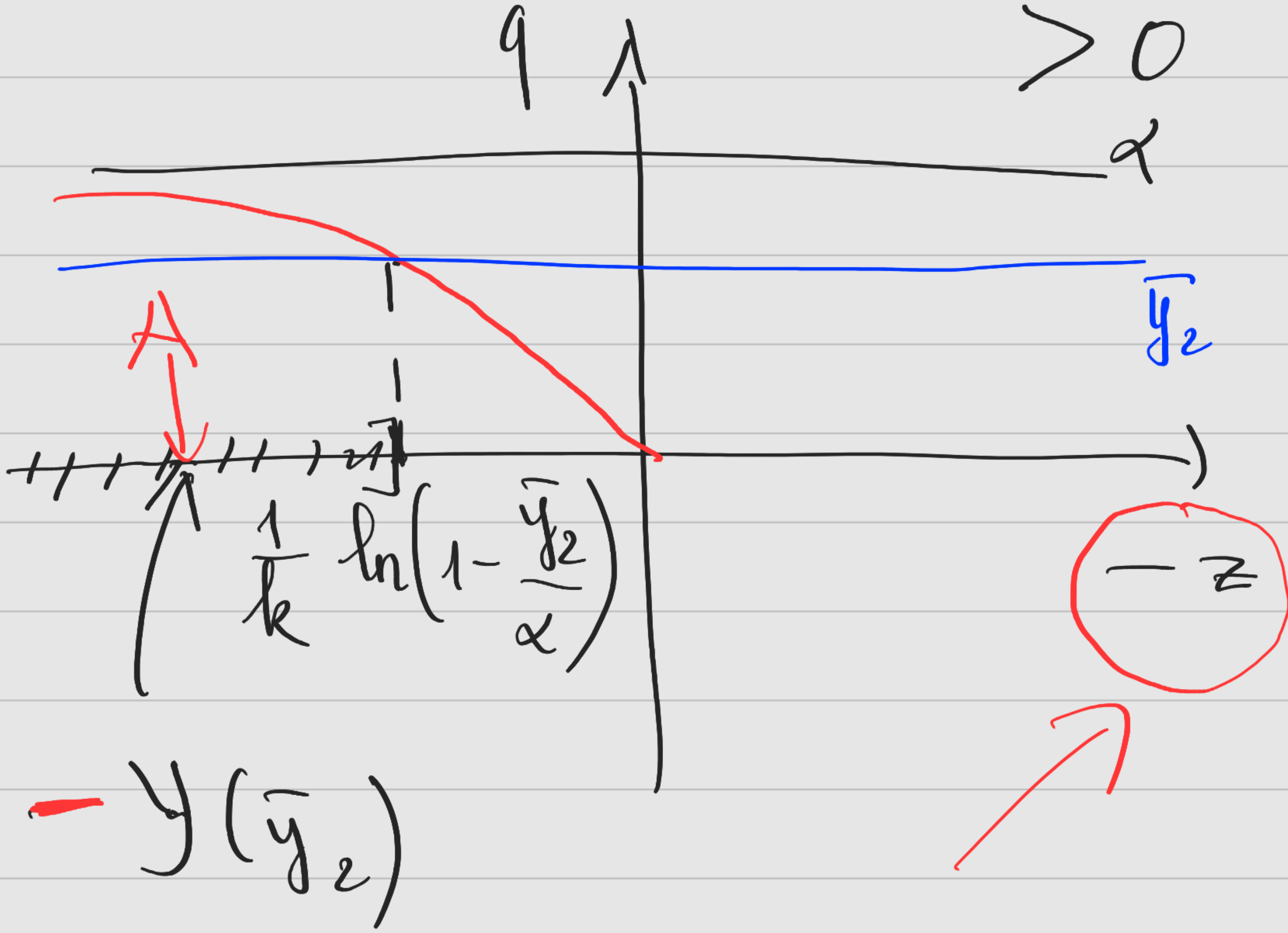
$$(1) \Leftrightarrow \underbrace{\frac{\alpha - \bar{y}_2}{\alpha}}_{>0} \geqslant \underbrace{e^{-kz}}_{>0}$$

$$\Leftrightarrow \quad \ln \frac{\alpha - \bar{y}_2}{\alpha} \geq \ln e^{-kz}$$

$$\Leftrightarrow \ln \frac{\alpha - \bar{y}_2}{\alpha} \geq -kz$$

$$\Leftrightarrow \quad \frac{1}{k} \ln \frac{\alpha - \bar{y}_2}{\alpha} \geq -z$$

$$\Leftrightarrow \quad z \geq \underbrace{-\frac{1}{k} \ln\left(1 - \frac{\bar{y}_2}{\alpha}\right)}_{>0}$$



$(CMP)(p_1, \bar{y}_2)$:

$$\text{Min } p_1 \cdot z$$

st :

$$f(z) \geq \bar{y}_2$$

$$\Leftrightarrow \quad \text{Min } p_1 \cdot z$$

$$\text{st: } z \in Y(\bar{y}_2)$$

Recall :

$$Y(\bar{y}_2) = \begin{cases} \emptyset & \text{if } \bar{y}_2 \geq \alpha \\ \left[-\frac{1}{k} \ln\left(1 - \frac{\bar{y}_2}{\alpha}\right), +\infty\right) & \text{if } 0 \leq \bar{y}_2 < \alpha \end{cases}$$

If $\bar{y}_2 \geq \alpha$

$\Rightarrow$ the domain of the (CMP) is empty.

If $\bar{y}_2 < \alpha$

$$\Rightarrow Y(\bar{y}_2) = \left[-\frac{1}{k} \ln\left(1 - \frac{\bar{y}_2}{\alpha}\right), +\infty \right)$$

(CMP) :

$$\text{Min } z$$
$$\text{st} \quad z \geq -\frac{1}{k} \ln\left(1 - \frac{\bar{y}_2}{\alpha}\right)$$

$$\Rightarrow z(p_1, \bar{y}_2) = -\frac{1}{k} \ln\left(1 - \frac{\bar{y}_2}{\alpha}\right)$$

$$c(p_1, \bar{y}_2) = -p_1 \frac{1}{k} \ln\left(1 - \frac{\bar{y}_2}{\alpha}\right)$$