Ex 22

$u_1(x_{11}, x_{12}) = x_{11} + x_{12}$

$u_2(x_{21}, x_{22}) = a x_{21} + b x_{22}$.

$e_1 = (2, 2)$, $e_2 = (2, 1)$

1 Definition of a Pareto optimal allocat°

$\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a P.O.A iff

(i) $\bar{x}$ is a feasible allocation:

$$\bar{x}_1 + \bar{x}_2 = e_1 + e_2$$

(ii) There doesn't exist any feasible allocation $x' = (x_1', x_2')$ such that.

$\forall\ i = 1,2 \ : \ u_i(x_i') \geqslant u_i(\bar{x}_i)$

and $\exists\ j \in \{1,2\}$ st : $u_j(x_j') > u_j(\bar{x}_j)$

2) $a = b = 1$

- We know that the set of P.O.A is a subset of the set of feasible allocation (F.A).

$$POA \subset FA$$

- Now we have to prove:

$$FA \subset POA$$

Let $x = (x_1, x_2) \in FA$,

i.e, $x_1 + x_2 = e_1 + e_2$

We have to prove that x is a P.O.A

Assume that x is NOT a P.O.A.

There exists $x' \in FA$ such that

$$u_i(x'_i) \geq u_i(x_i) \quad \forall i = 1,2$$

$\exists\, j \in \{1,2\}$ such that

$$u_j(x'_j) > u_j(x_j)$$

$$\Leftarrow) \quad x'_{i1} + x'_{i2} \geq x_{i1} + x_{i2}, \quad \forall i = 1,2$$

$$x'_{j1} + x'_{j2} > x_{j1} + x_{j2}$$

$$\Rightarrow (x'_{11} + x'_{12}) + (x'_{21} + x'_{22})$$

$$> (x_{11} + x_{12}) + (x_{21} + x_{22})$$

$\Leftrightarrow (x'_{11} + x'_{21}) + (x'_{12} + x'_{22})$

$> (x_{11} + x_{21}) + (x_{12} + x_{22})$

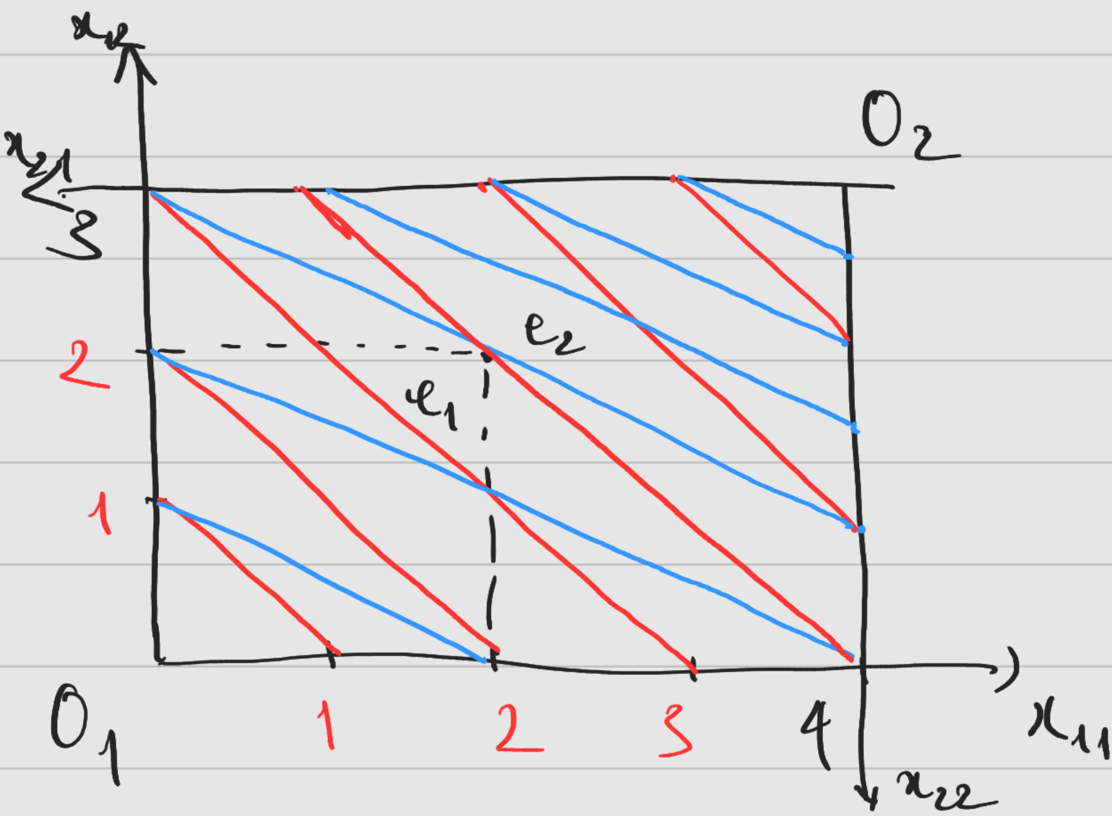
$\Leftrightarrow \quad 4 + 3 > 4 + 3$ (Impossible !)

$\Rightarrow x$ is a POA

$\Rightarrow$ PA = POA.

3) $a = 1$, $b = 2$

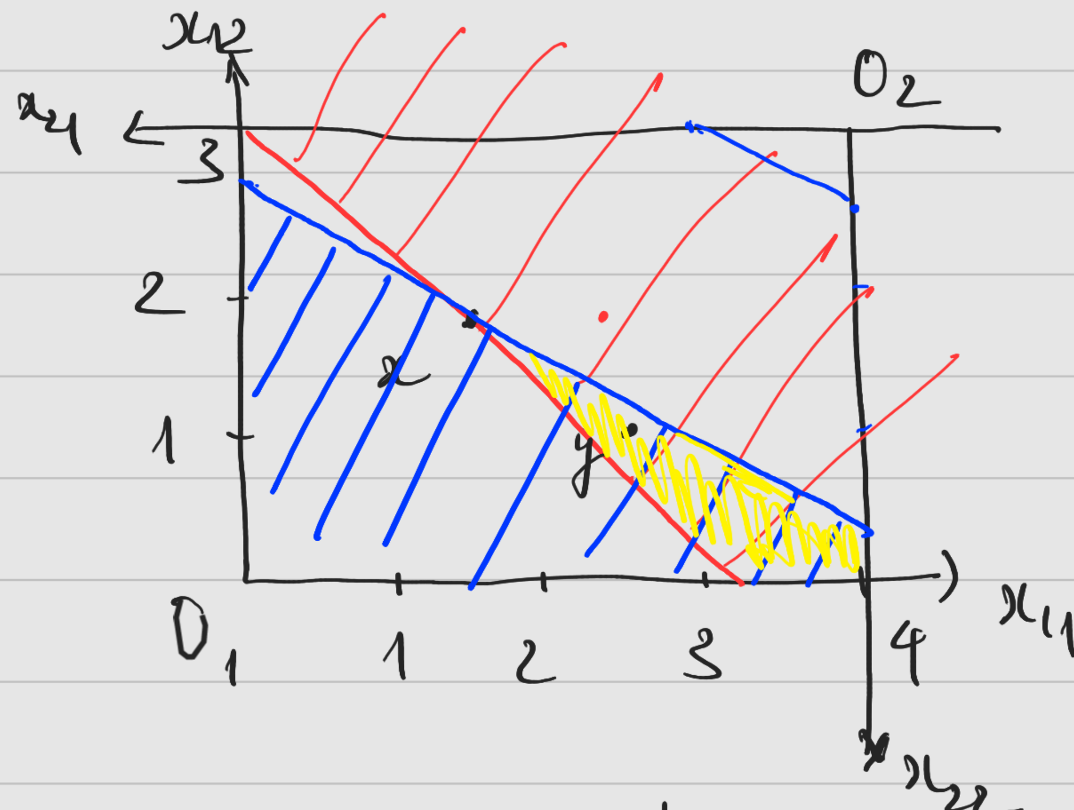
$u_2(x_{21}, x_{22}) = x_{21} + 2x_{22}$.



red lines : indifferent curves of consumer 1

blue lines : indifferent curves of consumer 2

4)



Consider a feasible allocation x that belongs to the interior of the Edgeworth box.

- The yellow area is the intersection of the upper contour sets of both consumers
- Any point in the interior of the yelow area is a strict improvemen

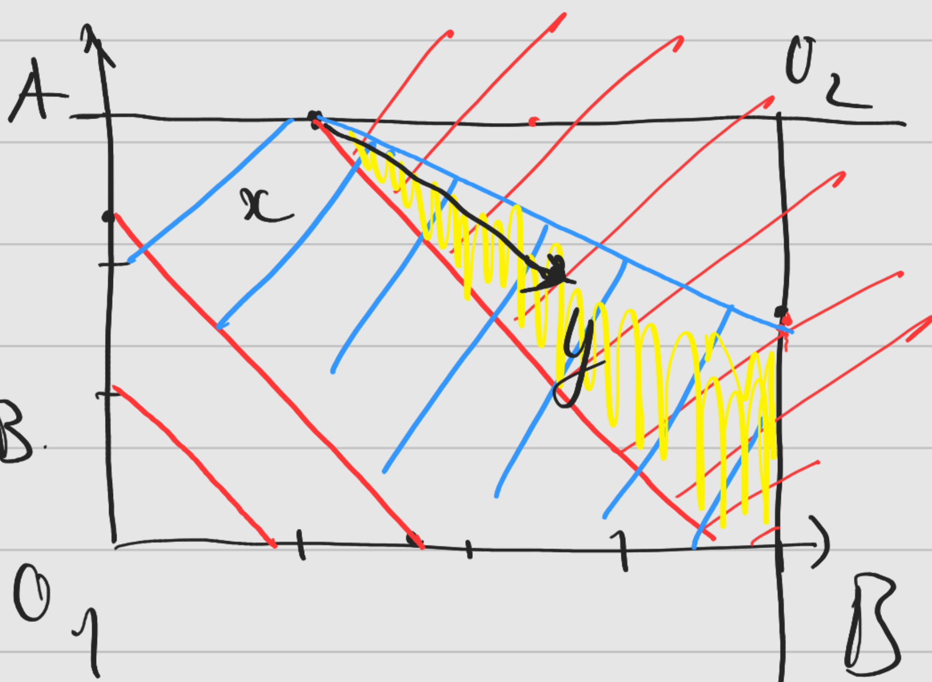
of x. (e.g. point y)

$\Rightarrow$ x is NOT a POA.

$\Rightarrow$ All interior points of EB are NOT POA.

- If $x \in \underline{bd(EB)}$

boundary of the EB.

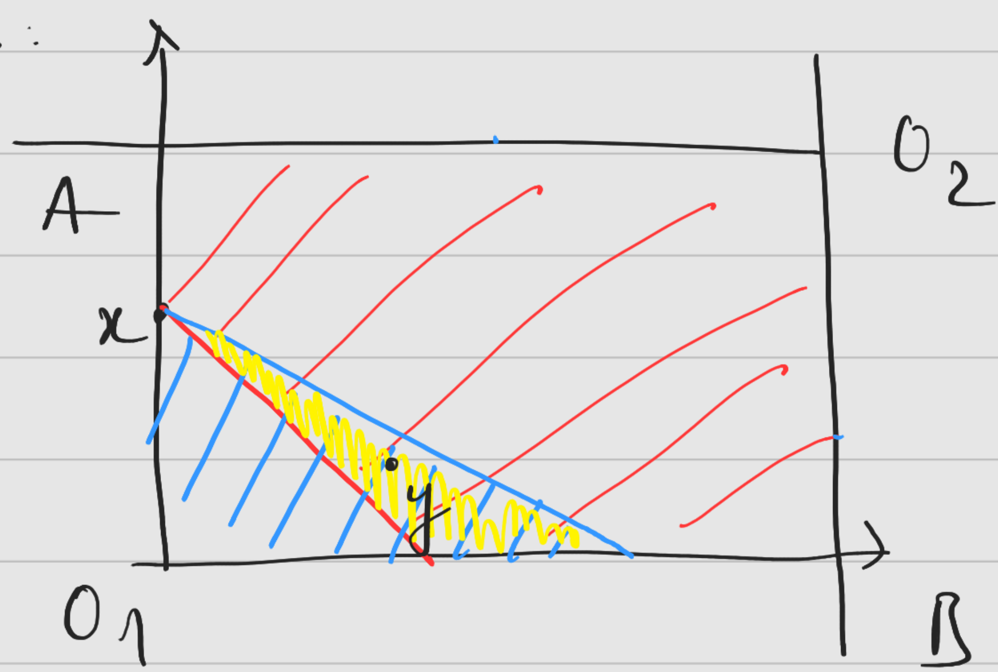


Case 1: $x \in O_2 A$.

x is not a POA.

Case 2:

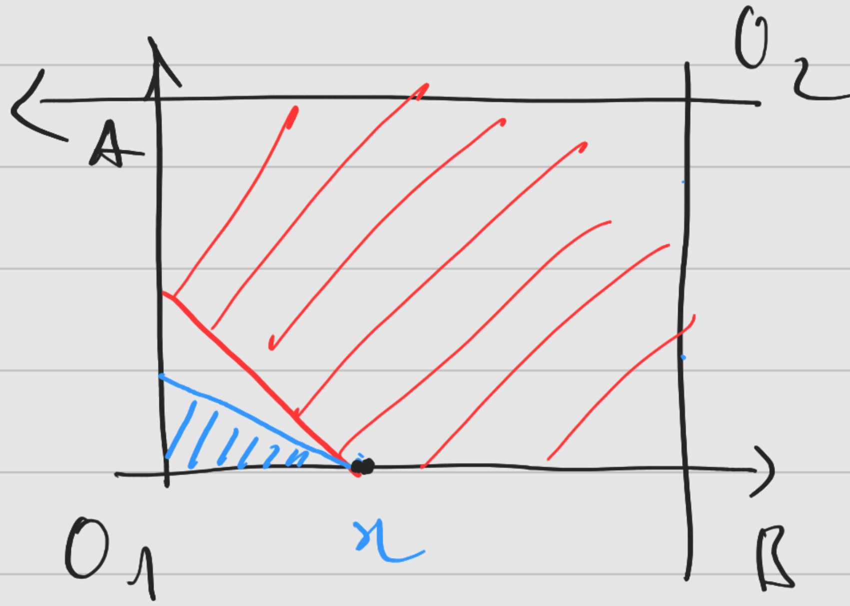
$x \in AO_1$



x is not a POA.

Case 3:

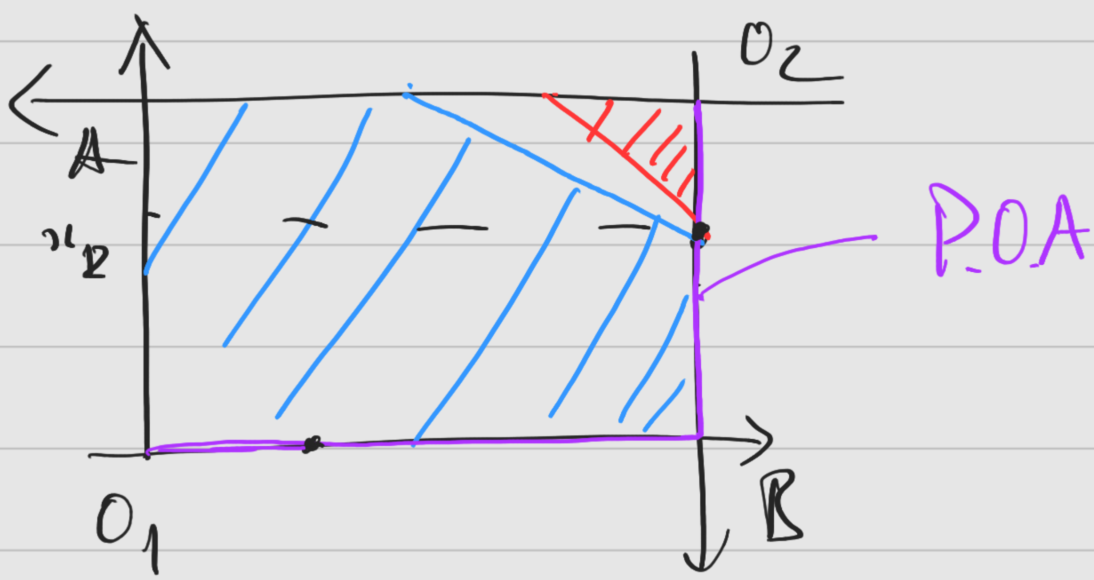
$x \in O_1B$



x is a POA

Case 4:

$x \in O_2B$



x is a POA

The set of POA is

$$\{(x_u, 0), (4 - x_u, 3) \mid x_u \in [0,4]\}$$

$$\cup \left\{ (4, x_{12}), (0, 3 - x_{12}) \mid x_{12} \in [0,3] \right\}$$

Ex 25 :

$$u_1(x_{11}, x_{12}) = \frac{1}{3} \ln x_{11} + \frac{2}{3} \ln x_{12}$$

$$u_2(x_{21}, x_{22}) = \frac{1}{4} \ln x_{21} + \frac{3}{4} \ln x_{22}$$

1 Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be a PO* of the economy.

It is straightforward that $\bar{x} \gg 0$

There exists $\lambda > 0$ such that

$$\begin{cases} \nabla u_1(\bar{x}_1) = \lambda \nabla u_2(\bar{x}_2) \\ \bar{x}_1 + \bar{x}_2 = e_1 + e_2 = (2,2) \end{cases}$$

$$\boxed{\nabla u_1(\bar{x}_1) = \left(\frac{1}{3\bar{x}_{11}}, \frac{2}{3\bar{x}_{12}} \right)}$$

$$\nabla u_2(\bar{x}_2) = \left(\frac{1}{4\bar{x}_{21}}, \frac{3}{4\bar{x}_{22}} \right)$$

$$\Rightarrow \begin{cases} \dfrac{1}{3\,\bar{x}_{11}} = \lambda \cdot \dfrac{1}{4\,\bar{x}_{21}} \\[2ex] \dfrac{2}{3\,\bar{x}_{12}} = \lambda\,\dfrac{3}{4\,\bar{x}_{22}} \end{cases}$$

$$\Rightarrow \quad \frac{\bar{x}_{12}}{2\,\bar{x}_{11}} = \frac{\bar{x}_{22}}{3\,\bar{x}_{21}} \qquad (*)$$

We have:

$$\begin{cases} \bar{x}_{11} + \bar{x}_{21} = 2 \\ \bar{x}_{12} + \bar{x}_{22} = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \bar{x}_{21} = 2 - \bar{x}_{11} \\ \bar{x}_{22} = 2 - \bar{x}_{12} \end{cases}$$

$$(*) \Leftrightarrow \frac{\bar{x}_{12}}{2\,\bar{x}_{11}} = \frac{2 - \bar{x}_{12}}{3\,(2 - \bar{x}_{11})}$$

$$\Leftrightarrow \bar{x}_{12}\,(6 - 3\bar{x}_{11}) = \bar{x}_{11}\,(4 - 2\bar{x}_{12})$$

$$\Leftrightarrow 6\,\bar{x}_{12} - 3\,\bar{x}_{11}.\bar{x}_{12} = 4\bar{x}_{11} - 2\,\bar{x}_{11}.\bar{x}_{12}$$

$$\Leftrightarrow \quad 6\bar{x}_{12} = 4\bar{x}_{11} + \bar{x}_{11} \cdot \bar{x}_{12}$$

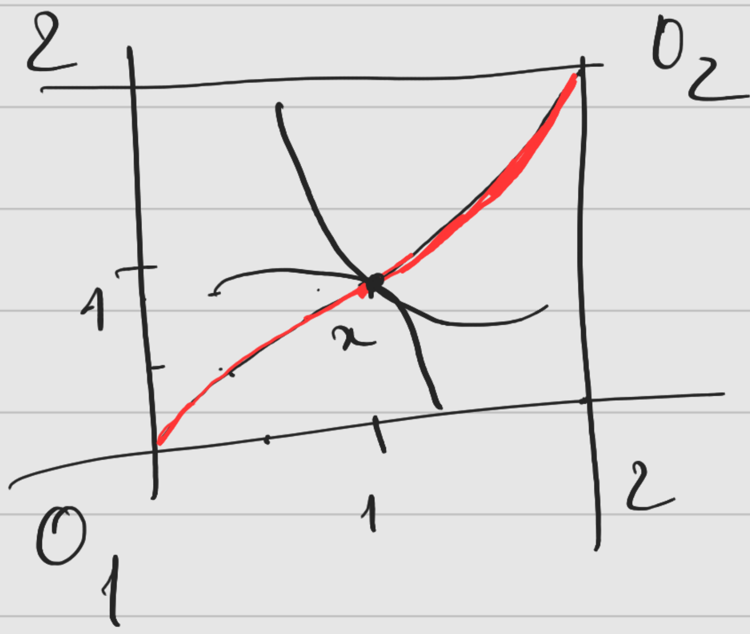
$$\Leftrightarrow \quad \bar{x}_{12}(6 - \bar{x}_{11}) = 4\bar{x}_{11}$$

$$\Leftrightarrow \quad \bar{x}_{12} = \frac{4\bar{x}_{11}}{6 - \bar{x}_{11}}$$

$$\Rightarrow \quad \bar{x}_1 = \left(\bar{x}_{11}, \frac{4\bar{x}_{11}}{6 - \bar{x}_{11}}\right)$$

$$\bar{x}_2 = \left(2 - \bar{x}_{11}, 2 - \frac{4\bar{x}_{11}}{6 - \bar{x}_{11}}\right)$$

$$POA = \left\{ \left(\bar{x}_{11}, \frac{4\bar{x}_{11}}{6 - \bar{x}_{11}}\right); \left(2 - \bar{x}_{11}, 2 - \frac{4\bar{x}_{11}}{6 - \bar{x}_{11}}\right), \ \forall \bar{x}_{11} \in (0, 2) \right\}$$



2) let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be a POA st. $\bar{x}_{11} = \bar{x}_{21}$

We have : $\bar{x}_{21} = 2 - \bar{x}_{11} = \bar{x}_{11}$

$\Rightarrow \bar{x}_{11} = \bar{x}_{21} = 1$

$\Rightarrow \bar{x}_1 = \left(1, \frac{4}{5}\right)$

$\bar{x}_2 = \left(1, \frac{6}{5}\right)$

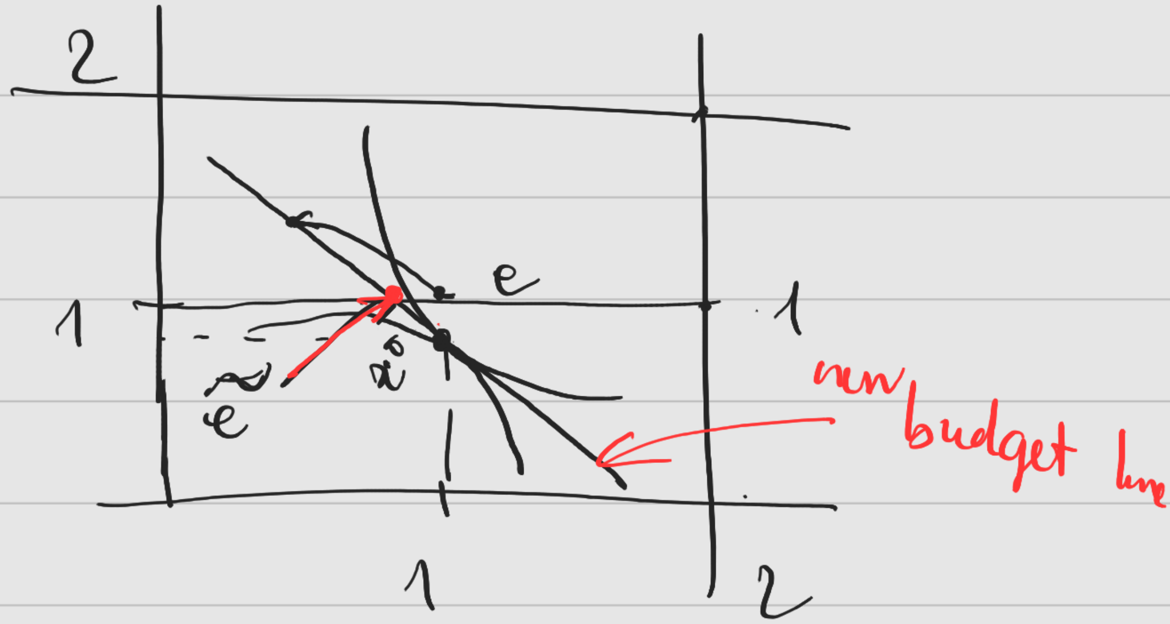
3) $x^* = \bar{x} = \left(\left(1, \frac{4}{5}\right), \left(1, \frac{6}{5}\right)\right)$

let us call the transfer of commodity 1 : t

$e_1 \longrightarrow (1+t, 1) = \tilde{e}_1$

$e_2 \longrightarrow (1-t, 1) = \tilde{e}_2$

Find t such that: x^* is a C.E.



We have

$$p^* = \nabla u_1(x_1^*)$$

$$\Leftrightarrow p^* = \left(\frac{1}{3}, \frac{8}{15}\right)$$

We must have:

$$\boxed{p^* \tilde{e}_i = p^* \cdot x_i^* \quad \forall i = 1,2}$$

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$$\Leftrightarrow \begin{cases} \frac{1}{3}\cdot(1+t) + \frac{8}{15} = \frac{1}{3} + \frac{8}{15}\cdot\frac{4}{5} \\ \frac{1}{3}(1-t) + \frac{8}{15} = \frac{1}{3} + \frac{8}{15}\cdot\frac{6}{5} \end{cases}$$

$\Rightarrow \boxed{t = \ ?}$