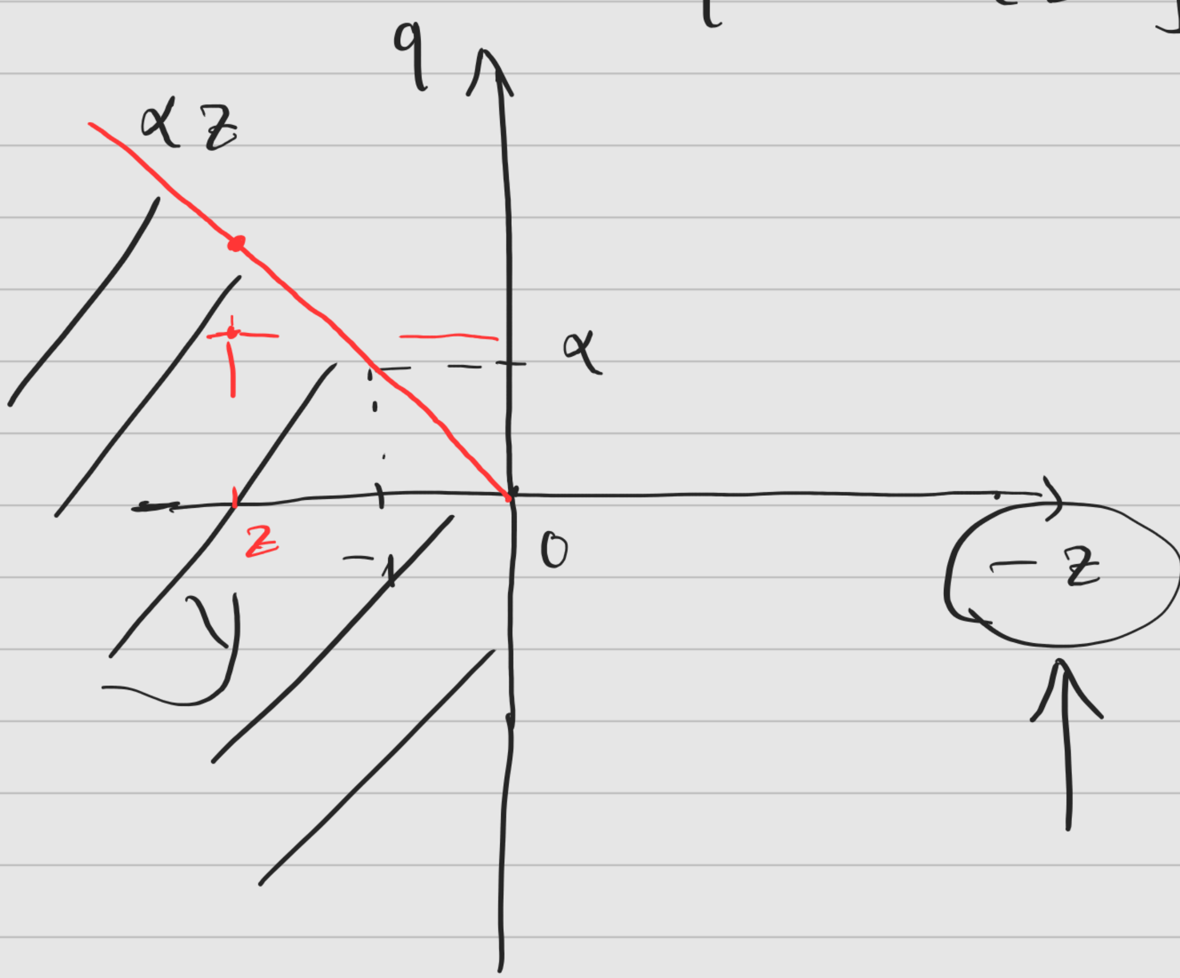
Ex 8: (Problem sets)

1 $\quad Y = \{ (-z, q) \mid z \geq 0 \text{ and } q \leq f(z) \}$

$f = \alpha z \quad$ for $\alpha > 0$ and $z \geq 0$

$\Rightarrow Y = \{ (-z, q) \mid z \geq 0 \text{ and } q \leq \alpha z \}$



2) Some basic properties of production sets.

① Possibility of inaction : $0 \in Y$

② Impossibility of free production :

$$Y \cap \mathbb{R}^L_+ \subset \{0\}$$

③ Irreversibility : $Y \cap -Y \subset \{0\}$

$$(-1, 3) \in Y \Rightarrow (+1, -3) \notin Y$$

④. Free disposability

$$Y - \mathbb{R}^L_+ \subset Y$$

If $(y_1, \dots, y_L) \in Y$

$$\Rightarrow (y_1 - \varepsilon_1, \dots, y_L - \varepsilon_L) \in Y$$

⑤ Convexity : Y is convex

⑥. Closedness : Y is a closed subset of $\mathbb{R}^L$.

$$Y = \left\{ (-z, q) \;\middle|\; \begin{array}{l} z \geq 0 \text{ and} \\ q \leq \alpha z \end{array} \right\}$$

*) It is clear that $0 \in Y$

$\Rightarrow$ Y satisfies ①

*) let $(-z, q) \in Y \cap \mathbb{R}^2_+$

$$\Rightarrow \begin{cases} (z, q) \in Y \\ (-z, q) \in \mathbb{R}^2_+ \end{cases}$$

$$\Leftrightarrow \begin{cases} z \geq 0 \text{ and } q \leq \alpha z \\ -z \geq 0 \text{ and } q \geq 0 \end{cases}$$

$\Rightarrow \quad z = 0$

$$\Rightarrow \quad \begin{cases} q \leq \alpha . 0 = 0 \\ q \geq 0 \end{cases}$$

$\Rightarrow \quad q = 0$

$\Rightarrow \quad Y \cap \mathbb{R}^2_+ = \{0\} \subseteq \{0\}$

$\Rightarrow$ Y satisfies ②

*) Let $(-z, q) \in Y \cap -Y$

$$\Leftrightarrow \begin{cases} (-z, q) \in Y \\ (z, -q) \in Y \end{cases}$$

$\Leftrightarrow \begin{cases} z \geq 0 & \text{and} \quad q \leq \alpha z. \\ -z \geq 0 & \text{and} \quad -q \leq \alpha(-z) \end{cases}$

$\Rightarrow \begin{cases} z = 0 \\ q = 0 \end{cases}$

$\Rightarrow \mathcal{Y} \cap -\mathcal{Y} = \{0\} \subseteq \{0\}$

$\Rightarrow \mathcal{Y}$ satisfies 3

*) Let $\begin{cases} (-z, q) \in \mathcal{Y} \\ (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+ \end{cases}$

$(-z - \varepsilon_1, q - \varepsilon_2) \in \mathcal{Y}$?

We have:

$$q - \varepsilon_2 \leq q \leq \alpha z \leq \alpha(z + \varepsilon_1)$$

$\Rightarrow (-z - \varepsilon_1, q - \varepsilon_2) \in \mathcal{Y}$

$\Rightarrow \mathcal{Y}$ satisfies 4

* Let $(-z, q) \in \mathcal{Y}$ and $(-z', q') \in \mathcal{Y}$ and

Check: $\lambda(-z, q) + (1-\lambda)(-z', q')$

$\Leftrightarrow \left(-\left[\lambda z + (1-\lambda) z'\right], \lambda q + (1-\lambda) q'\right) \in Y$

$\Leftrightarrow \begin{cases} \lambda z + (1-\lambda) z' \geq 0 \\ \lambda q + (1-\lambda) q' \leq \alpha \cdot \left[\lambda z + (1-\lambda) z'\right] \end{cases}$

$\lambda z + (1-\lambda) z' \geq 0$ holds true since

$z \geq 0$ and $z' \geq 0$.

$\begin{cases} q \leq \alpha z \\ q' \leq \alpha z' \end{cases} \Rightarrow \lambda q + (1-\lambda) q' \leq \lambda \alpha z + (1-\lambda) \alpha z'$

$\Leftrightarrow \lambda q + (1-\lambda) q' \leq \alpha \left[\lambda z + (1-\lambda) z'\right]$

holds true

$\Rightarrow Y$ is convex

* Let $\{(-z^n, q^n)\}_{n \in \mathbb{N}} \subset Y$

and $(-z^n, q^n) \xrightarrow{n \to +\infty} (-z, q)$

We have:

$$\begin{cases} z^n \geq 0 & \forall n \\ q^n \leq \alpha z^n & \forall n \end{cases}$$

$$\Rightarrow \begin{cases} z \geq 0 \\ q \leq \alpha z. \end{cases}$$

$\Rightarrow$ $(-z, q) \in Y$

$\Rightarrow$ Y is closed.

$$f(z) = \alpha \sqrt{z}$$
$$f(z) = \alpha z^2 + \beta z$$

Exercises for you

$\Uparrow$

Do it and submit it to me ~~x~~ before this Sunday.

Ex 9: $L = 3$

The production function:

$$f(z_1, z_2) = z_1^{\alpha} \cdot z_2^{\beta} \quad , \alpha, \beta > 0$$

$$z_1, z_2 \geq 0$$

1. $Y = \{ (-z_1, -z_2, q) \mid z_1 \geq 0, z_2 \geq 0 \text{ and } q \leq f(z_1, z_2) \}$

$\Leftrightarrow Y = \{ (-z_1, -z_2, q) \mid z_1, z_2 \geq 0 \text{ and } q \leq z_1^{\alpha} \cdot z_2^{\beta} \}$

2) *) $0 \in Y$ because:

$0 \geq 0$, $0 \geq 0$ and $0 \leq 0^{\alpha} \cdot 0^{\beta}$

$\Rightarrow$ Y satisfies possibility of inaction

- Let $(-z_1, -z_2, q) \in Y \cap \mathbb{R}^3_+$

$\Leftrightarrow \begin{cases} (-z_1, -z_2, q) \in Y \\ (-z_1, -z_2, q) \in \mathbb{R}^3_+ \end{cases}$

$\Rightarrow \begin{cases} z_1 \geq 0, \; z_2 \geq 0 \text{ and } q \leq z_1^\alpha \cdot z_2^\beta \\ -z_1 \geq 0, \; -z_2 \geq 0, \; q \geq 0 \end{cases}$

$\Rightarrow \quad z_1 = 0, \; z_2 = 0 \text{ and } q = 0$

$\Rightarrow \quad Y \cap \mathbb{R}^3_+ = \{0\} \subseteq \{0\}.$

$\Rightarrow$ Y satisfies impossibility of free production

* let $(-z_1, -z_2, q) \in Y \cap -Y$

$\Leftrightarrow \begin{cases} (-z_1, -z_2, q) \in Y \\ (z_1, z_2, -q) \in Y \end{cases}$

$\Leftrightarrow \begin{cases} z_1 \geq 0, \; z_2 \geq 0 \text{ and } q \leq z_1^\alpha \cdot z_2^\beta \\ -z_1 \geq 0, \; -z_2 \geq 0 \text{ and } -q \leq (-z_1)^\alpha \cdot (-z_2)^\beta \end{cases}$

$\Rightarrow \quad z_1 = 0, \; z_2 = 0 \text{ and } q = 0$

$\Rightarrow$ Y satisfies irreversibility.

* let $(-z_1, -z_2, q) \in Y$
and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3_+$.

We have to check:

$$(-z_1 - \varepsilon_1, -z_2 - \varepsilon_2, q - \varepsilon_3) \in Y,$$

$$\Leftrightarrow \left\{ \begin{array}{l} z_1 + \varepsilon_1 \geq 0, \quad z_2 + \varepsilon_2 \geq 0 \\ q - \varepsilon_3 \leq (z_1 + \varepsilon_1)^\alpha \cdot (z_2 + \varepsilon_2)^\beta \end{array} \right.$$

Since: $z_1, z_2 \geq 0$, $z_1 + \varepsilon_1 \geq 0$
and $z_2 + \varepsilon_2 \geq 0$.

$$q - \varepsilon_3 \leq q \leq z_1^\alpha z_2^\beta \leq (z_1 + \varepsilon_1)^\alpha \cdot (z_2 + \varepsilon_2)^\beta$$

$$\Rightarrow (-z_1 - \varepsilon_1, -z_2 - \varepsilon_2, q - \varepsilon_3) \in Y$$

$\Rightarrow$ Y satisfies Free disposability.

*) Let $(-z_1, -z_2, q) \in Y$ and $(-z_1', -z_2', q') \in Y$

+) $\begin{cases} \lambda z_1 + (1-\lambda) z_1' \geq 0 \\ \lambda z_2 + (1-\lambda) z_2' \geq 0 \end{cases}$

Check :

$$\lambda q + (1-\lambda) q' \leq f(\lambda z_1 + (1-\lambda) z_1', \lambda z_2 + (1-\lambda) z_2') \quad ?$$

$$\Leftarrow \lambda q + (1-\lambda) q' \leq (\lambda z_1 + (1-\lambda) z_1')^{\alpha} \cdot (\lambda z_2 + (1-\lambda) z_2')^{\beta}$$

We have :

$$\begin{cases} q \leq z_1^{\alpha} z_2^{\beta} \\ q' \leq (z_1')^{\alpha} (z_2')^{\beta} \end{cases}$$

If $\alpha + \beta \leq 1$, then $f(\cdot,\cdot)$ is concave

Then, $f(\lambda z_1 + (1-\lambda) z_1', \lambda z_2 + (1-\lambda) z_2')$

$\geqslant \lambda f(z_1, z_2) + (1-\lambda) f(z_1', z_2')$

$\Rightarrow (\lambda z_1 + (1-\lambda) z_1')^{\alpha} \cdot (\lambda z_2 + (1-\lambda) z_2')^{\beta}$

$\geqslant \lambda z_1^{\alpha} z_2^{\beta} + (1-\lambda)(z_1')^{\alpha}(z_2')^{\beta}$

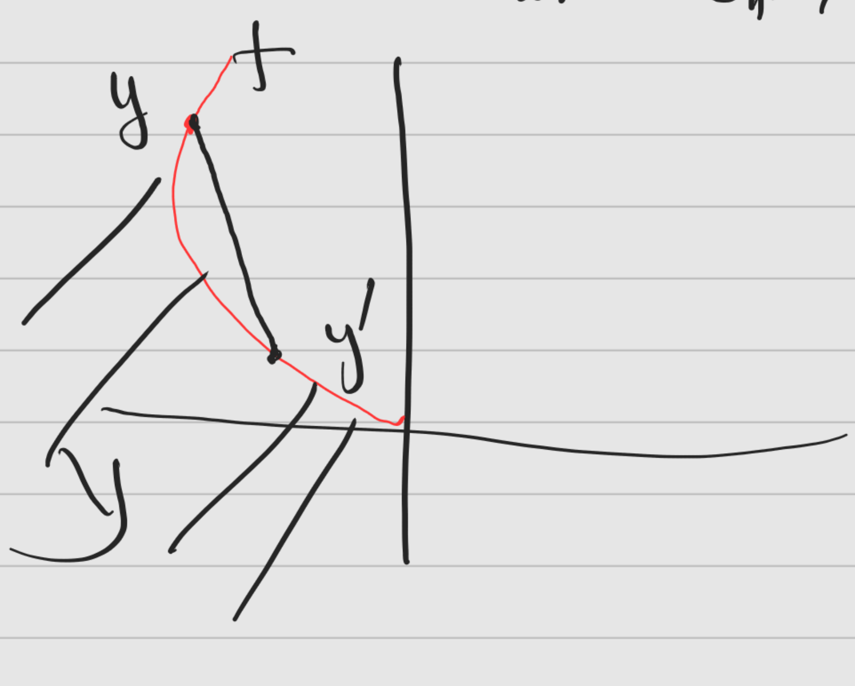
$\geqslant \lambda q + (1-\lambda) q'$

$\Rightarrow \Big(-(\lambda z_1 + (1-\lambda) z_1'), -(\lambda z_2 + (1-\lambda) z_2'),$

$\lambda q + (1-\lambda) q' \Big) \in Y$

$\Rightarrow$ Y is convex

+) If $\alpha + \beta > 1$, then $f(\cdot,\cdot)$ is neither concave nor convex.



In general, $\mathcal{Y}$ is not convex.

f) Let $(-z_1^n, -z_2^n, q^n) \subset \mathcal{Y}$

and $(-z_1^n, -z_2^n, q^n) \longrightarrow (-z_1, -z_2, q)$

We have: $\begin{cases} z_1^n \geq 0 \ \forall \ n \\ z_2^n \geq 0 \ \forall \ n \\ q^n \leq f(z_1^n, z_2^n) \ \forall n \end{cases}$

Let $n \to \infty$

$\begin{cases} z_1 \geq 0 \\ z_2 \geq 0 \\ q \leq f(z_1, z_2) \end{cases}$ (Since $f(\cdot,\cdot)$ is continuous)

$\Rightarrow (-z_1, -z_2, q) \in \mathcal{Y}$

$\mathcal{Y}$ is closed.